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THE

DOCTRINE
AND
APPLICATION
OF
FLUXIONS.

CONTAINING
(Beſides what is common on the Subject)
A Number of NEW IMPROVEMENTS
in the THEORY,

AND
The SOLUTION of a Variety of New, and very
Intereſting, Problems in different Branches of
the MATHEMATICKS.

PART I.

By THOMAS SIMPSON, F.R.S.

THE SECOND EDITION.
Reviſed and carefully corrected.

LONDON:
Printed for JOHN NOURSE, in the Strand,
BOOKSELLER TO HIS MAJESTY.
MDCCLXXVI.

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TO THE

RIGHT HONOURABLE

George Earl of *Macclesfield.*

MY LORD,

AS I eſteem it a very great Honour to be permitted to place the following Sheets under your Lordſhip's Protection, who are not only an Encourager of, but an Ornament to, Mathematical Learning; I have taken more than ordinary Pains, that, *What* is here uſhered into the World, with ſuch Advantage, may not be found altogether unworthy of ſo diſtinguiſhed a Patron.

1-4-35 McK

I am not vain enough to imagine, that, to One ſo deeply read in *theſe* abſtruſe and curious Speculations, as your Lordſhip is univerſally allowed to be, this Work will appear without Faults: But then, I have the Satiſfaction to think, on the other hand, that, whatever is Here to be met with capable of bearing the Teſt of an exact and ſolid Judgment, will *alſo* have its due Weight, and not fail of receiving your Lordſhip's Approbation: And if, upon the Whole, there is Merit enough found to entitle me to a favourable Reception, it will gratify the higheſt Ambition of,

MY LORD,

Your LORDSHIP'S

Moſt Obedient Humble Servant,

Tho. Simpſon,

each ſucceeding Moment, are greater and greater. Therefore the Fluxion muſt be leſs than any Space that can be deſcribed, in the given Time, when the Line increaſes. And, in the ſame Manner, the Fluxion will appear to be greater than any Space that can be deſcribed, in the ſame Time, when the Line decreaſes. *It* muſt, therefore, be equal to that Space, which will ariſe, when the Length of the generating Line, from the given *Poſition, is ſuppoſed* neither to increaſe nor decreaſe: Agreeable to *Art.* 4.

Thus much it ſeem'd proper to offer Here with regard to the Firſt Principles—I ſhall now proceed to ſay ſomething concerning the Order obſerv'd in treating, and putting together, the ſeveral Parts of the Work; wherein the Eaſe and Benefit of the younger Beginner have been particularly conſulted: To load ſuch an One with a *Multitude of* Rules and Precepts, before giving him any Taſte of their Uſe and Application, would, certainly, be very diſcouraging; and like obliging a Traveller to aſcend an high Mountain, without allowing him to ſtop by the Way, to take Breath, and refreſh his Spirits with a Proſpect of the agreeable and extenſive View he has to expect when he arrives at the Summit: I have therefore, after demonſtrating the Firſt Principles, proceeded immediately to exemplify their Utility in ſeveral entertaining Enquiries, before touching at all upon the Inverſe Method, or the more difficult Parts of the Direct. And, ſince that Branch of the Inverſe Method which treats of the Compariſon of Fluents is, naturally, ſomewhat difficult, it is referred to the Second Part of the Work, together with ſuch other Matters in the Theory as might appear, either, too tedious or hard to a Learner at firſt ſetting out. The like Care has been taken in the Diſpoſal of the reſt of the Work——As to the ſeveral Particulars whereof *It* is compoſed, I muſt refer to the Book itſelf, They being too many to be here enumerated: One Thing, however, I muſt not omit to take notice of, relating to that Part which treats of the aforeſaid Buſineſs of Fluents: To which it may, perhaps, be objected, That, notwithſtanding my having inſiſted ſo largely on the Subject, there are a Number of Forms of Fluxions and Fluents to be met with in Authors, that I have not ſo much as touch'd upon. This is granted; but then they are moſt of them ſuch as, I dare pronounce, can never ariſe in any Inquiry into Nature: And it would, doubtleſs, be Time and Labour miſapply'd, to ſwell the Work, and embarraſs the Learner with a Number of unneceſſary Difficulties, and empty Speculations; when what is, really, proper and uſeful, in the Subject, is ſufficient (it is well known) to exerciſe his utmoſt Attention and Reſolution.

I Cannot put an End to this Preface without acknowledging my Obligations to a ſmall Tract, in-

PREFACE.

HAVING, in the Year 1737, publiſhed a Piece, on this ſame Subject, under the Title of *A Treatiſe of Fluxions* (whereof the whole Impreſſion hath been long ſince ſold) it may be proper here, firſt of all, to aſſign the Reaſons why this Work is ſent abroad into the World as a New Book, rather than a Second Edition of the ſaid Treatiſe. Which, in ſhort, are theſe two: Firſt, becauſe the preſent Work is vaſtly more full and comprehenſive; and, ſecondly, becauſe the principal Matters in it which are alſo to be met with in that Treatiſe, are handled in a different Manner.

BESIDES the Preſs-Errors with which the ſaid Treatiſe abounds, there are ſeveral Obſcurities and Defects (which the Author's Want of Experience, and the many Diſadvantages he then laboured under, in his firſt Sally, may, it is hoped, in ſome meaſure excuſe.) But what is now offered to the Publick, being a Performance of more mature Conſideration and Judgment, it will, I flatter myſelf, be found much more correct, and claim a favourable Reception; eſpecially, as particular Care and Pains have been taken to put every Thing in a clear Light, and to oblige the lower, as well as the more experienced, Claſs of Readers.

The Notion and Explication *Here* given of the firſt Principles of Fluxions, are not eſſentially different from what they are in the abovementioned Treatiſe, tho' expreſſed in other Terms. The Conſideration of Time, which I have introduced into the General Definition, will, perhaps, be diſliked by *Thoſe* who would have Fluxions to be *meer Velocities*: But the Advantage of conſidering them *otherwiſe* (not as the Velocities Themſelves, but the Magnitudes *They* would, uniformly, generate in a given finite Time) appear to me ſufficient to obviate any Objection on that Head.

By taking Fluxions as *meer Velocities*, the Imagination is confined, as it were, to a Point, and, without proper Care, inſenſibly involved in metaphyſical Difficulties: But according to our Method of conceiving and explaining the Matter, leſs Caution in the Learner is neceſſary, and the higher Orders of Fluxions are rendered much more eaſy and intelligible——Beſides, tho' Sir

Iſaac Newton defines Fluxions to be *the Velocities of Motions*, yet He hath Recourſe to the Increments, or Moments, generated in equal Particles of Time, in order to determine thoſe Velocities; which he afterwards teaches us to expound by finite Magnitudes of other Kinds: Without which (as is already hinted above) we could have but very obſcure Ideas of the higher Orders of *Fluxions*: For if Motion in (or at) a Point be ſo difficult to conceive, that, *Some* have, even, gone ſo far as to diſpute the very Exiſtence of Motion, how much more perplexing muſt it be to form a Conception, not only, of the Velocity of a Motion, but alſo in infinite Changes and Affections of *It*, in one and the ſame Point, where all the Orders of Fluxions are to be conſidered?

SEEING the Notion of a Fluxion, according to our Manner of defining It, ſuppoſes an uniform Motion, it may, perhaps, ſeem a Matter of Difficulty, at firſt View, how the Fluxions of Quantities, generated by Means of accelerated and retarded Motions, can be rightly aſſigned; ſince not any, the leaſt, Time can be taken during which the generating Celerity continues the ſame: Here, indeed, we cannot expreſs the Fluxion by any Increment or Space, *actually*, generated in a given Time (as in uniform Motions.) But, then, we can eaſily determine, what the contemporary Increment, or generated Space *would be*, if the Acceleration, or Retardation, was to ceaſe at the propoſed Poſition in which the Fluxion is to be found: Whence the true Fluxion, itſelf, will be obtained, without the Aſſiſtance of infinitely ſmall Quantities, or any metaphyſical Conſiderations.

THUS, for Example, the Motion of a Ball, deſcending by the Force of its own Gravity, is continually accelerated; but to have the Fluxion of the Diſtance fall'n thro' at any given Poſition of the Ball, we muſt find how far the Ball *would*, uniformly, deſcend, from that Point, in a given Time, if the Gravity, or the Earth's Attraction, from thence, was to ceaſe acting. By which Means we ſhall have as clear an Idea of the Fluxion and the true Meaſure of the Velocity of the Ball, at any Point aſſigned, as in thoſe Caſes where the Motion is, *actually*, uniform.

AGAIN, if a Right-line be ſuppoſed to move parallel to itſelf with an equable Motion, and to increaſe in Length, at the ſame Time; the Area generated thereby, will increaſe with an accelerated Velocity: But the Fluxion thereof, at any given Poſition of the Line, will be had by taking that Part of the Increment which *would*, uniformly, ariſe, was the Length (as well as the Velocity) of the Line to continue invariable from the propoſed Poſition. For, if the Length be ſuppoſed to increaſe, from the ſaid Poſition, the Area generated, from thence, will be, evidently, greater than That which would uniformly ariſe in the ſame Time; ſince the new Parts, produced intitled, *An Explanation of Fluxions in a Short Eſſay on the Theory*; printed for *W. Innys*: Wrote by a worthy Friend of mine (who was too modeſt to put his Name to that, his firſt, Attempt) whoſe Manner of determining the Fluxion of a Rectangle, and illuſtrating the higher Order of Fluxions, I have, in particular, follow'd, with little or no Variation.

The following BOOKS are all written by Mr. *Thomas Simpſon*, F. R. S. and printed for *J. Nourſe*.

1. THE Elements of Geometry; with their Application to the Menſuration of Superficies and Solids, to the Determination of the Maxima and Minima of Geometrical Quantities, and to the Conſtruction of a great Variety of Geometrical Problems, 8vo. the third Edition, 5*s*.

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THE

THE DOCTRINE and APPLICATION OF FLUXIONS.

PART the Firſt.

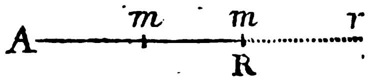
SECTION I.

Of the Nature, and Inveſtigation, of Fluxions.

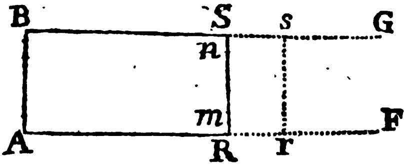
1. IN order to form a proper Idea of the Nature of Fluxions, all Kinds of Magnitudes are to be conſidered as generated by the *continual* Motion of ſome of their Bounds or Extremes; as a Line by the Motion of a Point; a Surface by the Motion of a Line; and a Solid by the Motion of a Surface.

2. Every Quantity ſo generated is called a variable, or flowing Quantity: *And the Magnitude by which any flowing Quantity* WOULD BE *uniformly increaſed in a given Portion of Time, with the generating Celerity at any propoſed Poſition, or Inſtant (was it from thence to continue invariable) is the Fluxion of the ſaid Quantity at that Poſition, or Inſtant.*

Thus, let the Point *m* be conceived to move from A, and generate the variable Right-line A*m*, by a Motion any how regulated; and let the Celerity thereof, when it arrives at any propoſed Poſition R, be ſuch *as would*, was it to continue uniform from that Point, be ſufficient to deſcribe the Diſtance, or Line R*r*, in the given Time allotted for the Fluxion: Then will R*r* be the Fluxion of the variable Line A*m*, in that Poſition.

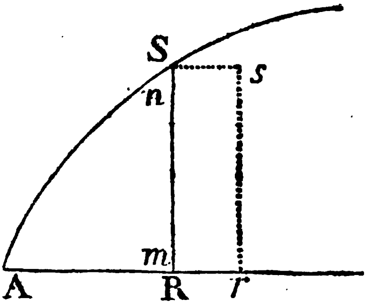


3. The Fluxion of a plane Surface is conceived in like Manner, by ſuppoſing a given Right-line *mn* to move parallel to itſelf, in the Plane of the parallel, and immoveable Lines AF and BG: For, if (as above) Rr be taken to expreſs the Fluxion of the Line A*m*, and the Rectangle RrsS be completed; then that Rectangle, being the Space which *would be* uniformly deſcribed by the generating Line *mn*, in the Time that A*m* *would be* uniformly increaſed by *mr*, is therefore the Fluxion of the generated Rectangle B*m*, in that Poſition, according to the true Meaning of the Definition.



4. If the Length of the generating Line *mn* continually varies, the Fluxion of the Area will *ſtill* be expounded by a Rectangle (under that Line and the Fluxion of the Abſciſſa, or Baſe:) For let the curvilineal Space A*mn* be generated by the continual, and parallel, Motion of the (now) variable Line *mn*, and let Rr be the Fluxion of the Baſe, or Abſciſſa, *Am* (as before); then the Rectangle RrsS will, here alſo, be the Fluxion of the generated Space A*mn*: Becauſe, if the Length and Velocity of the generating Line *mn* were

to continue invariable from the Poſition RS, the Rectangle R*rs*S would then be uniformly generated, with the very Celerity wherewith it begins to be generated, or with which the Space A*mn* is increaſed in that Poſition.



5. From what has been hitherto ſaid it will appear, that *the Fluxions of Quantities are, always, as the Celerities by which the quantities themſelves increaſe in Magnitude:* Whence it will not be difficult to form a Notion of the Fluxions of Quantities otherwiſe generated; as well ſuch as ariſe from the Revolution of Right-lines and Planes, as thoſe by parallel Motion: But of this hereafter. I come now to ſhew the Manner of determining the Fluxions of algebraic Quantities; by which all others, of what Kind ſoever, are explicable. But firſt of all it will be requiſite to premiſe the following Obſervations.

I. *That the final Letters* u, w, x, y, z *of the Alphabet are commonly put for variable Quantities; and the initial Letters* a, b, c, d, &c. *for invariable ones:* Thus the Diameter of a given Circle may be denoted by a, and the Sine of any Arch thereof (conſidered as variable) by x.

II. *That the Fluxion of a Quantity repreſented by a ſingle Letter, is uſually expreſſed by the ſame Letter with a Dot or Full-point over it:* Thus the Fluxion of x is repreſented by $\dot{x}$, and that of y by $\dot{y}$.

III. *That the Fluxion of a Quantity which decreaſes, inſtead of increaſing, is to be conſidered as negative.*

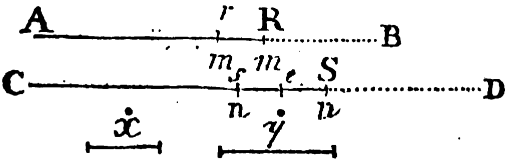
PROPOSITION I.

6. *The Fluxion of a Quantity being given, it is propoſed to find the Fluxion of any Power of that Quantity.*

As a clear underſtanding of this Problem will be of great Importance throughout the whole Work, it may not be improper to conſider it firſt in one or two of its moſt ſimple Caſes.

Caſe 1. Let $\dot{x}$ expreſs the Fluxion of x, (according to the foregoing Notation) and let the Fluxion of x^2 be required.

Conceive two Points m and n to proceed, at the ſame time, from two other Points A and C, along the Right-lines AB and CD, in ſuch ſort, that the Meaſure of the Diſtance CS (y), deſcribed by the latter, may be, *always*, equal to the Square of that AR (x), deſcribed by the former moving uniformly.



Furthermore, let r, s, and R, S, be any contemporary Poſitions of the generating Points, and let the Lines $\dot{x}$ and $\dot{y}$ repreſent the reſpective Diſtances that *would be* uniformly deſcribed, in the ſame time, with the Celerities of thoſe Points at R and S, then thoſe Lines will expreſs the Fluxions of Am and Cn in this Poſition, (*by the Definition*, Art. 2 and 5).

Moreover, ſince $Cs = Ar^2$ and $CS = AR^2$ (*by Hypotheſis*), if Rr be denoted by v, we ſhall have CS $(y) = x^2$, and $Cs\ (= \overline{x - v}|^2) = x^2 - 2xv + v^2$, and conſequently $Ss\ (= CS - Cs) = 2xv - v^2$; from whence we gather, that, while the Point m moves over the Diſtance v, the Point n moves over the Diſtance

$2xv-v^2$. But this laſt Diſtance (ſince the Square of any Quantity is known to increaſe faſter, in Proportion, than the Root) is not deſcribed with an uniform Motion (like the former), but an accelerated one; and therefore is equal to, and may be taken to expreſs, the uniform Space that might be deſcribed with the mean Celerity at ſome intermediate Point *e*, in the ſame time. Therefore, ſeeing the Diſtances that might be deſcribed, in equal times, with the uniform Celerity of *m*, and the mean Celerity at *e*, are to each other as v to $2xv-v^2$, or as 1 to $2x-v$, or, laſtly, as $\dot{x}$ to $2x\dot{x}-v\dot{x}$, (all which are in the ſame Proportion) it is evident, that, in the time the Point *m* would move uniformly over the Diſtance $\dot{x}$, the other Point *n*, with its Celerity at *e*, *would* move uniformly over the Diſtance $2x\dot{x}-v\dot{x}$. This being the Caſe, let *r*, R, and *s*, S, be now ſuppoſed to coincide, by the Arrival of the generating Points at R and S, then *e* (being always between *s* and S, will likewiſe coincide with S; and the Diſtance, $2x\dot{x}-\dot{x}$, which might be uniformly deſcribed in the aforeſaid time, with the Velocity at *e*, (now at S), will become barely equal to $2x\dot{x}$; which *(by the Defin.)* is equal to $(\dot{y})$, the true Fluxion of C*n* or x^2 [a].

$v\dot{x}$

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[a] *It may, perhaps, ſeem inaccurate, that the Fluxions of x and x^2 are compared together, and expreſſed both by Lines, when the flowing Quantities themſelves, conſidered as a Right Line and a Square, admit of no Compariſon.——This Objection would, indeed, be of force, were the Expreſſions reſtrained to a geometrical Signification; but here our Notions are more abſtracted and univerſal, not obliging us to regard what Kind of Extenſion, may be defined by this or that Expreſſion, but only the Values of the algebraic Quantities thereby ſignified; to which the Meaſures of all other Quantities whatever are ultimately referred.——And, though Quantities of different Kinds cannot be compared with each other, their Meaſures, in Numbers, may.——Thus, for Inſtance, though it would be wrong to affirm, that a Square whoſe Area is 9 Inches is equal to a Line of 9 Inches long, yet it is no Impropriety at all to ſay the Numbers expreſſing their Meaſures, in Inches, are equal.*

7. *Caſe* 2. Let the Fluxion of x^3 be required.

Suppoſe every Thing to remain as in the preceding Caſe; only let C*n* be here equal to the Cube of A*m* (inſtead of the Square).

Then, in the very ſame manner, we have S*s* ($=$CS$-$C*s*$=x^3-\overline{x-v}|^3$)$=3x^2v-3xv^2+v^3$: From whence it appears, that the Diſtances which *might be* deſcribed, in the ſame time, with the uniform Celerity of *m*, and the mean Celerity at *e*, will, in this Caſe, be to each other as v to $3x^2v-3xv^2+v^3$, or as $\dot{x}$ to $3x^2\dot{x}-3xv\dot{x}+v^2\dot{x}$: Which laſt Expreſſion, when *s*, *e*, and S coincide (as before) will become $3x^2\dot{x}$, the true Fluxion of x^3 required.

8. *Univerſally.* Let C*n* be, *always*, equal to $\overline{\mathrm{A}m}|^n$; alſo let $\overline{x-v}|^n$ (or $x-v$ raiſed to the Power whoſe Exponent is n) be repreſented by $x^n-ax^{n-1}v+bx^{n-2}v^2-cx^{n-3}v^3$, *&c.* and let every Thing elſe be ſuppoſed as above.

Then, ſince S*s* ($x^n-\overline{x-v}|^n$) is $=ax^{n-1}v-bx^{n-2}v^2+cx^{n-3}v^3$, *&c.* it is plain that the Spaces which might be deſcribed, in the ſame time, with the uniform Celerity of *m*, and the mean Celerity at *e*, will, here, be to each other as v to $ax^{n-1}v-bx^{n-2}v^2+cx^{n-3}v^3$, *&c.* or as $\dot{x}$ to $ax^{n-1}\dot{x}-bx^{n-2}v\dot{x}+cx^{n-3}v^2\dot{x}$, *&c.*

Therefore, all the Terms, wherein v is found, vaniſhing, when *s*, *e*, and S coincide, we have $ax^{n-1}\dot{x}$ for the required Fluxion of C*n*, or x^n; which Fluxion, becauſe the numeral Co-efficient of the ſecond Term of a Binomial involved is known to be, *univerſally*, equal to the Exponent of the Power, will alſo be truly expreſſed by $nx^{n-1}\dot{x}$. Q. E. I.

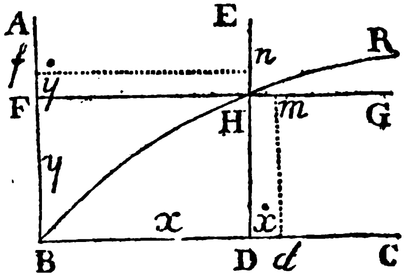
9. If the Quantity A*m* (or x) be generated with an accelerated, or a retarded Motion, inſtead of an uniform

form one, the Fluxion of x^n (or C*n*) will come out exactly the ſame:

For the Spaces *r*R and *s*S, actually deſcribed in the ſame time, being always, to each other, in the Ratio of $\dot{x}$ to $ax^{n-1}\dot{x} - bx^{n-2}v\dot{x}$, &c. the mean Celerities, at certain intermediate Points between *r*, R and *s*, S muſt, alſo, be in that Ratio: Which, when *v* vaniſhes (as above) will become that of $\dot{x}$ to $ax^{n-1}\dot{x}$, (or $nx^{n-1}\dot{x}$) the very ſame as before.

PROPOSITION II.

10. *To find the Fluxion of the Product or Rectangle of two variable Quantities.*



Conceive two Right-lines DE and FG, perpendicular to each other, to move, from two other Right-lines, BA and BC, continually parallel to themſelves, and thereby generate the Rectangle DF. Let the Path of their Interſection, or the Loci of the Angle H, be the Line BHR; alſo let D*d* ($\dot{x}$) and F*f* ($\dot{y}$) be the Fluxions of the Sides BD (x) and BF (y), and let *dm* and *fn*, parallel to DH and FH, be drawn. Therefore, becauſe the Fluxion of the Space or Area BDH is truly expreſſed by the Rectangle D*m* ($= y\dot{x}$ *) and that of the Area, or Space BFH, by the Rectangle F*n*, and equal Quantities have equal Fluxions, it follows that the Fluxion of the Rectangle $xy = \mathrm{DF}$ ($= \mathrm{BDH} + \mathrm{BFH}$) is truly expreſſed by $\dot{y}x + \dot{x}y$. Q. E. I.

* Art. 4.

The ſame otherwiſe.

11. Let xy be the given Rectangle (as before); and put $z = x + y$, then z^2 being $= x^2 + 2xy + y^2$, we have $\frac{1}{2}z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2 = xy$. But the Fluxion of $\frac{1}{2}z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2$, (and conſequently that of its Equal xy) is $z\dot{z} - x\dot{x} - y\dot{y}$ (by *Art.* 6): Which, becauſe $z = x + y$ and $\dot{z} = \dot{x} + \dot{y}$, is alſo equal to $\overline{x+y} \times \overline{\dot{x}+\dot{y}} - x\dot{x} - y\dot{y} = y\dot{x} + x\dot{y}$. Q. E. I.

COROLLARY I.

12. Hence the Fluxion of the Product of three variable Quantities (yzu) may be derived: For, if x be put $= zu$; then yzu will become $= yx$, and its Fluxion $= y\dot{x} + x\dot{y}$ (*as above*:) But x being $= zu$, and, therefore, $\dot{x} = z\dot{u} + u\dot{z}$, if theſe Values be ſubſtituted in $y\dot{x} + \dot{x}y$, it will become $y \times \overline{\dot{z}u + \dot{u}z} + z\dot{u}y = yz\dot{u} + yu\dot{z} + zu\dot{y}$ the Fluxion of yzu required. In like Manner the Fluxion of $xyzu$ will appear to be $xy\dot{z}u + xyz\dot{u} + \dot{x}yzu + x\dot{y}zu$, and that of $xyzuw = xyzu\dot{w} + xyz\dot{u}w + xy\dot{z}uw + x\dot{y}zuw + \dot{x}yzuw$.

COROLLARY 2.

13. Hence, alſo, the Fluxion of a Fraction $\frac{u}{z}$ may be determined. For, putting $x = \frac{u}{z}$, we have $xz = u$, and therefore $x\dot{z} + z\dot{x} = \dot{u}$ (*as above*); whence, by Tranſpoſition and Diviſion, $\dot{x} = \frac{\dot{u}}{z} - \frac{x\dot{z}}{z} = \frac{\dot{u}}{z} - \frac{u\dot{z}}{z^2}$ (by writing $\frac{u}{z}$ for x) $= \frac{z\dot{u} - u\dot{z}}{z^2}$; which is the true Fluxion of x, or its Equal $\frac{u}{z}$, the Fraction propoſed.

14. Now, from the foregoing Propoſitions, and their ſubſequent Corollaries, the following practical Rules,

for determining the Fluxions of algebraic Quantities, are obtained.

RULE I.

To find the Fluxion of any given Power of a variable Quantity.

Multiply the Fluxion of the Root by the Exponent of the Power, and the Product by that Power of the ſame Root whoſe Exponent is leſs by Unity than the given Exponent.

This Rule is inveſtigated in Prop. 1, and is nothing more than $nx^{n-1}\dot{x}$ (the Fluxion of x^{n}) expreſſed in Words.

Hence the Fluxion of x^{3} is $3x^{2}\dot{x}$; that of x^{5} is $5x^{4}\dot{x}$; and that of $\overline{a+y}|^{7}$ is $7\dot{y} \times \overline{a+y}|^{6}$, (becauſe, a being conſtant, $\dot{y}$ is the true Fluxion of the Root $a+y$, in this Caſe).

Moreover the Fluxion of $\overline{a^{2}+z^{2}}|^{\frac{3}{2}}$, will be $\frac{3}{2} \times 2z\dot{z} \times \overline{a^{2}+z^{2}}|^{\frac{1}{2}}$, or $3z\dot{z}\sqrt{a^{2}+z^{2}}$: For here, x being put $= a^{2}+z^{2}$, we have $\dot{x} = 2z\dot{z}$, and therefore $\frac{3}{2}x^{\frac{1}{2}}\dot{x}$, the Fluxion of $x^{\frac{3}{2}}$ (or $\overline{a^{2}+z^{2}}|^{\frac{3}{2}}$) is $= 3z\dot{z}\sqrt{a^{2}+z^{2}}$, as above.

RULE II.

15. To find the Fluxion of the Product of ſeveral variable Quantities multiplied together.

Multiply the Fluxion of each, by the Product of the reſt of the Quantities, and the Sum of the Products thus ariſing will be the Fluxion ſought *.

Thus the Fluxion of xy, is $\dot{x}y + y\dot{x}$; that of xyz, is $xy\dot{z}+xz\dot{y}+yz\dot{x}$; and that of $xyzu$, is $xyz\dot{u}+xyu\dot{z}+xzu\dot{y} + yzu\dot{x}$.

*Art. 12.

RULE III.

16. To find the Fluxion of a Fraction.

From the Fluxion of the Numerator drawn into the Denominator, ſubtract the Fluxion of the Denominator drawn into the Numerator, and divide the Remainder by the Square of the Denominator *.

*Art. 13.

Thus, the Fluxion of $\frac{x}{y}$ is $\frac{y\dot{x}-x\dot{y}}{y^2}$; that of $\frac{x}{x+y}$, is $\frac{\dot{x}\times\overline{x+y}-\overline{\dot{x}+\dot{y}}\times x}{\overline{x+y}|^{2}}=\frac{\dot{y}x-\dot{x}y}{\overline{x+y}|^{2}}$; and that of $\frac{x+y+z}{x+y}$, or $1+\frac{z}{x+y}$, is $\frac{\dot{z}\times\overline{x+y}-\overline{\dot{x}+\dot{y}}\times z}{\overline{x+y}|^{2}}$; and ſo of others.

17. In the Examples hitherto given, each is reſolved by its own particular Rule; but in thoſe that follow, the Uſe of two, and ſometimes of all the three, Rules is requiſite.

Thus (*by Rule* 1. *and* 2.) the Fluxion of x^2y^2 is $2x^2y\dot{y}+2y^2x\dot{x}$; that of $\frac{x^2}{y^2}$ is $\frac{2y^2x\dot{x}-2x^2y\dot{y}}{y^4}$, (*by Rule* 1. *and* 3.) and that of $\frac{x^2y^2}{z}$ is $\frac{\overline{2x^2y\dot{y}+2y^2x\dot{x}}\times z-x^2y^2\dot{z}}{z^2}$; where all the three Rules are neceſſary.

When the propoſed Quantity is affected by a Co-efficient, or conſtant Multiplicator, the Fluxion found as above, muſt be multiplied by that Co-efficient or Multiplicator.

Thus, the Fluxion of $5x^3$ is $15x^2\dot{x}$. For, the Fluxion of x^3 being $3x^2\dot{x}$, that of $5x^3$, which is 5 times as great, muſt conſequently be $5\times 3x^2\dot{x}$, or $15x^2\dot{x}$.

And, in the very ſame Manner the Fluxion of ax^n will appear to be $nax^{n-1}\dot{x}$. Moreover, the Fluxion of $\frac{a}{\overline{x^2+y^2}|^{\frac{1}{2}}}$, or $a\times\overline{x^2+y^2}|^{-\frac{1}{2}}$, will be expreſſed by

a ×

$a \times -\frac{1}{2} \times \overline{2x\dot{x}+2y\dot{y}} \times \overline{x^2+y^2}|^{-\frac{3}{2}}$, or $-\frac{a \times \overline{x\dot{x}+y\dot{y}}}{\overline{x^2+y^2}|^{\frac{3}{2}}}$; that of $\sqrt{x+y^{\frac{1}{2}}}$, or $\overline{x+y^{\frac{1}{2}}}|^{\frac{1}{2}}$, by $\overline{\frac{1}{2}\dot{x}+\frac{1}{2}\times\frac{1}{2}\dot{y}y^{-\frac{1}{2}}} \times \overline{x+y^{\frac{1}{2}}}|^{-\frac{1}{2}}$, (*Rule* 1.) or $\frac{\frac{1}{2}\dot{x}+\frac{1}{4}y\dot{y}^{-\frac{1}{2}}}{\sqrt{x+y^{\frac{1}{2}}}}$, or $\frac{\frac{1}{2}\dot{x}y^{\frac{1}{2}}+\frac{1}{4}\dot{y}}{\sqrt{xy+y^{\frac{3}{2}}}}$; $\times y^{-\frac{1}{2}}$

and that of $\frac{\overline{x+a}|^2}{\sqrt{x^2-a^2}}$, or $\frac{\overline{x+a}|^2}{\overline{x^2-a^2}|^{\frac{1}{2}}}$, by

$$\frac{2\dot{x} \times \overline{x+q} \times \overline{x^2-a^2}|^{\frac{1}{2}} - x\dot{x} \times \overline{x^2-a^2}|^{-\frac{1}{2}} \times \overline{x+a}|^2}{x^2-a^2}\text{; which}$$

by Reduction, is $= \frac{2\dot{x} \times \overline{x^2-a^2}|^{\frac{1}{2}} - \dot{x}x \times \overline{x^2-a^2}|^{-\frac{1}{2}} \times \overline{x+a}}{x-a}$

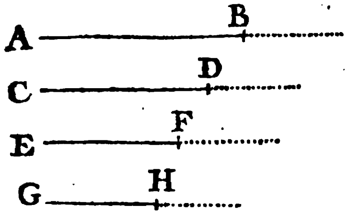
$$= \frac{2\dot{x} \times \overline{x^2-a^2} - x\dot{x} \times \overline{x+a}}{\overline{x-a} \times \overline{x^2-a^2}|^{\frac{1}{2}}} = \frac{2\dot{x} \times \overline{x-a} \times \overline{x+a} - \dot{x}x \times \overline{x+a}}{\overline{x-a} \times \sqrt{x^2-a^2}}$$

$$= \frac{\overline{x+a} \times \overline{x\dot{x}-2a\dot{x}}}{\overline{x-a} \times \sqrt{x^2-a^2}}.$$

Having explained the Manner of conſidering and determining the firſt Fluxions of variable or flowing Quantities, it will be proper to ſay ſomething, now, concerning the higher Orders, as Second, Third, Fourth, &c. Fluxions.

18. *The Second Fluxion of a Quantity is the Fluxion of the variable or algebraic Quantity expreſſing the Firſt Fluxion already defined* *. *By the Third Fluxion is meant the Fluxion of the variable Quantity expreſſing the Second: And by the Fourth, the Fluxion of the variable Quantity expreſſing the Third Fluxion: And ſo on.* *Art. 2.

Thus, for Example, let the Line AB repreſent a variable Quantity, generated by the Motion of the Point B, and let the (firſt) Fluxion thereof (or the Space that *might be*, uniformly deſcribed in a given Time, with the Celerity of B) be always expreſſed by the Diſtance of the Point D from a given, or fixed Point C: Then, if the Celerity of B be not every where the ſame; the Diſtance CD, expreſſing the Meaſure of that Celerity, muſt alſo vary, by the Motion of D, from, or towards C, according as the Celerity of B is an increaſing or a decreaſing one: And the Fluxion of the Line CD, ſo varying (or the Space (EF) that *might be* uniformly deſcribed in the aforeſaid given Time, with the Celerity of D) is the ſecond Fluxion of AB. Again, if the Motion of B be ſuch that neither it, nor that of D, (which depends upon it) be equable, then EF, expreſſing the Celerity of D, will alſo have its Fluxion GH; which is the third Fluxion of AB, and the ſecond Fluxion of CD.



And thus are the Fluxions of every other Order to be conſidered, *being the Meaſures of the Velocities by which their reſpective flowing Quantities, the Fluxions of the preceding Order, are generated* *.

19. Hence it appears, that a ſecond Fluxion always ſhews the rate of the Increaſe, or Decreaſe, of the firſt Fluxion; and that Third, Fourth, &c. Fluxions, differ in Nothing (except their Order and Notation) from Firſt Fluxions, being actually ſuch to the Quantities from whence they are immediately derived; and therefore are alſo determinable, in the very ſame Manner, by the general Rules already delivered.

Thus, *by Rule* 3. the (firſt) Fluxion of x^3 is $3x^2\dot{x}$: And, if $\dot{x}$ be ſuppoſed conſtant, that is, if the Root x be generated with an equable Celerity, the Fluxion of $3x^2\dot{x}$ (or $3\dot{x} \times x^2$) again taken, by the ſame Rule, will be $3\dot{x} \times 2x\dot{x}$, or $6x\dot{x}^2$; which therefore is the ſecond Fluxion of x^3: Whoſe Fluxion, found in like Sort, will be $6\dot{x}^3$, the third Fluxion of x^3. Farther than which we cannot go in this Caſe, becauſe the laſt Fluxion $6\dot{x}^3$ is here a conſtant Quantity.

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* Art. 2.

20. In the preceding Example the Root x is ſuppoſed to be generated with an equable Celerity: But, if the Celerity be an increaſing or a decreaſing one, then $\dot{x}$, expreſſing the Meaſure thereof, being variable, will alſo have its Fluxion; which is uſually denoted by $\ddot{x}$: Whoſe Fluxion, according to the ſame Method of Notation, is again deſigned by $\dddot{x}$; and ſo on, with reſpect to the higher Orders.

21. Here follow a few Examples, wherein the Root x, (or y) is ſuppoſed to be generated with a variable Celerity.

Thus, the firſt Fluxion of x^3 is $3x^2\dot{x}$ (or $3x^2 \times \dot{x}$). And, if the Fluxion of $3x^2 \times \dot{x}$ (conſidered as a Rectangle) be, again, found (*by Rule* 2.) we ſhall have $6x\dot{x} \times \dot{x} + 3x^2 \times \ddot{x} = 6x\dot{x}^2 + 3x^2\ddot{x}$, for the ſecond Fluxion of x^3.

Moreover, from the Fluxion laſt found we ſhall in like manner get $6\dot{x} \times \dot{x}^2 + 6x \times 2\dot{x}\ddot{x} + 6x\dot{x} \times \ddot{x} + 3x^2 \times \dddot{x}$ (or $6\dot{x}^3 + 18x\dot{x}\ddot{x} + 3x^2\dddot{x}$) for the third Fluxion of x^3.

Thus alſo, if $\dot{y} = nx^{n-1}\dot{x}$, then will $\ddot{y} = n \times \overline{n-1} \times x^{n-2}\dot{x}^2 + n\ddot{x}x^{n-1}$; and if $\dot{z}^2 = \dot{x}\dot{y}$, then will $2\dot{z}\ddot{z} = \dot{x}\ddot{y} + \dot{y}\ddot{x}$: And ſo of others. But, in the Solution of Problems, it will be convenient to make the firſt Fluxion of ſome one of the ſimple Quantities (x or y) invariable, not only to avoid Trouble, but that it may ſerve as a Standard to which the variable Fluxions of the other Quantities, depending thereon, may be always referred. The Reader is alſo deſired here (once for all) to take particular Notice, that *the Fluxions of all Kinds and Orders, whatever, are contemporaneous, or ſuch as may be generated together, with their reſpective Celerities, in one and the ſame Time.*

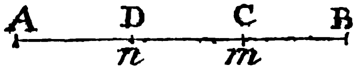
SECTION II.

On the Application of Fluxions to the Solution of Problems DE MAXIMIS ET MINIMIS.

22. IF a Quantity, conceived to be generated by Motion, increaſes, or decreaſes, till it arrives at a certain Magnitude or Poſition, and then, on the contrary, grows leſſer or greater, and it be required to determine the ſaid Magnitude or Poſition, the Queſtion is called a Problem *de Maximis & Minimis*.

GENERAL ILLUSTRATION.

Let a Point *m* move uniformly in a Right Line, from A towards B, and let another Point *n* move after it, with a Velocity either increaſing, or decreaſing, but ſo that it may, at a certain Poſition, D, become equal to that of the former Point *m*, moving uniformly.



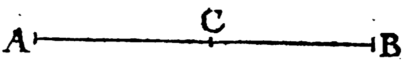
This being premiſed, let the Motion of *n* be firſt conſidered as an increaſing one; in which Caſe the Diſtance of *n* behind *m* will continually increaſe, till the two Points arrive at the cotemporary Poſitions C and D; but afterwards it will, again, decreaſe; for the Motion of *n*, till then, being ſlower than at D, it is alſo ſlower than that of the preceding Point *m* (by Hypotheſis) but becoming quicker, afterwards, than that of *m*, the Diſtance *mn* (as has been already ſaid) will again decreaſe: And therefore is a *Maximum*, or the greateſt of all, when the Celerities of the two Points are equal to each other.

But, if *n* arrives at D with a decreaſing Celerity; then its Motion being firſt ſwifter, and afterwards ſlower, than that of *m*, the Diſtance *mn* will firſt decreaſe and then increaſe; and therefore is a *Minimum*, or the leaſt of all, in the forementioned Circumſtance.

Since then the Diſtance *mn* is a *Maximum* or a *Minimum*, when the Velocities of *m* and *n* are equal, or when that Diſtance increaſes as faſt through the Motion of *m*, as it decreaſes by that of *n*, its Fluxion at that Inſtant is evidently equal to Nothing *. Therefore, as the Motion of the Points *m* and *n* may be conceived ſuch that their Diſtance *mn* may expreſs the Meaſure of any variable Quantity whatever, it follows, that the Fluxion of any variable Quantity whatever, when a Maximum or Minimum, is equal to Nothing.

EXAMPLE I.

23. *To divide a given Right-line* AB *into two ſuch Parts,* AC, BC, *that their Product, or Rectangle, may be the greateſt poſſible.*



Put the given Line AB $= a$, and let the Part AC, conſidered as variable (by the Motion of C from A towards B) be denoted by x: Then BC being $= a - x$, we have $AC \times BC = ax - x^2$: Whoſe Fluxion $a\dot{x} - 2x\dot{x}$ being put $= 0$, according to the preſcript, we get $a\dot{x} = 2x\dot{x}$, and conſequently $x = \frac{1}{2}a$. Therefore AC and BC, in the required Circumſtance, are equal to each other: Which we alſo know from other Principles.

EXAMPLE II.

24. *To find the Fraction which ſhall exceed its Cube by the greateſt Quantity poſſible.*

Let x denote a variable Quantity, expreſſing Number in general; then the Exceſs of x above x^3 being univerſally repreſented by $x - x^3$, if the Fluxion thereof be taken, &c. we ſhall have $\dot{x} - 3x^2\dot{x} = 0$; and therefore $x = \sqrt{\frac{1}{3}}$, the Fraction required.

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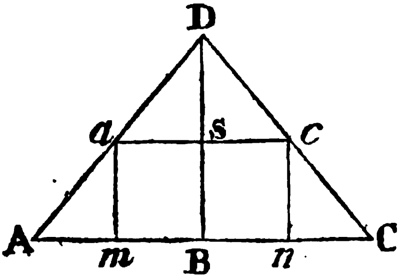
* Art. 2 and 5.

EXAMPLE III.

25. *To determine the greateſt Rectangle that can be inſcribed in a given Triangle.*



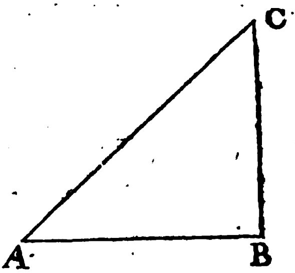
Put the Baſe AC of the given Triangle = b, and its Altitude BD = a; and let the Altitude (BS) of the inſcribed Rectangle *mc* (conſidered as variable) be denoted by x: Then, becauſe of the parallel Lines AC, and *ac*, it will be as BD (a) : AC (b) :: DS ($a-x$) : $\frac{ba-bx}{a}$ $= ac$: Whence the Area of the Rectangle, or $ac \times BS$ will be $= \frac{bax-bx^2}{a}$: Whoſe Fluxion $\frac{ba\dot{x}-2bx\dot{x}}{a}$ being (as before) put $= 0$, we ſhall get $x = \frac{1}{2}a$. Whence the greateſt inſcribed Rectangle is that whoſe Altitude is juſt half the Altitude of the Triangle.

26. It will be proper to obſerve *here*, that the Value of a Quantity, when a *Maximum* or *Minimum*, may oftentimes be determined with more Facility by taking the Fluxion of ſome given Part, Multiple, or Power, thereof, than from the Fluxion of the Quantity itſelf. Thus, in the preceding Example, where the general Expreſſion is $\frac{bax-bx^2}{a} = \frac{b}{a} \times \overline{ax-x^2}$, if the conſtant Multiplicator $\frac{b}{a}$ be rejected, we ſhall have $ax-x^2$; whoſe Fluxion $a\dot{x}-2x\dot{x}$ being put $= 0$, we get $x = \frac{1}{2}a$, *the very ſame as before.*

The Reaſon of which is obvious; becauſe when the Quantity itſelf (be it of what Kind it will) is the greateſt, or leaſt poſſible, any *given* Part, Power, or Multiple of it is alſo the greateſt or leaſt poſſible.

EXAMPLE IV.

27. *Of all right-angled plain Triangles having the ſame given Hypothenuſe, to find that* (ABC) *whoſe Area is the greateſt.*



Let AC $= a$, AB $= x$, and BC $= y$: Then, $x^2 + y^2$ being $= a^2$, we ſhall have $y = \sqrt{a^2 - x^2}$, and conſequently $\frac{xy}{2} = \frac{x}{2}\sqrt{a^2 - x^2} =$ the Area of the Triangle; whoſe Square $\frac{a^2x^2}{4} - \frac{x^4}{4}$ being, *alſo*, a *Maximum* *, the Fluxion thereof $\frac{a^2x\dot{x}}{2} - x^3\dot{x}$ muſt therefore be $= 0$, †: Whence x is found $= a\sqrt{\tfrac{1}{2}}$, and y $(\sqrt{a^2 - x^2}) = a\sqrt{\tfrac{1}{2}}$.

*Art. 26.

†Art. 22.

The ſame otherwiſe.

Since $\frac{1}{2}xy$ is a *Maximum*, and $x^2 + y^2 = a^2$, let the Fluxions of both be taken, and you will have $\frac{1}{2}x\dot{y} + \frac{1}{2}y\dot{x} = 0$, and $2x\dot{x} + 2y\dot{y} = 0$; from the former of which $\dot{y}$ will be $= -\frac{y\dot{x}}{x}$; and from the latter, it will be $= -\frac{x\dot{x}}{y}$: Therefore $\frac{y\dot{x}}{x}$ and $\frac{x\dot{x}}{y}$ are equal to each other, and conſequently $x = y$, (the ſame as before.)

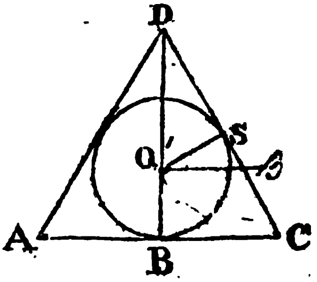
EXAMPLE V.

28. *Of all right-angled plain Triangles containing the ſame given Area, to find that whereof the Sum of the two Legs* AB + BC *is the leaſt poſſible.* (See the preceding Figure.)

Let one Leg, AB, be denoted by x, and the Area of the Triangle by a; then the other Leg will be denoted by $\frac{2a}{x}$, and the Sum of the two Legs will be $x + \frac{2a}{x}$; whereof the Fluxion is $\dot{x} - \frac{2a\dot{x}}{x^2}$; which, put $= 0$, gives x (AB) $= \sqrt{2a}$: Whence BC $\left(\frac{2a}{x}\right)$ is alſo $= \sqrt{2a}$. Therefore the two Legs are equal to each other.

EXAMPLE VI.

29. *To determine the Dimenſions of the leaſt Iſoſceles Triangle* ACD *that can circumſcribe a given Circle.*



Let the Diſtance (OD) of the Vertex of the Triangle from the Center of the Circle, be denoted by x, and let the remaining Part of the Perpendicular, which is the Radius of the Circle, be repreſented by a: Then, if OS, perpendicular to DC, be drawn, we ſhall have DS $= \sqrt{x^2 - a^2}$; and therefore, ſince DS : OS : : DB : BC, we likewiſe have BC $= \frac{a \times \overline{x + a}}{\sqrt{x^2 - a^2}}$; which multiplied by $\overline{x + a}$ (BD) gives $\frac{a\times\overline{x+a}|^2}{\sqrt{x^2-a^2}}$ for the Area of the Triangle: Which being a *Minimum*, its Square muſt be a *Minimum*, and conſequently $\frac{\overline{x+a}|^4}{x^2-a^2}$, or its Equal $\frac{\overline{x+a}|^3}{x-a}$, a *Minimum* alſo *. Whoſe Fluxion, therefore, which is *Art.26. $\frac{3\dot{x}\times\overline{x+a}|^2\times\overline{x-a}-\dot{x}\times\overline{x+a}|^3}{\overline{x-a}|^2}$, being put $=0$, and the Whole divided by $\frac{\dot{x}\times\overline{x+a}|^2}{\overline{x-a}|^2}$, we alſo get $3\times\overline{x-a}-\overline{x+a}=0$; whence $x=2a$: Therefore, OD being $=2$OS, and the Triangles ODS and BDC equiangular, it is evident that DG is likewiſe $=2$BC $=$ AC; and ſo the Triangle ACD, when the leaſt poſſible, is equilateral.

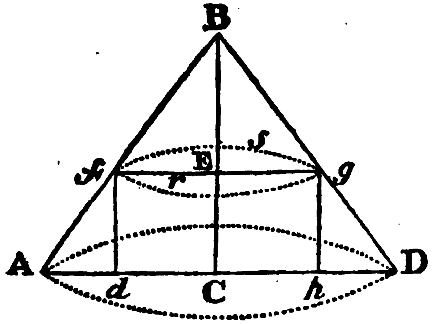
EXAMPLE VII.

30. *To determine the greateſt Cylinder, dg, that can be inſcribed in a given Cone* ADB.

Let $a=$ BC, the Altitude of the Cone;
$b=$ AD, the Diameter of its Baſe;
$x=fg$ (db) the Diameter of the Cylinder, conſidered as variable;
$p=\left(\frac{3{,}14159,\ \&c.}{4}\right)$ the Area of the Circle whoſe Diameter is Unity.

Then, the Areas of Circles being to one another as the Squares of their Diameters, we have, $1^2 : x^2 :: p : (px^2)$ the Area of the Circle $fsgr$: Moreover, from the Similarity of the Triangles ABC and Adf, we have $\frac{1}{2}b$ (AC) : a (BC) :: $\frac{1}{2}b-\frac{1}{2}x$ (Ad) : $df=\frac{ab-ax}{b}$, which multiplied by the Area px^2 (found above) gives

$pabx^2$

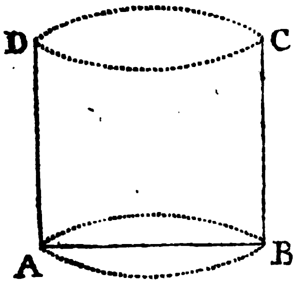


$\frac{pabx^2 - pax^3}{b}$ for the ſolid Content of the Cylinder: Which being a *Maximum*, its Fluxion $\frac{2pabx\dot{x}}{b} - \frac{3pax^2\dot{x}}{b}$ muſt

*Art.22. be $= 0$ *, conſequently $x = \frac{2b}{3}$ and $df = \frac{a}{3}$: From whence it appears, that the inſcribed Cylinder will be the greateſt poſſible, when the Altitude thereof is juſt $\frac{2}{3}$ of the Altitude of the whole Cone.

EXAMPLE VIII.

31. *To determine the Dimenſions of a cylindric Meaſure* ABCD, *open at the Top, which ſhall contain a given Quantity (of Liquor, Grain,* &c.) *under the leaſt internal Superficies poſſible.*

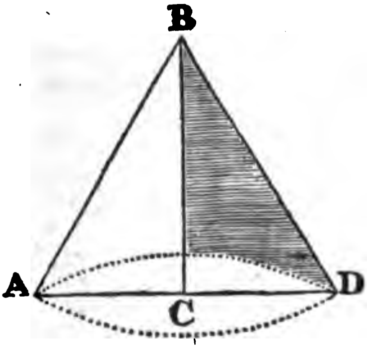


Let the Diameter $AB = x$, and the Altitude $AD = y$; moreover let p (3,14159, *&c.*) denote the Periphery of the Circle whoſe Diameter is Unity, and let c be the given Content of the Cylinder. Then it will be $1 : p :: x : (px)$ the Circumference of the Baſe; which, multiplied by the Altitude y, gives pxy for the concave Superficies of the Cylinder. In like Manner, the Area of the Baſe, by multiplying the ſame Expreſſion into $\frac{1}{4}$ of the Diameter x, will be found $= \frac{px^2}{4}$; which drawn into the Altitude y, gives $\frac{px^2y}{4}$ for the ſolid Content of the Cylinder; which being made $= c$, the concave Surface pxy will be found $= \frac{4c}{x}$, and conſequently the whole Surface $= \frac{4c}{x} + \frac{px^2}{4}$: Whereof the Fluxion, which is, $-\frac{4c\dot{x}}{x^2} + \frac{px\dot{x}}{2}$, being put $= 0$, we ſhall get $-8c + px^3 = 0$; and therefore $x = 2\sqrt[3]{\frac{c}{p}}$: Further, becauſe $px^3 = 8c$, and $px^2y = 4c$, it follows, that $x = 2y$; whence y is alſo known, and from which it appears, that the Diameter of the Baſe muſt be juſt the Double of the Altitude.

EXAMPLE IX.

32. *Of all Cones under the ſame given Superficies* (s) *to find that* (ABD) *whoſe Solidity is the greateſt.*

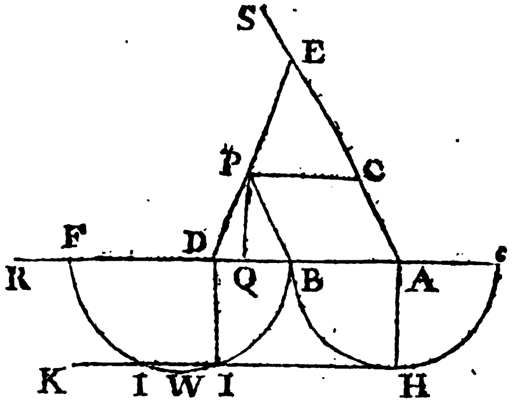
Let the Semidiameter of the Baſe, AC $= x$, and the Length of the ſlant Side AB $= y$; and let p (as in the preceding Examples) denote the Periphery of the Circle whoſe Diameter is Unity.



Then the Circumference of the Baſe will be $= 2px$, the Area of the Baſe $= px^2$, and the convex Superficies of the Cone $= pxy$, (which laſt is found by multiplying half the Periphery of the Baſe by the Length of the ſlant Side): Wherefore, ſince the whole Superficies is $= px^2 + pxy = s$, we have $y = \frac{s}{px} - x$; whence the Altitude CB $(\sqrt{AB^2 - AC^2}) = \sqrt{\frac{s^2}{p^2x^2} - \frac{2s}{p}}$; which multiplied by $\left(\frac{px^2}{3}\right)$ $\frac{1}{3}$ of the Area of the Baſe, gives $\frac{px^2}{3}\sqrt{\frac{s^2}{p^2x^2} - \frac{2s}{p}}$ for the ſolid Content of the Cone. Which being a *Maximum*, its Square $\frac{s^2x^2}{9} - \frac{2psx^4}{9}$ muſt alſo be a *Maximum*; and therefore $\frac{2s^2x\dot{x}}{9} - \frac{8psx^3\dot{x}}{9} = 0$; whence $s - 4px^2 = 0$, and conſequently $x = \sqrt{\frac{s}{4p}}$: From which $y \left(= \frac{s}{px} - x = \frac{s - px^2}{px} = \frac{3px^2}{px} = 3x\right)$ will likewiſe be known; and from whence it will appear that the greateſt Cone under a given Surface, (or a given Cone under the leaſt Surface) will be when the Length of the ſlant Side is to the Semi-diameter of the Baſe in the Ratio of 3 to 1, or, (which comes to the ſame) when the Square of the Altitude is to the Square of the whole Diameter in the Ratio of 2 to 1.

EXAMPLE X.

33. *To determine the Poſition of a Right-line* DE, *which, paſſing through a given Point* P, *ſhall cut two Right-lines* AR *and* AS, *given by Poſition, in ſuch ſort that the Sum of the Segments,* AD *and* AE, *made thereby, may be the leaſt poſſible.*



Make PB, parallel to AS, $= a$, and PC, parallel to AR, $= b$; and let BD $= x$: Then, by reaſon of the parallel Lines, it will be, $x : a :: b : \text{CE} = \frac{ab}{x}$: Therefore $\text{AD} + \text{AE} = b + x + a + \frac{ab}{x}$, and its Fluxion, $\dot{x} - \frac{ab\dot{x}}{x^2}$, which, in the required Circumſtance, being $= 0$, we have $x^2 - ab$ alſo $= 0$, and conſequently $x = \sqrt{ab}$; whence the Poſition of DE is known. But the ſame Thing may be otherwiſe determined, independent of Fluxions, from the general Solution of the Problem for finding the Poſition of DE, when the Sum of the Segments AD and AE (inſtead of being a *Minimum*) ſhall be equal to a given Quantity. Of which Problem, the geometrical Conſtruction may be as follows.

Compleat

Compleat the Parallelogram ABPC (as before) and, in RA produced, take $\mathrm{A}c=\mathrm{AC}$, and let $c\mathrm{F}$ be equal to the given Sum of the two Segments: Alſo let two Semi-circles be deſcribed upon $\mathrm{B}c$ and BF, and let AH, perpendicular to $\mathrm{B}c$, interſect the former in H; likewiſe let HK, parallel to $\mathrm{F}c$, interſect the latter in I; draw ID perpendicular to $\mathrm{F}c$, and, through P and D draw DE; which will be the Poſition required. For $\mathrm{AB}\times\mathrm{A}c$ being $=\mathrm{AH}^2=\mathrm{DI}^2=\mathrm{BD}\times\mathrm{DF}$, we have $\mathrm{BD} : \mathrm{AB} :: \mathrm{A}c\ (\mathrm{AC}) : \mathrm{DF}$; alſo, becauſe of the parallel Lines, we have $\mathrm{BD} : \mathrm{AB} :: \mathrm{AC} : \mathrm{CE}$; whence $\mathrm{DF}=\mathrm{CE}$, and conſequently $\mathrm{AD}+\mathrm{AE}\ (\mathrm{AD}+\mathrm{AC}+\mathrm{FD})$ is equal to $c\mathrm{F}$, which Conſtruction is more neat than that in *p*. 155. of my *Geometry*. But to ſhew how far this may conduce to the Matter firſt propoſed; we are to obſerve, that, as the Problem here conſtructed appears to be impoſſible, when the Right-line HK (inſtead of cutting or touching) falls wholly below the Circle BWF, the leaſt poſſible Value of BF (and conſequently of AD + AE) muſt, therefore, be when that Right-line touches the Circle; that is, when $\mathrm{BD}=\mathrm{DI}=\mathrm{AH}=\sqrt{\mathrm{AB}\times\mathrm{AC}}$; which Value is the very ſame with that found above.

The ſame Concluſion may alſo be deduced from the algebraic Solution of the foreſaid Problem: For, putting $b+x+a+\frac{ab}{x}$ (AD + AE) $=s$, and ſolving the Equation, x will be found $=\frac{s-a-b}{2}\pm\sqrt{\frac{\overline{s-a-b}|^2}{4}-ab}$: Which Equation being no longer poſſible than till $\frac{\overline{s-a-b}|^2}{4}-ab$ is $=0$, we have x, in that Circumſtance, $=\frac{s-a-b}{2}=\sqrt{ab}$; *ſtill as before*. In like Manner the *Maxima* and *Minima* may be determined in other Caſes, by finding the Poſition or Circumſtance wherein the general Problem begins to be impoſſible, (ſuppoſing the Quantity ſought to be given). But the Operation by

Fluxions is, for the general Part, much more ſhort and expeditious.

EXAMPLE XI.

34. *The ſame being given as in the preceding Example, to determine the Poſition, when the Line* DE, *itſelf, is the leaſt poſſible.*

Upon AF let fall the perpendicular PQ; make BQ $= c$, and, the reſt, as before: Then DP² being ($=$ DB² + BP² — 2BQ × DB) $= x^2 + a^2 - 2cx$, and DB² : DP² :: DA² : DE², we have $x^2 : x^2 + a^2 - 2cx :: \overline{b+x}|^2 : \text{DE}^2 = \frac{\overline{b+x}|^2 \times \overline{x^2 - 2cx + a^2}}{x^2} = \overline{b+x}|^2 \times \overline{1 - \frac{2c}{x} + \frac{a^2}{x^2}}$; whoſe Fluxion, which is $2\dot{x} \times \overline{b+x} \times \overline{1 - \frac{2c}{x} + \frac{a^2}{x^2}} + \overline{b+x}|^2 \times \overline{\frac{2c\dot{x}}{x^2} - \frac{2a^2\dot{x}}{x^3}}$, being put $= 0$, and the whole Equation divided by $2\dot{x} \times \overline{b+x}$, there will come out $1 - \frac{2c}{x} + \frac{a^2}{x^2} + \overline{b+x} \times \frac{c}{x^2} - \frac{a^2}{x^3} = 0$; whence $x^3 - 2cx^2 + a^2x + \overline{b+x} \times \overline{cx - a^2} = 0$; that is, (by Reduction) $x^3 - cx^2 + bcx - a^2b = 0$: From the Reſolution of which Equation, the Poſition of DE is determined.

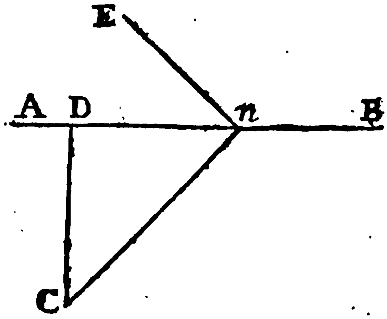
LEMMA.

rec)

35. *If a Body or Point* (*n*) *be ſuppoſed to move in a Right-line* AB, *its abſolute Celerity, in the Direction of that Line, will be to the relative Celerity, whereby it tends to, or from, a given Point* C, *any where out of the Line, as the Diſtance* C*n*, *is to the Diſtance* D*n*, *intercepted by* n *and the Perpendicular* CD; *or, as Radius to the Co-ſine of the Angle of Inclination* D*n*C.

For, putting CD $= a$, D*n* $= x$, and C*n* $= y$, we have $a^2 + x^2 = y^2$, and conſequently $2x\dot{x} = 2y\dot{y}$ *: Whence

* Art. 2 and 5.



Whence $\dot{x}$: $\dot{y}$ :: y (Cn) : x (Dn) :: Radius : Co-ſine DnC: But, the Fluxions of Quantities are as the Celerities of their Increaſe *, therefore the Truth of the Propoſition is manifeſt.

Art. 2 and 5.

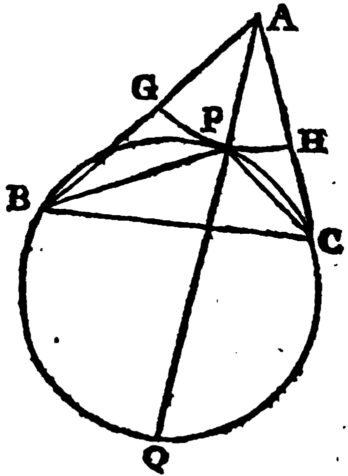
COROLLARY.

It follows from hence, that the relative Celerities in any two different Directions nE and nC, are directly as the Co-ſines of the correſponding Angles DnE and DnC. Therefore, when nE is perpendicular to Cn, (and the Angle DnE therefore equal to C) the Celerity in the Direction nE, will be to that in the Direction nC, as the Sine of DnC is to its Co-ſine. From whence it appears, that the Celerities in the Directions Dn, Cn, and En (perpendicular to nC) are to each other as Cn, Dn, and CD reſpectively.

EXAMPLE XII.

36. *To determine the Poſition of a Point, from whence, if three Right-lines be drawn to ſo many given Points* A, B, C, *their Sum ſhall be the leaſt poſſible.*

Let HPG be the Periphery of a Circle deſcribed about the Point A, as a Center, at any Diſtance AG; in which let the Point P be conceived to move with an uniform Celerity, from G towards H. Then, becauſe the relative Celerity thereof, in the Direction PC, is to that in the Direction BP produced, as the Co-ſine of the Angle CPH to the Co-ſine of the Angle BPG, *(by the preceding Lemma)*; and, ſince theſe Celerities, when the Sum of CP and BP is a *Minimum*, muſt be equal *, it follows, therefore, that the ſaid Angles CPH and BPG, as well as their Co-ſines, will in that Circumſtance become equal to each other; and conſequently APC alſo equal to APB. From whence it appears, that (take AG what you will) the Sum of the three Lines, AP, BP, and CP, cannot be the leaſt poſſible when the Angles APB and APC are unequal. And, by the ſame Argument, it alſo appears that their Sum cannot be the leaſt poſſible, when the Angles BPA and BPC are unequal: Therefore, their Sum muſt be the leaſt poſſible, when all the three Angles about the Point P are equal to one another; provided the Caſe will admit of ſuch an Equality, or that no one of the Angles of the Triangle ABC is equal to, or greater than $\frac{1}{3}$ of 4 Right Angles (for otherwiſe, the Point P will fall in the obtuſe Angle): Hence this

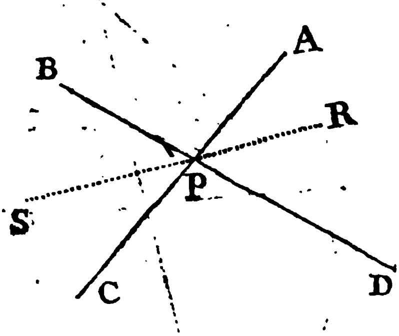


CONSTRUCTION.

Deſcribe, upon BC, a Segment of a Circle, to contain an Angle of 120°, and let the whole Circle BCQ be compleated; and from A, to the Middle (Q) of the Arch BQC, draw AQ interſecting the Circumference of the Circle in P; which will be the Point required. For, the Angles BPQ and CPQ, ſtanding upon the equal Arches BQ and CQ, have their Complements APB and APC equal to each other; and therefore, the Angle BPC being 120° (by Conſtruction) each of the ſaid

* Art. 2 and 22.

ſaid Angles APB, APC, will, likewiſe be 120 Degrees.

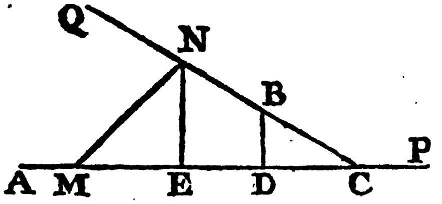


After the ſame Manner, it will appear that the Sum of all the Lines AP, BP, CP, *&c.* drawn from any Number of given Points A, B, C, *&c.* to meet in another Point P, will be the leaſt poſſible, when the Co-ſines of the Angles RPA, RPB, RPC, *&c.* that the ſaid Lines make with any other Line RS, paſſing through the Point of Concourſe, deſtroy each other: Which will be when all the Angles APB, BPC, CPD, *&c.* are equal, in all Caſes where the Poſition of the given Points will admit of ſuch an Equality. But, if the Number of given Points be four, the required Point will be in the Interſection of the two Right-lines joining the oppoſite Points: For, ſuppoſing APC and BPD to be continued Right-lines, the Co-ſine of RPA will be equal and contrary to that of RPC, and that of RPB equal and contrary to that of RPD.

EXAMPLE XIII.

37. *If two Bodies move at the ſame Time, from two given Places* A *and* B, *and proceed uniformly from thence in given Directions,* AP *and* BQ, *with Celerities in a given Ratio; it is propoſed to find their Poſition, and how far each has gone, when they are the neareſt poſſible to each other.*

Let M and N be any two cotemporary Poſitions of the Bodies, and upon AP let fall the Perpendiculars NE and BD; alſo let QB be produced to meet AP in



in C, and let MN be drawn: Moreover, let the given Celerity in BQ be to that in AP, as n to m, and let AC, BC, and CD, (which are alſo given) be denoted by a, b, and c reſpectively, and make the variable Diſtance CN $= x$: Then, by reaſon of the parallel Lines NE and BD, we ſhall have b (CB) : x (CN) :: c (CD) : CE $= \frac{cx}{b}$. Alſo, becauſe the Diſtances, BN and AM, gone over in the ſame Time, are as the Celerities, we likewiſe have, $n : m :: x - b$ (BN) : AM $= \frac{mx - mb}{n}$, and conſequently CM (AC—AM) $= a + \frac{mb}{n} - \frac{mx}{n} = d - \frac{mx}{n}$, (by writing $d = a + \frac{mb}{n}$). Whence MN^2 ($= \text{CM}^2 + \overline{\text{CN}^2 - \text{CM} \times 2\text{CE}}$) will alſo be found $= \overline{d - \frac{mx}{n}}^2 + x^2 - \overline{d - \frac{mx}{n}} \times \frac{2cx}{b} = d^2 - \frac{2dmx}{n} + \frac{m^2x^2}{n^2} + x^2 - \frac{2cdx}{b} + \frac{2cmx^2}{nb}$; whoſe Fluxion $- \frac{2dm\dot{x}}{n} + \frac{2m^2x\dot{x}}{n^2} + 2x\dot{x} - \frac{2cd\dot{x}}{b} + \frac{4cmx\dot{x}}{nb}$ being made $= 0$ (becauſe MN is to be a *Minimum*) we get $- bdmn + m^2bx + n^2bx - n^2cd + 2mncx = 0$; and conſequently $x = \frac{mnbd + n^2cd}{m^2b + n^2b + 2mnc} = \frac{nd \times \overline{mb + nc}}{b \times \overline{m^2 + n^2} + 2mnc}$; from whence BN, AM, and MN are alſo given.

The

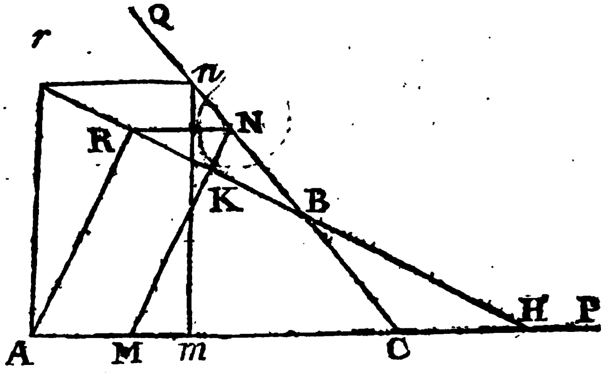
The ſame otherwiſe.

Becauſe the relative Celerities of the two Bodies, at M and N, in the Direction of the Line MN (produced) are truly expreſſed by $\frac{\textit{Co-ſine}\ \mathrm{M}}{\textit{Radius}} \times m$, and $\frac{\textit{Co-ſ.}\mathrm{N}}{\textit{Rad.}} \times n$, reſpectively *; and as theſe Celerities, when the Diſtance MN is a *Minimum*, do become equal to each other †, it follows that, in this Circumſtance, $m : n ::$ Co-ſ. N. : Co-ſ. M : : Secant of M : Secant of N (*by plane Trig.*)

* Art. 35.

† Art. 22.

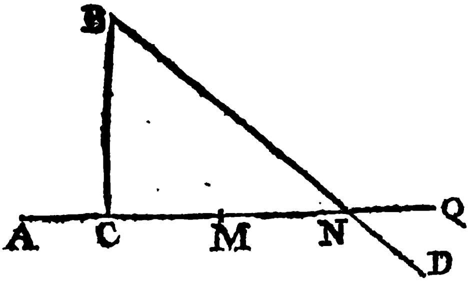
Whence this Conſtruction. Take CH to CB in the given Ratio of *m* to *n*, and draw HB; upon which produced (if neceſſary) let fall the Perpendicular AR; draw RN parallel to AH, meeting CQ in N; laſtly, draw NM parallel to AR, and it will give the Poſition required. For, firſt, it is plain, becauſe AM (RN) : BN (: : CH : CB) : : *m* : *n*, that M and N are cotemporary Poſitions: It is likewiſe plain, that RN and BN will be Secants of the Angles KNR (CMN) and KNB (CNM) to the Radius NK; becauſe the Angle NKR (=ARK) is a Right-one. Which Lines or Secants are in the propoſed Ratio of *m* to *n*, as has been already ſhewn.



But the ſame Solution may be, yet, otherwiſe derived, independent of Fluxions, from Principles intirely geometrical. For, let *m* and *n* be any two cotemporary Poſitions at Pleaſure, and let CH (as before) be to CB, as the Celerity in AP to that in CQ; moreover, let *nr*, parallel to AP, be drawn, meeting HB produced in *r*, and let A, *r* be joined. Then, ſince CB : CH : : B*n* : *nr* (*by ſim. Triangles*) and CB : CH : : B*n* : A*m*, (*by Hyp.*) it follows, that *nr* and A*m*, (which are parallel) will alſo be equal to each other; and therefore A*r* and *mn*, likewiſe equal and parallel. But A*r* is the leaſt poſſible when perpendicular to H*r*. Whence the Solution is manifeſt.

EXAMPLE XIV.

38. *Let the Body* M *move, uniformly, from* A *towards* Q, *with the Celerity m, and let another Body* N *proceed from* B, *at the ſame time, with the Celerity n. Now it is propoſed to find the Direction* (BD) *of the latter, ſo that the Diſtance* MN *of the two Bodies, when the latter arrives in the Way or Direction* AQ *of the former, may be the greateſt poſſible.*



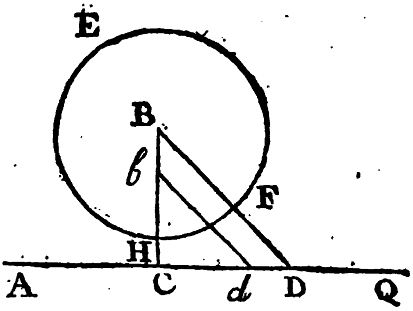
Let BC be perpendicular to AQ, and make AC = *a*, BC = *b*, and BN = *x*. Therefore, if the Poſition M be ſuppoſed cotemporary with N, we ſhall have $n : m :: x : AM = \frac{mx}{n}$; whence $CM = \frac{mx}{n} - a$, and conſequently MN (CN—CM) $= \sqrt{x^2 - b^2} - \frac{mx}{n} + a$; whereof the Fluxion being taken, and made $= 0$, we get $\frac{x}{\sqrt{x^2 - b^2}} = \frac{m}{n}$; therefore $x = \frac{mb}{\sqrt{m^2 - n^2}}$, and CN $(\sqrt{x^2 - b^2}) = \frac{nb}{\sqrt{m^2 - n^2}}$: Whence, $m : n$ (:: BN : CN :: Radius : Co-ſine N. The ſame Concluſion is otherwiſe derived, thus,

Let the Right-line BD be ſuppoſed to revolve about the Point B, as a Center, with a Motion ſo regulated, that the intercepted Part thereof BN may increaſe with the uniform Celerity n: Then, the Celerity with which *Art. 35. CN is increaſed being $= \frac{n \times \textit{Radius}}{\textit{Co-ſine}\ \mathrm{N}}$*, this Expreſſion, when MN is a *Maximum*, muſt, conſequently, be equal †Art. 22. to (m) the Velocity of the other Body † M; and therefore $m : n$ :: Radius : Co-ſine N, as before.

EXAMPLE XV.

39. *Suppoſing a Ship to ſail from a given Place* A, *in a given Direction* AQ, *at the ſame time that a Boat, from another given Place* B, *ſets out in order (if poſſible) to come up with her, and ſuppoſing the Rate at which each Veſſel runs to be given; it is required to find in what Direction the latter muſt proceed, ſo that, if it cannot come up with the former, it may, however, approach it as near as poſſible.*

Let the Celerity of the Ship be to that of the Boat in the given Ratio of m to n; alſo let D and F be the Places of the two Veſſels when neareſt poſſible to each other, and, from the Center B, through F, ſuppoſe the Circumference of a Circle to be deſcribed. Then (the Diſtance DF being the leaſt poſſible), the Point F muſt be in the Right-line (DB) joining the Point D and the Center



Center B; becauſe no other Point in the whole Periphery, at which the Boat from B might arrive in the ſame time, is ſo near to D as that wherein the Line DB interſects the ſaid Periphery.—But now, to get an Expreſſion for DF, in algebraic Terms, let BC be perpendicular to AQ, and make AC $=a$, BC $=b$, and CD $=x$; and then BD ($\sqrt{BC^2+CD^2}$) will be $=\sqrt{b^2+x^2}$; moreover, becauſe $m:n::$ AD $(a+x)$: BF, you will have BF $=\frac{na+nx}{m}$, and conſequently, DF $=\sqrt{b^2+x^2}-\frac{na+nx}{m}$; whoſe Fluxion, $\frac{x\dot{x}}{\sqrt{b^2+x^2}}-\frac{n\dot{x}}{m}$, being made $=0$, we find $x=\frac{nb}{\sqrt{m^2-n^2}}$; whence the Direction BD is known: And, if the Value of x, thus found, be ſubſtituted in that of DF, (found above) we ſhall have DF $=\frac{b\sqrt{m^2-n^2}-na}{m}$; whence the Poſition of F is known. And from which it is obſervable, that, as DF muſt be a *real*, *poſitive* Quantity (by the Queſtion) this Method of Solution can only obtain when m is greater than n, and $b\sqrt{m^2-n^2}$, alſo greater than na: For in all other Caſes the Boat will be able to come up with the Ship.

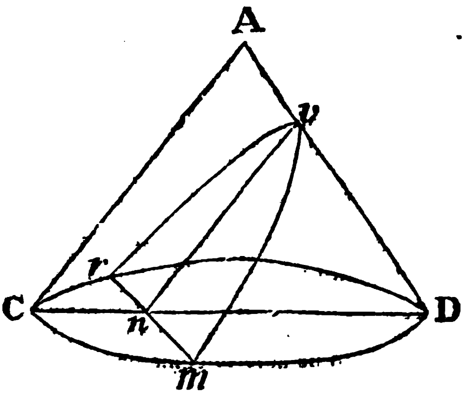
The ſame otherwiſe.

Let the Radius of the Circle EFH be conceived to increaſe uniformly, with the Celerity n, whilſt the Point n

D moves uniformly along AQ, with the Celerity m: Then, the Celerity at D, in the Direction of BD produced, being $= \frac{m \times \text{Co-ſine D}}{\text{Radius}}$, the relative Celerity with which the Point D recedes from the Periphery of the ſaid variable Circle, will be univerſally expreſſed by $\frac{m \times \text{Co-ſine D}}{\text{Radius}} - n$; which being $= 0$, when DF is a *Minimum*, we have in this Caſe $m \times$ *Co-ſine* D $= n \times$ *Radius*, and conſequently $m : n ::$ *Radius* : *Co-ſine* D. Therefore, if, at C, a right-angled Triangle C*bd* be conſtituted, whoſe Baſe C$d = n$, and its Hypothenuſe $db = m$, and parallel to the latter you draw BD, it will be the Direction required: In which, if there be taken BF, a Fourth-proportional to m, n, and AD, you will alſo have the Poſition required.

EXAMPLE XVI.

40. *To determine the greateſt Parabola that can be formed by cutting a given Cone* ACD.



Let *nv*, parallel to CA, be the Axis of the Parabola *rvm*, and *rm* the Baſe (or Ordinate) thereof; putting

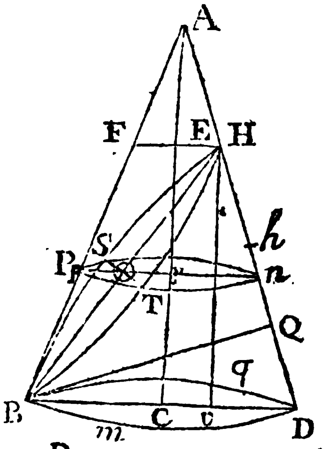
DC$=a$, CA$=b$, and D$n=x$; then, becauſe of the parallel Lines, it will be, $a:b::x:\frac{bx}{a}=nv$: Moreover, by the Property of the Circle, we have rn^2 $(=nm^2=\text{D}n\times\text{C}n)=ax-x^2$, and conſequently $rm=2\sqrt{ax-x^2}$; which multiplied by $\frac{2}{3}\times\frac{bx}{a}$ (becauſe every Parabola is $\frac{2}{3}$ of a Parallelogram of the ſame Baſe and Altitude) gives $\frac{4bx}{3a}\sqrt{ax-x^2}$ for the Content of the Parabola: Whoſe Fluxion, or that of ax^3-x^4 * being put equal to Nothing; we find $x=\frac{3a}{4}$: Whence $nv=\frac{3}{4}\times\text{AC}$, $rm=\text{CD}\times\sqrt{\frac{3}{4}}$, and the Area of the greateſt, or required, Parabola$=\text{AC}\times\text{CD}\times\frac{\sqrt{3}}{4}$.

*Art. 26.

EXAMPLE XVII.

41. *To determine the greateſt Ellipſis* BTES *that can be formed by cutting a given Cone* ABD.

Let BE be the greater, and TS the leſſer, Axis of the Ellipſis BTES, conſidered as variable by the Motion of (the End of the Tranſverſe) E, along the Line AD; moreover let Ev be parallel to AC the Axis of the Cone, meeting the Diameter BD in v, and let the Diameters EF and np be parallel to BD; whereof the latter np is ſuppoſed to



paſs through O the Center of the Ellipſis: Then, putting AC$=a$, CD$=b$, and C$v=x$, we ſhall have B$v=b+x$; alſo, becauſe of the parallel Lines we have CD (b) : CA (a) :: Dv $(b-x)$: $\frac{a\times\overline{b-x}}{b}=Ev$; whence BE $(\sqrt{\mathrm{B}v^2+\mathrm{E}v^2})$ $=\frac{\sqrt{b^2\times\overline{b+x}|^2+a^2\times\overline{b-x}|^2}}{b}$. Furthermore, ſince the Triangles EOn, EBD, and BOp, BEF are equiangular, and EO $(=$BO$)$ $=\frac{1}{2}$ BE, we likewiſe have O$n=\frac{1}{2}$BD$=b$, and O$p=\frac{1}{2}$EF$=$C$v=x$; and conſequently O$n\times$Op ($=$OT^2, *by the Property of the Circle*) $=bx$; whence ST $=2\sqrt{bx}$, and therefore BE$\times$ST $=\frac{\sqrt{b^2\times\overline{b+x}|^2+a^2\times\overline{b-x}|^2\times 4bx}}{b}$.

Now the Area of any Ellipſis being in a conſtant Ratio to the Rectangle of its greater and leſſer Axes (namely as 3,14159, &c. to 4) the laſt general Expreſſion muſt therefore be a *Maximum*, when the Area is ſo; and therefore its Fluxion, or that of $b^2x\times\overline{b+x}|^2+a^2x\times\overline{b-x}|^2$ $(=b^4x+2b^3x^2+b^2x^3+a^2b^2x$
Art.22. $-2a^2bx^2+a^2x^3)$ equal to Nothing; that is, $\dot{x}b^4$ $+4b^3x\dot{x}+3b^2x^2\dot{x}+a^2b^2\dot{x}-4a^2bx\dot{x}+3a^2x^2\dot{x}=0$: Whence $x^2-\frac{4bx\times\overline{a^2-b^2}}{3a^2+3b^2}=-\frac{b^2}{3}$, and $x=\frac{2b\times\overline{a^2-b^2}\pm b\sqrt{a^4-14a^2b^2+b^4}}{3a^2+3b^2}$; from which the Ellipſis is known.

M But it is obſervable, that, when $a^4-14a^2b^2+b^4$ is negative, this Solution fails, becauſe the Square Root of a negative Quantity is to be extracted. Therefore, to determine the Limit, put $a^4-14a^2b^2+b^4=0$; then, by ordering the Equation, you will get $a^2=b^2\times\overline{7+\sqrt{48}}$, and $a=b\times\overline{2+\sqrt{3}}$; and therefore $a:b::2+\sqrt{3}:1$. Hence, if the Ratio of AC to CD be not greater than that of $2+\sqrt{3}$ to 1, or (which comes to the ſame thing) if the Angle DAC be not leſs than 15 Degrees, the Fluxion of the Ellipſis can never become equal to Nothing; but the Ellipſis itſelf will increaſe continually, from the Vertex till it coincides with the Baſe of the Cone; and therefore is greater at the Baſe than in any other Poſition.

But it is further to be obſerved, that this Problem is confined to, yet, narrower Limits. For, either the Ellipſis will increaſe, continually, from the Vertex, to the Baſe, of the Cone, (which is ſhewn to be the Caſe when the Angle DAC is greater than 15°) or elſe it *will increaſe till the Point E arrives at a certain Poſition* H, *and afterwards decreaſe to another certain Poſition b*, and then increaſe again till it coincides with the Baſe of the Cone, (for it muſt always increaſe again before it coincides with the Baſe, becauſe, after the Point E is got below the Perpendicular BQ, both the Axes of the Ellipſis increaſe at the ſame time).

The ſame thing alſo appears from the foregoing Equation $x = \frac{2b\times\overline{a^2-b^2} \pm b\sqrt{a^4-14a^2b^2+b^4}}{3a^2+3b^2}$; whoſe two Roots expreſs the two Values of x (or Cv) at the Times of the *Maximum* (at H) and its ſucceeding *Minimum* (at *b*). Hence it is manifeſt, that the Ellipſis may admit of a *Maximum* between the Vertex of the Cone and the Perpendicular BQ, and yet, that *Maximum* be leſs than the Baſe of the Cone, unleſs the foreſaid Angle DAC be ſo much leſs than 15° (above found) that the Increaſe from *b* to D, be leſs than the Decreaſe from H to *b*. Now therefore, to determine the exact Limit, let the foreſaid Increment and Decrement be ſuppoſed equal to each other, or, which is the ſame in Effect, let the Ellipſis BTESB = the Circle BqDm, or BE × ST = BD², that is, let $\frac{\sqrt{b^2\times\overline{b+x}|^2+a^2\times\overline{b-x}|^2\times\overline{4bx}}}{b}=4b^2$: From which

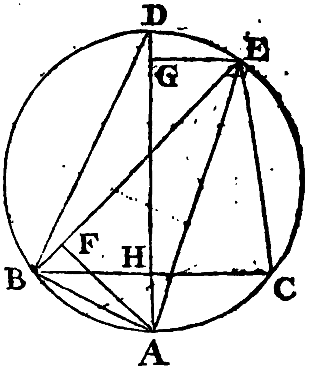
Equation you will get $a^2 = \frac{b^2}{x} \times \frac{4b^3 - b^2x - 2bx^2 - x^3}{\overline{b-x}|^2}$ $= \frac{b^2}{x} \times \frac{4b^2 + 3bx + x^2}{b - x}$: Moreover, from the Equation $b^4\dot{x} + 4b^3x\dot{x} + 3b^2x^2\dot{x} + a^2b^2\dot{x} - 4a^2bx\dot{x} + 3a^2x^2\dot{x} = 0$, (given above) you will, again, get $a^2 = \frac{b^2 \times \overline{b^2 + 4bx + 3x^2}}{-b^2 + 4bx - 3x^2}$ $= \frac{b^2 \times \overline{b^2 + 4bx + 3x^2}}{\overline{b-x} \times \overline{3x-b}}$; Whence, by comparing these equal Values, there ariſes $\frac{4b^2 + 3bx + x^2}{x} = \frac{b^2 + 4bx + 3x^2}{3x - b}$, which, ordered, gives $x^2 + 2bx - b^2 = 0$, and therefore $x = b\sqrt{2} - b$.

Moreover, $\frac{a^2}{b^2}$ being $= \frac{4b^2 + 3bx + x^2}{bx - x^2}$, if $b^2 - 2bx$ be ſubſtituted herein for, its Equal, x^2, it will become $\frac{a^2}{b^2} = \frac{5b^2 + bx}{bx - x^2} = \frac{5b + x}{3x - b} = \frac{5b + b\sqrt{2} - b}{3b\sqrt{2} - 3b - b} = \frac{4 + \sqrt{2}}{-4 + 3\sqrt{2}}$ $= \frac{\overline{4 + \sqrt{2}} \times \overline{4 + 3\sqrt{2}}}{\overline{-4 + 3\sqrt{2}} \times \overline{4 + 3\sqrt{2}}} = \frac{22 + 16\sqrt{2}}{2} = 11 + 8\sqrt{2}$. Hence we have, $1 : \sqrt{11 + 8\sqrt{2}} :: b$ (DC) : a (AC) :: Radius to the Tangent of the Angle ADC = 78° 3′: Whoſe Complement DAC = 11° 57′, is the leaſt Limit poſſible. Therefore, unleſs the Angle which the ſlant Side makes with the Axis be leſs than 11° 57′, the greateſt Ellipſis will be leſs than the Baſe of the Cone.

EXAMPLE XVIII.

42. *Of all Triangles, having the ſame given Perimeter, and inſcribed in the ſame given Circle; to determine the greateſt.*

Let the Diameter DA biſect the Baſe BC of the required Triangle BEC in H, draw AE, AB and BD; alſo draw AF perpendicular to BE, and GE, parallel to BC,



BC, meeting AD in G: Then, putting AD = a, half the given Perimeter of the Triangle = b, and DH = y; we have BH = $\sqrt{ay-y^2}$, and therefore EF = $b-\sqrt{ay-y^2}$[a]. Moreover DH (y) : AD (a) :: DB² : DA² :: EF² $\left(\overline{b-\sqrt{ay-y^2}}\,^2\right)$: EA² $= \frac{a}{y} \times \overline{b-\sqrt{ay-y^2}}\,^2$; therefore AG $\left(\frac{AE^2}{AD}\right) = \frac{\overline{b-\sqrt{ay-y^2}}\,^2}{y}$, and HG = (AG—AH =) $\frac{b^2-2b\sqrt{ay-y^2}}{y}$; whence the Area of the Triangle BEC (BH × HG) $= \frac{b^2\sqrt{ay-y^2}}{y} - 2ba + 2by$, whoſe Fluxion $2b\dot{y} - \frac{\frac{1}{2}ab^2\dot{y}}{y\sqrt{ay-yy}}$ being put = 0, gives $y\sqrt{ay-yy} = \frac{1}{4}ba$; whence y, and from thence the Sides of the Triangle may be determined.

EXAMPLE XIX.

43. *To determine the greateſt Area that can be contained under four given Right-lines.*

Though it is demonſtrable from common Geometry that the Area will be a *Maximum*, when the Trapezium ABCD, formed by the given Lines, may be inſcribed in a Circle[b], yet I ſhall here give the Solution from the Principles of Fluxions, (whoſe Uſes I am now illuſtrating).

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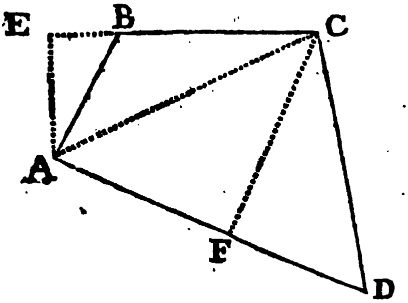
[a] By Prop. 13. Page 62. *Elem. Trig.*

[b] See Page 117. of *Elem. Geometry.*



illuſtrating. In order to which, let the Diagonal AC be drawn, and upon CB and AD let fall the Perpendiculars AE and CF; putting AB $=a$, BC $=b$, CD $=c$, DA $=d$, BE $=x$, and DF $=y$: Then AE being $=\sqrt{a^2-x^2}$, and CF $=\sqrt{c^2-y^2}$, the Area of the Trapezium ($\frac{1}{2}$BC x AE + $\frac{1}{2}$AD x CF) will be $=\frac{1}{2}b\sqrt{a^2-x^2}+\frac{1}{2}d\sqrt{c^2-y^2}$; and its Fluxion $\frac{-\frac{1}{2}bx\dot{x}}{\sqrt{a^2-x^2}}-\frac{\frac{1}{2}dy\dot{y}}{\sqrt{c^2-y^2}}=0$ *; and therefore $\frac{-dy\dot{y}}{\sqrt{c^2-y^2}}=\frac{bx\dot{x}}{\sqrt{a^2-x^2}}$. Moreover, ſince b^2+a^2+2bx ($=$AC^2) $=d^2+c^2-2dy$, by taking the Fluxion thereof, we have $2b\dot{x}=-2d\dot{y}$, or $-d\dot{y}=b\dot{x}$; which, ſubſtituted for $-d\dot{y}$ in the foregoing Equation, gives $\frac{b\dot{x}y}{\sqrt{c^2-y^2}}=\frac{bx\dot{x}}{\sqrt{a^2-x^2}}$, and $\frac{y}{\sqrt{c^2-y^2}}=\frac{x}{\sqrt{a^2-x^2}}$; and conſequently, $\sqrt{c^2-y^2}$ (CF) : y (DF) :: $\sqrt{a^2-x^2}$ (AE) : x (BE) : From which it appears that the Triangles DCF and ABE are ſimilar, and that (D + ABC being = 2 Right-angles) the Trapezium may be inſcribed in a Circle; but this by the Bye. We are now to get an Expreſſion for the Area in known Terms, and in order thereto we have $b^2+a^2+2bx=dd+c^2-2dy$, $y=\frac{cx}{a}$, and CF $=\frac{c\sqrt{a^2-x^2}}{a}$ (becauſe AB : BE :: DC : DF, &c.): Therefore, by Subſtitution, $b^2+a^2+2bx=d^2+c^2-\frac{2cdx}{a}$, and the Area ($\frac{1}{2}$BC x AE + $\frac{1}{2}$AD

*Art. 22.

$+\frac{1}{2}\mathrm{AD}\times\mathrm{CF}) = \frac{1}{2}b\sqrt{a^2-x^2} + \frac{cd}{2a}\sqrt{a^2-x^2} = \frac{ab+cd}{2a}\sqrt{a^2-x^2}$; and therefore the Square thereof $= \frac{\overline{ab+cd}|^2}{4a^2}\times\overline{a^2-x^2} = \frac{\overline{ab+cd}|^2}{4a^2}\times\overline{a+x}\times\overline{a-x} = \frac{\overline{ab+cd}|^2}{4}\times\overline{1+\frac{x}{a}}\times\overline{1-\frac{x}{a}}$. But ſince $b^2+a^2+2bx = d^2+c^2-\frac{2cdx}{a}$, we have $\frac{x}{a} = \frac{d^2+c^2-b^2-a^2}{2ab+2cd}$, $1+\frac{x}{a} = 1+\frac{dd+c^2-b^2-a^2}{2ab+2cd} = \frac{2ab+2cd+dd+c^2-b^2-a^2}{2ab+2cd} = \frac{\overline{d+c}|^2-\overline{b-a}|^2}{2ab+2cd}$; and $1-\frac{x}{a} = \frac{2ab+2cd-dd-c^2+b^2+a^2}{2ab+2cd} = \frac{\overline{b+a}|^2-\overline{d-c}|^2}{2ab+2cd}$; and conſequently, the Square of the Area $= \frac{\overline{ab+cd}|^2}{4}\times\frac{\overline{d+c}|^2-\overline{b-a}|^2}{2ab+2cd}\times\frac{\overline{b+a}|^2-\overline{d-c}|^2}{2ab+2cd} = \frac{\overline{\overline{d+c}|^2-\overline{b-a}|^2}\times\overline{\overline{b+a}|^2-\overline{a-c}|^2}}{16}$ which (becauſe the Difference of the Squares of any two Quantities is equal to a Rectangle under their Sum and Difference) will alſo be $= \frac{\overline{d+c+b-a}\times\overline{d+c-b+a}\times\overline{b+a+a-c}}{4}\times\frac{\overline{b+a-d+c}}{4} = \overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-a}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-b}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-c}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-d}$. Whence it appears, that, if from $\frac{1}{2}$ the Sum of all the four Sides each particular Side be ſubtracted, the continual Product of the Remainders will be the Square, or ſecond Power, of the Area.

From this Theorem, the Rule in common Practice, for finding the Area of a Triangle, having the three Sides given, is deduced, as a Corollary: For, making

$a = 0$, the Trapezium becomes a Triangle, and the second Power of its Area $= \overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b} \times \overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-b} \times \overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-c} \times \overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-d}$: Which, in Words, is the common Rule.

EXAMPLE XX.

44. *To find the greateſt Value of y in the Equation* $a^4x^2 = \overline{xx+yy}|^3$.

By putting the whole Equation into Fluxions, &c. we have $2a^4x\dot{x} = \overline{2x\dot{x}+2y\dot{y}} \times 3 \times \overline{x^2+y^2}|^2$; which in the *Art.22. required Circumſtance, when $\dot{y} = 0$ *, becomes $2a^4x\dot{x} = 6x\dot{x} \times \overline{x^2+y^2}|^2$; whence $x^2+y^2 = \frac{a^2}{\sqrt{3}}$, and $\overline{x^2+y^2}|^3 = \frac{a^6}{3\sqrt{3}}$: But, by the given Equation $\overline{x^2+y^2}|^3 = a^4x^2$; conſequently $a^4x^2 = \frac{a^6}{3\sqrt{3}}$, and therefore $x = a\sqrt{\frac{1}{3\sqrt{3}}}$; whence $y^2 \left(= \frac{a^2}{\sqrt{3}} - x^2\right) = \frac{2a^2}{3\sqrt{3}}$, and $y = a\sqrt{\frac{2}{3\sqrt{3}}}$.

The ſame otherwiſe.

Since $\overline{xx+yy}|^3$ is given $= a^4x^2$, we have $x^2+y^2 = a^{\frac{4}{3}} \times x^{\frac{2}{3}}$, and therefore $y^2 = a^{\frac{4}{3}} \times x^{\frac{2}{3}} - x^2$; whoſe Fluxion, $\frac{2}{3}a^{\frac{4}{3}} \times x^{-\frac{1}{3}}\dot{x} - 2x\dot{x}$, being put $= 0$, we alſo get $\frac{a^{\frac{4}{3}} \times x^{-\frac{1}{3}}}{3} = x$; whoſe Cube is $\frac{a^4 \times x^{-1}}{27} = x^3$, or $\frac{a^4}{27x} = x^3$; whence $27x^4 = a^4$, and conſequently $x = a\sqrt{\frac{1}{3\sqrt{3}}}$, the ſame as before.

45. When, in the general Expreſſion, whoſe *Maximum* or *Minimum* is ſought, there are two or more indeterminate Quantities, independent of each other, their reſpective Values, in the required Circumſtance, will be determined, by making them flow, one by one, while the others are ſuppoſed invariable; as in the following

EXAMPLE XXI.

Wherein it is propoſed to find three ſuch Values of x, y, *and* z, *as ſhall make the Value of* $\overline{b^3 - x^3} \times \overline{x^2z - z^3} \times \overline{xy - y^2}$ *the greateſt poſſible.*

Firſt, conſidering y as variable, and the reſt conſtant, we have $x\dot{y} - 2y\dot{y} = 0$ *; whence $y = \frac{1}{2}x$, and $xy - y^2 = \frac{1}{4}x^2$. By making z variable, we have $x^2\dot{z} - 3z^2\dot{z} = 0$; whence $z = \frac{x}{\sqrt{3}}$, and $x^2z - z^3 = \frac{2x^3}{3\sqrt{3}}$. Now let theſe Values of $xy - y^2$ and $x^2z - z^3$ be ſubſtituted in the given Expreſſion, and it will become $\frac{x^2}{4} \times \frac{2x^3}{3\sqrt{3}} \times \overline{b^3 - x^3} = \frac{b^3x^5 - x^8}{6\sqrt{3}}$; therefore $5b^3x^4\dot{x} - 8x^7\dot{x} = 0$: Whence $x = \frac{1}{2}b \times \sqrt[3]{5}$, $y \left(= \frac{1}{2}x\right) = \frac{1}{4}b \times \sqrt[3]{5}$, and $z \left(= \frac{x}{\sqrt{3}}\right) = \frac{1}{2}b \times \frac{\sqrt[3]{5}}{\sqrt{3}}$. *Art. 22.

The Reaſon of the foregoing Proceſs is obvious: For, if the Fluxion of the given Expreſſion, when any one of the indeterminate Quantities is made variable, be not equal to Nothing, that Expreſſion may become greater, without altering the Values of the reſt, which are conſidered as conſtant †: And therefore cannot be the greateſt poſſible, unleſs the ſaid Fluxion is equal to Nothing. †Art. 22.

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EXAMPLE XXII.

46. *To determine the different Values of x, when that of* $3x^4-28ax^3+84a^2x^2-96a^3x+48b^4$ *becomes a* Maximum *or* Minimum.

The Fluxion of the given Expreſſion being *(as uſual)* put equal to Nothing, we have $12\dot{x}^3-84ax^2+168a^2x-96a^3=0$, or $x^3-7ax^2+14a^2x-8a^3=0$: From whence (by the Method of Diviſors) we get $x-a=0$, $x-2a=0$, or $x-4a=0$: Therefore, the Roots of the Equation, or the three Values of x, are a, $2a$, and $4a$.

SCHOLIUM.

47. It appears, from the laſt Example, that a Quantity may admit of as many *Maxima* and *Minima* (according to the Meaning of the Definition *) as there are poſſible Roots in the Equation, ariſing from aſſuming its Fluxion equal to Nothing. Now to know which of thoſe Roots point out a *Maximum*, and which a *Minimum*; find whether the Value of the ſaid Fluxion, a little before it becomes equal to Nothing, be poſitive or negative; if *poſitive*, the ſucceeding Root gives a *Maximum*; but if *negative*, a *Minimum*: The Reaſon of which is extremely obvious; becauſe ſo long as any Quantity increaſes, its Fluxion is poſitive, but when it decreaſes the Fluxion is negative.

As an Example hereof, let the Quantity $3x^4-28ax^3+84a^2x^2-96a^3x+48b^4$, be again reſumed; whoſe Fluxion is $12\dot{x}\times\overline{x^3-7ax^2+14a^2x-8a^3}=12\dot{x}\times\overline{x-a}\times\overline{x-2a}\times\overline{x-3a}$: Whereof the Value, before it becomes equal to Nothing, the firſt time (or before $x=a$) being negative (becauſe the Product of three negative Factors is negative) its firſt Root (a) therefore indicates a *Minimum*: Whence we may conclude, without conſidering farther, that the ſecond Root $(2a)$ gives a *Maximum*, and the third $(4a)$ another *Minimum*. But, if you

* Art. 22.

you would know whether the firſt or third Root gives the leſſer Value of the two; it is but ſubſtituting in the given Quantity, which will come out $48b^4-37a^4$, and $48b^4-64a^4$ reſpectively; therefore the latter is the leſſer, and the very leaſt Value the propoſed Expreſſion can admit of.

When all the Roots prove impoſſible, the Quantity propoſed (as its Fluxion can never become $=0$) muſt either increaſe, or decreaſe, continually; and therefore can neither admit of a *Maximum nor a Minimum.*

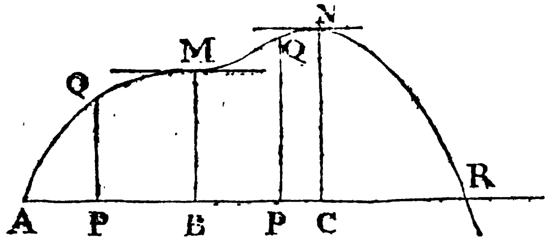
Moreover, it may ſo happen, that the Roots are poſſible, the Fluxion $=0$, and yet the Quantity itſelf be neither a *Maximum* nor a *Minimum* in that Circumſtance.

For let us, again, ſuppoſe the Point n to move after m, as in the general Illuſtration, (*vid. Art.* 22.) only let the Velocity of n (*in the firſt Caſe*) increaſe no longer than 'till it arrives at D; after which let it again decreaſe: Then, though the Fluxion of the Diſtance mn is Nothing, at the Poſition CD, yet the Diſtance itſelf will not be a *Maximum*; becauſe n (having afterwards, as well as before, a leſs Velocity than m) will ſtill continue to loſe ground.—In the ſame manner the Matter may be explained with regard to a *Minimum.* And it is evident, that theſe Caſes will always happen when the Fluxion of the given Quantity is of the ſame Denomination (with regard to poſitive and negative) both before and after, it becomes equal to Nothing: Which, by the Rules of common Algebra, is known to be when the Equation admits of an even Number of equal Roots.—An Example hereof, however, may not be improper.

Let then the Quantity propoſed be $24a^3x-30a^2x^2+16ax^3-3x^4$; whoſe Fluxion is $24a^3\dot{x}-60a^2x\dot{x}+48ax^2\dot{x}-12x^3\dot{x}=12\dot{x}\times\overline{a-x}\times\overline{a-x}\times\overline{2a-x}$: Which being made $=0$, it appears that the two leaſt Roots are equal. Therefore there is neither a *Maximum* nor *Minimum* when $x=a$ (becauſe whether x be taken a little leſs, or a little greater, than a, the Value of the Fluxion will ſtill be affirmative.) The greateſt Root, however, not being affected with another equal one, indicates a *Maximum*, according to the Rule above preſcribed.

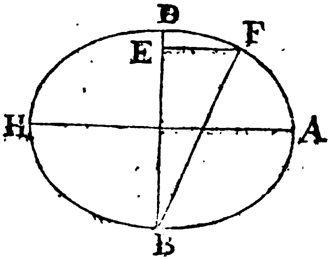
To render what has been obſerved above ſtill more conſpicuous, let the given Expreſſion, $24a^3x - 30a^2x^2 + 16ax^3 - 3x^4$, be repreſented by the variable Ordinate PQ of the Curve AQMNR; whoſe Abſciſſa AP is (as uſual) denoted by x.

Then, whilſt ($12\dot{x} \times \overline{a-x} \times \overline{a-x} \times \overline{2a-x}$) the Fluxion of the Ordinate continues poſitive, (or till x becomes $= a = AB$) the Ordinate itſelf will increaſe: But at the Poſition BM it becomes ſtationary (if I may be allowed the Expreſſion) the Fluxion being then $= 0$. After which, the Fluxion being again affirmative, the Ordinate will again increaſe, till x becomes $= 2a$ ($= AC$); when, the Fluxion becoming Nothing, a ſecond time,) and afterwards negative, CN will be a *Maximum*: Soon after which the Curve deſcends below its Axis, and continues to recede from it *in infinitum*.



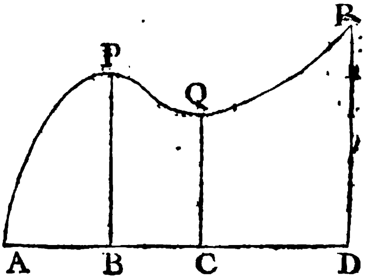
Another Thing there is that ought to be regarded in the Solution of theſe Kinds of Problems, and that is, whether the *Maxima* or *Minima*, found by aſſuming the Fluxion $= 0$, fall within the Limits preſcribed by the Nature of the Queſtion or Figure; which is often reſtrained by Conditions that do not enter into the algebraic Computation.

Thus, for Example; ſuppoſe it were required to find that Point (F) in a given Ellipſis ABHD which, of all others, is the moſt remote from the Extreme B of the conjugate Axis BD.



Then, drawing FE parallel to the Tranſverſe AH, and putting AH $= a$, BD $= b$, and BE $= x$, we have, by the Property of the Curve BF^2 $(= BE^2 + EF^2)$ $x^2 + \overline{bx - x^2} \times \frac{a^2}{b^2}$; from whence x is found $= \frac{\frac{1}{2}a^2b}{a^2 - b^2}$. But, from the Nature of the Figure, the greateſt Value that x $(= BE)$ can poſſibly admit of is b $(= BD)$, therefore if the Relation of a and b be ſuch, that $\frac{\frac{1}{2}a^2b}{a^2 - b^2}$ is greater than b, this Solution is manifeſtly impoſſible. —— To determine the Limit, therefore, make $\frac{\frac{1}{2}a^2b}{a^2 - b^2} = b$; then it will be found that $2b^2 = a^2$. Whence the foregoing Solution can only obtain when $2BD^2$ is equal to, or leſs than AH^2.

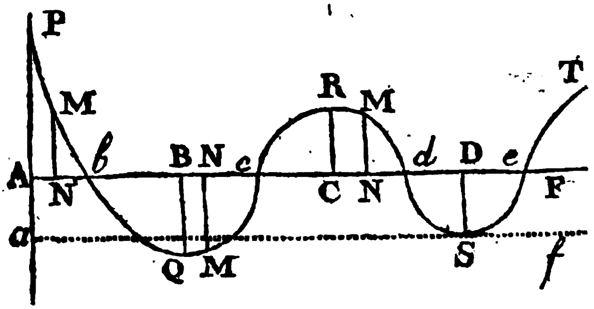
Again, it ought to be alſo conſidered whether the Value of x, found by the common Method, gives a leſs Quantity for the *Maximum*, and a greater for the *Minimum*, than will ariſe from the Extremes themſelves by which x is limited.



Thus, let it be required to determine the greateſt and leaſt Ordinates in a Curve, APR, whoſe Equation is $y^3 = 6a^2x - 9ax^2 + 4x^3$, and whoſe greateſt Abſciſſa AD is given equal $2a$.

Here we ſhall, by taking the Fluxion, &c. have $x = \frac{1}{2}a$, or $x = a$: The former of which Values gives the correſponding Ordinate BP $= a\sqrt[3]{\frac{5}{4}}$; and the latter, CQ $= a$: But the firſt of theſe is not the greateſt of all others, becauſe the Extreme DR exceeds it, being $= 2a$; nor is CQ the leaſt poſſible, becauſe the Ordinate at the other Extreme A is nothing at all.

Sometimes one, or more, of the Points Q, S, &c. determining the *Maxima* and *Minima*, will fall below the Axis AF, *(as in the annexed Figure)*. In which Caſe the correſponding Value of the general Expreſſion, repreſented by the Ordinate, will be negative: But at the Points b, c, d, &c. where the Curve interſects the Axis, it will be equal to nothing: Whence (by the Bye) the Reaſon why the Roots of an Equation ($x^n - ax^{n-1} + b^2x^{n-2} \ldots\ldots + q^n = 0$) are impoſſible by Pairs is evident. For, ſeeing Ab, Ac, Ad, Ae, &c. are the Roots of that Equation, or the different Values of x, when the Ordinate $x^n - ax^{n-1} + b^2x^{n-2} \ldots\ldots + q^n$ (MN) becomes equal to Nothing, it is plain, if PA, expreſſing the given Term q^n, be increaſed to Pa, ſo that AF (then coinciding with af) may touch the Curve in S, the adjacent Roots Ad and Ae will then become equal;



equal; and if q'' be farther increaſed, ſo that the Axis may fall wholly below the Curve, not only thoſe two, but alſo the other Roots, *Ab* and *Ac*, will become impoſſible.

Various other Obſervations might be made, relating to the Limits of Equations, determined by theſe *Maxima* and *Minima*; but this being foreign to the Matter in hand, I ſhall content myſelf with one Remark more, *viz.*

Any Expreſſion which, being put equal to Nothing, admits of two or more equal Roots, has as many ſucceeding Orders of Fluxions equal to Nothing, at the ſame time, as are expreſſed by the Number of thoſe Roots minus one.

Thus, an Equation, having three equal Roots, has both its firſt and ſecond Fluxions equal to Nothing, when the Fluent itſelf is equal to Nothing.

Hence we have another Way (beſides that given above) to know when a Quantity may have its Fluxion equal to Nothing, and yet neither admit of a *Maximum* nor a *Minimum*: For, ſince this Circumſtance always takes place when the Equation admits of an *even* Number of equal Roots (as has been already ſhewn) the Number of Orders of Fluxions, equal to Nothing, at the ſame time (including the Firſt) muſt alſo be even.

Hence, alſo, we have an eaſy Method for diſcovering when ſome of the Roots of an Equation are equal; and, if ſo, what they are.

Thus, let $x^3 - 3ax^2 + 4a^3 = 0$ be propounded; whereof the Fluxion $3x^2\dot{x} - 6ax\dot{x}$ being aſſumed equal to Nothing, we find $x = 2a$; which will alſo be a Root of the given Equation, if it admits of two equal ones: To try it, therefore, I ſubſtitute $2a$ for x, and find it anſwers.

Again, let $8x^4 - 28ax^3 + 18a^2x^2 + 27a^3x - 27a^4 = 0$; whereof the firſt and ſecond Fluxions being $32x^3\dot{x} - 84ax^2\dot{x} + 36a^2x\dot{x} + 27a^3\dot{x}$ and $96x^2\dot{x}^2 - 168ax\dot{x}^2 + 36a^2\dot{x}^2$, if the latter of them be aſſumed $= 0$, x will be found $=\frac{7a}{8}\pm\sqrt{\frac{25a^2}{64}}=\frac{3a}{2}$, or $\frac{a}{4}$: One of which Quantities, if the Equation propoſed admits of three equal Roots, will be the Value of each of them: By trying $\frac{3a}{2}$, it will be found to ſucceed. Whence, by a well known Rule, the fourth Root (being $=\frac{28a}{8}-\frac{3a}{2}\times 3=-a$) is alſo given.

The Reaſon of theſe Operations, as well as what is aſſerted above, may be thus demonſtrated.

Let $\overline{r-x}\times\overline{r-x}$ &c. $\times\overline{A+Bx+Cx^2}$ &c. $=0$, be any Equation, having two or more equal Roots, repreſented, each, by r: Put $y=r-x$, and let the Number of the equal Roots be denoted by n; then, by Subſtitution, we have $y^n\times\overline{A+B\times\overline{r-y}+C\times\overline{r-y}|^2}$ &c. $=0$; which, by expanding the Powers of $r-y$, and putting $a=A+Br+Cr^2$ &c. $b=\overline{B+2Cr+3Dr^2}$, &c. will be further transformed to $y^n\times\overline{a-by+cy^2-dy^3}$ &c. $=$: Whoſe Fluxion $na\dot{y}y^{n-1}-\overline{n+1}\cdot b\dot{y}y^n+\overline{n+2}\cdot c\dot{y}y^{n+1}$ &c. is evidently equal to Nothing, when y, or its Equal $r-x$, is Nothing (provided n be greater than Unity. It is equally plain, that the ſecond Fluxion $n\cdot\overline{n-1}\cdot a\dot{y}^2y^{n-2}-\overline{n+1}\cdot nb\dot{y}^2y^{n-1}+\overline{n+2}\cdot\overline{n+1}\cdot c\dot{y}^2y^n$ &c. will alſo be equal to Nothing, in the ſame Circumſtance, if n be greater than 2, &c. &c.

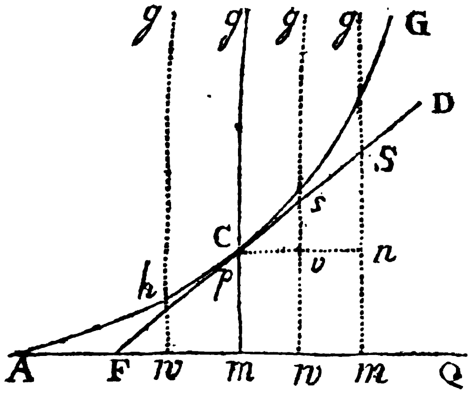
Hence, univerſally, let the Number (n) of equal Roots be what it will, that of the Orders of Fluxions equal to Nothing, at the ſame time, will be expreſſed by that Number *minus* one, as was to be ſhewn.

SECTION III.

The Uſe of FLUXIONS in drawing Tangents to Curves.

ILLUSTRATION.

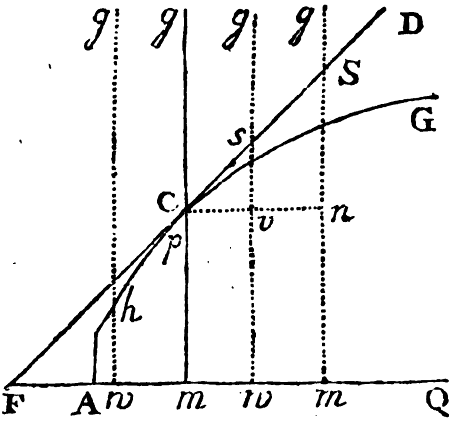
48. LET ACG be a Curve of any kind, and C the given Point from whence the Tangent is to be drawn.



Conceive a Right-line mg to be carried along uniformly, parallel to itſelf, from A towards Q, and let, at the ſame time, a Point p ſo move in that Line, as to deſcribe, or trace out, the given Curve ACG: Alſo let mm, or Cn (equal and parallel to mm) expreſs the Fluxion of Am, or the Celerity wherewith the Line mg is carried; and let nS expreſs the correſponding Fluxion of mp, in the Poſition mCg, or the Celerity of the Point p, in the Line mg. Moreover, through the Point C let the Right-line SF be drawn, meeting the Axis of the Curve (AQ) in F.

Now, it is evident, if the Motion of p, along the Line mg, was to become equable at C, the Point p would be at S, when the Line itſelf had acquired the Poſition mSg (becauſe, by Hypotheſis, Cn and nS expreſs the Diſtances that might be deſcribed by the two uniform Motions in the ſame time).

And, if wsg be aſſumed to repreſent any other Poſition of that Line, and s the contemporary Poſition of the Point p (ſtill ſuppoſing an equable Celerity of p); then the Diſtances Cv and vs, gone over, in the ſame time, by the two Motions, will, always, be to each other as the Celerities, or as Cn to nS: Therefore, ſince Cv : vs :: Cn : nS (which is a known Property of ſimilar Triangles) the Point s will, always, fall in the Right-line FCS: Whence it appears, that, if the Motion of the Point p along the Line mg was to become uniform at C, that Point would then move in the Right-line CS, inſtead of the Curve-line CG.



Now, ſeeing the Motion of p, in the Deſcription of Curves, muſt, either, be an accelerated or a retarded one, let it be, firſt, conſidered as an accelerated one: In which Caſe the Arch CG will fall, *wholly*, above the Right-line CD (as in Fig. 1.) becauſe the Diſtance of the Point p from the Axis AQ, at the End of any given Time, is greater than it would be if the Acceleration was to ceaſe at C; and, if the Acceleration had ceaſed at C, the Point p would (it is proved) have been always found in the ſaid Right-line FS.

But if the Motion of the Point p be a retarded one, it will appear, by reaſoning in the ſame manner, that the Arch CG will fall wholly below the Right-line CD (as in Fig. 2.)

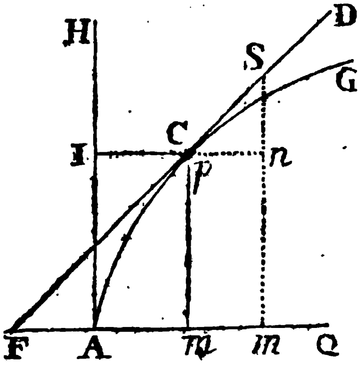
This being the Caſe, let the Line mg, and the Point p, along that Line, be now ſuppoſed to move back again, towards A and m, in the ſame manner they proceeded from thence: Then, ſince the Celerity of p (*Fig.* 1.) did before increaſe, it muſt now, on the contrary, decreaſe; and, therefore, as p, at the End of a given Time, after repaſſing the Point C, is not ſo near to AQ, as it would have been, had the Velocity continued the ſame as at C, the Arch Cb (as well as CG) muſt fall wholly above the Right-line FCD. And, by the ſame Method of arguing, the Arch Cb, *in the ſecond Caſe*, will fall, *wholly*, below FCD: Therefore FCD, *in both Caſes*, is a Tangent to the Curve at the Point C: Whence, the Triangles FmC and CnS being ſimilar, it appears, that the Sub-tangent mF is always a Fourth-proportional to (nS) the Fluxion of the ordinate (Cn), the Fluxion of the Abſciſſa, and the Ordinate (Cm).

Otherwiſe.

49. Let ACG repreſent the propoſed Curve, and let the Right-line FCD be a Tangent to it, at any Point C, meeting the Axis AQ (produced if neceſſary) in F: Suppoſe a Point p to move along the Curve, from A towards G, and let the abſolute Celerity thereof at C, in the Direction of the Tangent CD, or the Fluxion of the Line Ap ſo generated *, be denoted by CS, any Part of the ſaid Tangent: Then, if AH, mp and mS be made perpendicular, and Ipn parallel, to AQ, the relative Celerities of that Point, in the Directions Cn and nC, wherewith Ip (= Am) and mp increaſe in this

* Art. 2 and 5.

*Art. 35. Poſition, will be truly expreſſed by *Cn* and *nS* *: But the Celerities by which Quantities increaſe are as the Fluxions of thoſe Quantities; Therefore (CS being the Fluxion of the Curve-line *Ap*) *Cn* and *nS* are the correſponding Fluxions of the Abſciſſa *Am* and the Ordinate *mp*: And we have *Sn* : *nC* :: *mC* : *mF*, *the ſame as before.*

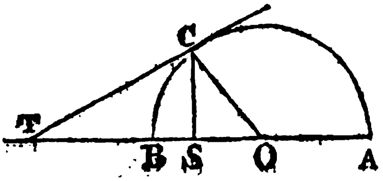


Hence, if the Abſciſſa *Am* be put = x, and the Ordinate *mp* = y, we ſhall have $mF = \frac{y\dot{x}}{\dot{y}}$: By means of which general Expreſſion, and the Equation expreſſing the Relation between x and y, the Ratio of the Fluxions $\dot{x}$ and $\dot{y}$ will be found, and from thence the Length of the Sub-tangent (*mF*) as in the following Examples.

EXAMPLE I.

50. *To draw a Right-line* CT, *to touch a given Circle* BCA, *in a given Point* C.

Let CS be perpendicular to the Diameter AB, and put AB = a, BS = x and SC = y: Then, by the Property of the Circle, y^2 (CS^2) = BS × AS ($= x \times \overline{a-x}$) $= ax - x^2$; whereof

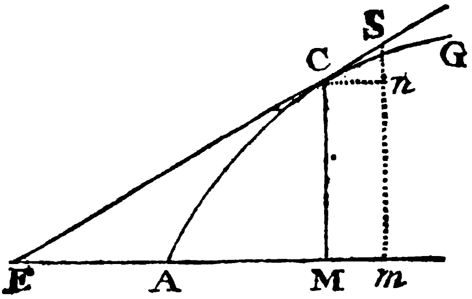


whereof the Fluxion being taken, in order to determine the Ratio of $\dot{x}$ and $\dot{y}$, we get $2y\dot{y} = a\dot{x} - 2x\dot{x}$; conſequently $\frac{\dot{x}}{\dot{y}} = \frac{2y}{a - 2x} = \frac{y}{\frac{1}{2}a - x}$, which, multiplied by y, gives $\frac{y\dot{x}}{\dot{y}} = \frac{y^2}{\frac{1}{2}a - x} =$ the Sub-tangent ST*. Whence (O being ſuppoſed the Center) we have OS $(\frac{1}{2}a - x)$: CS (y) :: CS (y) : ST; which we alſo know from other Principles. * Art. 48 and 49.

EXAMPLE II.

51. *To draw a Tangent to any given Point* C *of the conical Parabola* ACG.

If the *Latus Rectum* of the Curve be denoted by a, the Ordinate MC by y, and its correſponding Abſciſſa



AM by x; then the known Equation, expreſſing the Relation of x and y, being $ax = y^2$, we have, in this Caſe, $a\dot{x} = 2y\dot{y}$; whence $\frac{\dot{x}}{\dot{y}} = \frac{2y}{a}$, and conſequently $\frac{y\dot{x}}{\dot{y}}$† $= \frac{2y^2}{a} = \frac{2ax}{a} = 2x =$ MF. Therefore the Sub-tangent is juſt the double of its correſponding Abſciſſa AM: Which we likewiſe know from other Principles. † Art. 48 and 49.

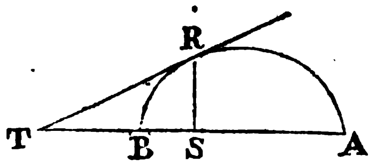
EXAMPLE III.

52. *To draw a Tangent to a Parabola of any kind.*

The general Equation of theſe ſort of Curves being $a^m x^n = y^{m+n}$, we have $na^m x^{n-1}\dot{x} = \overline{m+n} \times y^{m+n-1}\dot{y}$, and therefore $\frac{\dot{x}}{\dot{y}} = \frac{\overline{m+n} \times y^{m+n-1}}{na^m x^{n-1}}$; whence $\frac{y\dot{x}}{\dot{y}} = \frac{\overline{m+n} \times y^{m+n}}{na^m x^{n-1}} = \frac{\overline{m+n} \times a^m x^n}{na^m x^{n-1}}$ (becauſe $y^{m+n} = a^m x^n$) $= \frac{m+n}{n} \times x =$ the true Value of the Subtangent: Which, therefore, is to the Abſciſſa, in the conſtant Ratio of $m+n$ to n.

EXAMPLE IV.

53. *To draw a Tangent* RT, *to a given Point* R, *in a given Ellipſis* BRA.



If RS be an Ordinate to the principal Axis AB, and there be put *(as uſual)* BS $= x$, RS $= y$, AB $= a$, and the leſſer Axis $= b$; we ſhall, by the Property of the Curve, have $a^2 : b^2 :: ax - x^2$ (BS × AS) : y^2 (RS^2), and therefore $b^2 \times \overline{ax - x^2} = a^2 y^2$: Whence $b^2 \times \overline{a\dot{x} - 2x\dot{x}} = 2a^2 y\dot{y}$, and $\frac{\dot{x}}{\dot{y}} = \frac{2a^2 y}{b^2 \times \overline{a - 2x}}$; and conſequently the Sub-tangent

$$\text{ST} \left(\frac{y\dot{x}}{\dot{y}}\right)^* = \frac{2a^2y^2}{b^2 \times \overline{a - 2x}} = \frac{a^2y^2}{b^2 \times \overline{\frac{1}{2}a - x}} = \frac{b^2 \times \overline{ax - x^2}}{b^2 \times \overline{\frac{1}{2}a - x}} =$$

* Art. 49.

ax —

$\frac{ax-x^2}{\frac{1}{2}a-x}$. Whence the Point T being given, through which the Tangent muſt paſs, the Tangent itſelf may be drawn.

But if you would derive an Expreſſion for the Sub-tangent, in any other kind of Ellipſes (beſides the conical) let the Equation $\overline{a-x}|^m \times x^n = \frac{c}{a} \times y^{m+n}$, exhibiting the Nature of all Kinds of Ellipſes, be aſſumed: Then, by taking the Fluxion thereof, you will have $-m\dot{x} \times \overline{a-x}|^{m-1} \times x^n + n\dot{x}x^{n-1} \times \overline{a-x}|^m = \frac{c}{a} \times \overline{m+n} \times y^{m+n-1}\dot{y}$; and therefore $\frac{y\dot{x}}{\dot{y}} =$

$$\frac{\frac{c}{a} \times \overline{m+n} \times y^{m+n}}{-m \times \overline{a-x}|^{m-1} \times x^n + nx^{n-1} \times \overline{a-x}|^m}$$

$= \frac{\overline{m+n} \times \overline{a-x}|^m \times x^n}{-mx^n \times \overline{a-x}|^{m-1} + nx^{n-1} \times \overline{a-x}|^m}$ (becauſe $\frac{c}{a} \times y^{m+n} = \overline{a-x}|^m \times x^n$) $= \frac{\overline{m+n} \times \overline{a-x} \times x}{-mx + n \times \overline{a-x}} =$

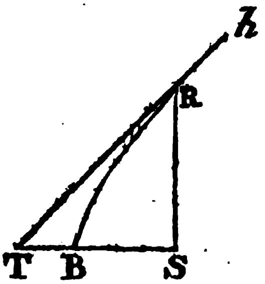
$\frac{\overline{m+n} \times \overline{ax-x^2}}{na-\overline{n+m} \times x}$; which is the Sub-tangent required.

EXAMPLE V.

54. *To draw a Tangent, to any given Point* R, *in a given Hyperbola* BR*b*.

If a and c be put to denote the two principal Diameters of the Hyperbola, the Equation of the Curve will be $c^2 \times \overline{ax+x^2} = a^2y^2$: From whence we have $c^2 \times \overline{a\dot{x} +}$

$\overline{a\dot{x}+2x\dot{x}} = 2a^2y\dot{y}$, $\therefore \frac{\dot{x}}{\dot{y}} = \frac{a^2y}{c^2\times\overline{\frac{1}{2}a+x}}$, and conſequently $\frac{y\dot{x}}{\dot{y}} = \frac{a^2y^2}{c^2\times\overline{\frac{1}{2}a+x}}$, $= \frac{c^2\times\overline{ax+x^2}}{c^2\times\overline{\frac{1}{2}a+x}} = \frac{ax+x^2}{\frac{1}{2}a+x} = ST$. Whence BT (ST − BS) $= \frac{\frac{1}{2}ax}{\frac{1}{2}a+x}$ is alſo known; and therefore the Point T being given the Tangent RT may be drawn.



The Manner of drawing Tangents to all Sorts of Hyperbolas, *univerſally*, will be the ſame as in the Ellipſes, the Equations of the two Kinds of Curves differing in Nothing but their Signs.

EXAMPLE VI.

55. *Let the propoſed Curve be that whoſe Equation is*

$$ax^2+xy^2+x^3-y^3=0.$$

Then we ſhall have $2ax\dot{x}+y^2\dot{x}+2xy\dot{y}+3x^2\dot{x}-3y^2\dot{y}=0$; therefore $2ax\dot{x}+y^2\dot{x}+3x^2\dot{x}=3y^2\dot{y}-2xy\dot{y}$, $\frac{\dot{x}}{\dot{y}}=\frac{3y^2-2xy}{2ax+y^2+3x^2}$, and conſequently $\frac{y\dot{x}}{\dot{y}}=\frac{3y^3-2xy^2}{2ax+y^2+3x^2}$*.

*Art. 48 and 49.

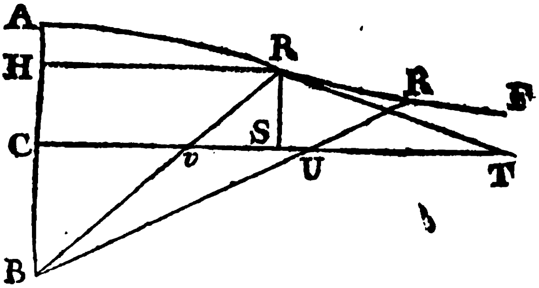
EXAMPLE VII.

56. *Let the given Curve be the Ciſſoid of* Diocles, *whoſe Equation is* $y^2 = \frac{x^3}{a-x}$.

Here we have $2y\dot{y} = \frac{3x^2\dot{x} \times \overline{a-x} + \dot{x}x^3}{\overline{a-x}|^2} = \frac{3ax^2\dot{x} - 2x^3\dot{x}}{\overline{a-x}|^2}$: Whence $\frac{\dot{x}}{\dot{y}} = \frac{2y \times \overline{a-x}|^2}{3ax^2 - 2x^3}$, and conſequently the Subtangent $\left(\frac{y\dot{x}}{\dot{y}}\right) = \frac{2y^2 \times \overline{a-x}|^2}{3ax^2 - 2x^3} = \frac{2x^3}{a-x} \times \frac{\overline{a-x}|^2}{3ax^2 - 2x^3} = \frac{2x \times \overline{a-x}}{3a-2x}$.

EXAMPLE VIII.

57. *Let the Conchoid of* Nicomedes *be propoſed*; whereof the Nature is ſuch, that, if from a Point B, called



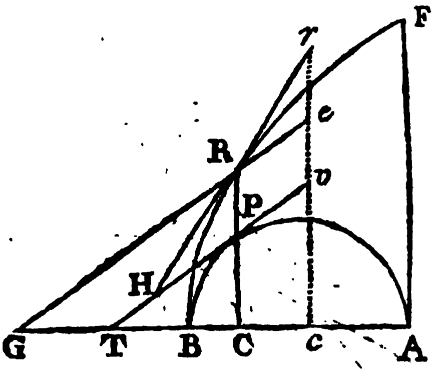
the Pole, any Number of Right-lines, BA, BR, BR, &c. be drawn, the Parts of thoſe Lines CA, vR, UR, &c. intercepted by the Curve and its Axis CT, ſhall be, all, equal to each other.

In this Caſe (ſuppoſing AB and RS perpendicular, and RH parallel, to CT; and putting BC $= a$, Rv (AC) $= b$, CS$= x$, and RS $= y$) we have, *per ſim. Triang.* $a+y$ (BH) : x (RH) :: y (RS) : $\frac{xy}{a+y} =$ Sv: But Sv ($\sqrt{Rv^2 - RS^2}$) is alſo $= \sqrt{b^2 - y^2}$; therefore $\frac{xy}{a+y} = \sqrt{b^2 - y^2}$, or $x^2 y^2 = \overline{a+y}|^2 \times \overline{b^2 - y^2}$ is the general Equation of the Curve; which, in Fluxions, gives $2x^2 y\dot{y} + 2y^2 x\dot{x} = 2\dot{y} \times \overline{a+y} \times \overline{b^2 - y^2} - 2y\dot{y} \times \overline{a+y}|^2 = 2\dot{y} \times \overline{a+y} \times \overline{b^2 - ay - 2y^2}$; and therefore $\frac{\dot{x}}{\dot{y}} = \frac{\overline{a+y} \times \overline{b^2 - ay - 2y^2} - x^2 y}{xy^2}$, conſequently $\frac{y\dot{x}}{\dot{y}} = \frac{\overline{a+y} \times y \times \overline{b^2 - ay - 2y^2} - x^2 y^2}{y \times xy} = \frac{\overline{a+y} \times y \times \overline{b^2 - ay - 2y^2} - \overline{a+y}|^2 \times \overline{b^2 - y^2}}{y \times \overline{a+y} \times \sqrt{b^2 - y^2}}$ (becauſe $x^2 y^2 = \overline{a+y}|^2 \times \overline{b^2 - yy}$) $= \frac{b^2 y - ayy - 2y^3 - abb + ayy - bby + y^3}{y\sqrt{bb - yy}} = \frac{-ab^2 - y^3}{y\sqrt{bb - yy}}$: Which being a negative Quantity, the Tangent will therefore fall on the contrary Side of the Ordinate, from the Vertex; and ſo, by changing the Signs we ſhall have $\frac{abb + y^3}{y\sqrt{bb - yy}}$ for the Sub-tangent ST in this Caſe.

After the Manner of theſe Examples the Sub-tangent, in Curves whoſe Abſciſſas are Right-lines, may be determined: But if the Abſciſſa, or Line terminating the Ordinate, on the lower Part, be another Curve, then the Tangent may be drawn as in the following

EXAMPLE IX.

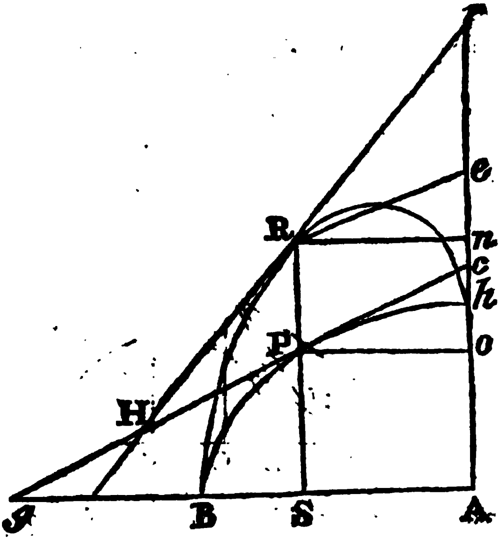
58. Let the Curve BRF be a Cycloid; whoſe Abſciſſa is here ſuppoſed to be the Semicircle BPA, to which let the Tangent PT be drawn (as above). Moreover let *r*RH be a Tangent to the Cycloid, at the corresponding Point R, and let GR*e* be parallel to TP*v*; putting the Arch (or Abſciſſa) BP$=z$, its Ordinate PR$=y$, AF$=b$, and BPA$=c$: Then, by the Property of the Curve, we ſhall have c (BPA) : b (AF) :: z (BP) : y (PR): Therefore $y=\frac{bz}{c}$, and $\dot{y}=\frac{b\dot{z}}{c}=re$: But, by ſimilar Triangles, $re\,(\dot{y})$: Re ($=$ P$v=\dot{z}$) :: PR (y) : PH $=\frac{y\dot{z}}{\dot{y}}=z$ (becauſe $y=\frac{bz}{c}$). Therefore, if in the Right-line PT, there be taken PH, equal to the Arch PB, you will have a Point H, through which the Tangent of the Cycloid muſt paſs.



EXAMPLE X.

59. Let BP*b* be a Curve of any Kind, to which the Method of drawing the Tangent *c*P*g* is known; let

BR*b* be another Curve of ſuch a Nature, that the Ordinate PR (y) ſhall always be a Mean-proportional be-

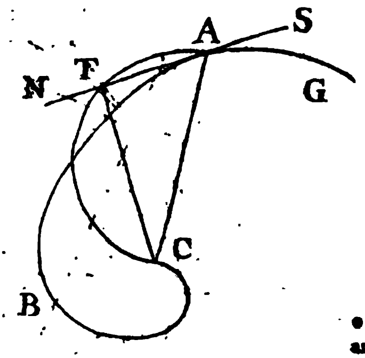


tween BS (x) and AS ($a-x$) ſuppoſing RPS perpendi- *Art.48 and 49. cular to AB: Put Po = $\dot{x}$, SP = v, oc = $\dot{v}$ *, and er = $\dot{y}$: Then, (as above) er ($\dot{y}$) : Rr (= Pc = $\sqrt{\dot{x}^2+\dot{v}^2}$) :: RP ($y$) : PH = $\frac{y\sqrt{\dot{x}^2+\dot{v}^2}}{\dot{y}}$: But, by the Equation of the Curve $y^2 = ax - xx$; whence $2y\dot{y} = a\dot{x} - 2x\dot{x}$, and $\frac{y}{\dot{y}} = \frac{2ax - 2x^2}{a\dot{x} - 2x\dot{x}}$, and therefore PH = $\frac{2ax - 2x^2 \times \sqrt{\dot{x}^2+\dot{v}^2}}{a\dot{x} - 2x\dot{x}}$: Which will be expreſſed independent of Fluxions, when the Property of the Curve BPh, or the Relation of x and v is given: Thus, let BPh be the common Parabola, and AB its *Latus Rectum*;

tion; then v being $= \sqrt{ax}$, $\dot{v}$ will be $= \frac{a\dot{x}}{2\sqrt{ax}}$, $\dot{x}^2 + \dot{v}^2 = \dot{x}^2 + \frac{a\dot{x}\dot{x}}{4x} = \frac{\dot{x}\dot{x} \times \overline{4x+a}}{4x}$; and therefore PH $\left(\frac{\overline{2ax - 2xx} \times \sqrt{\dot{x}^2 + \dot{v}^2}}{a\dot{x} - 2x\dot{x}}\right) = \frac{\overline{a-x} \times \sqrt{4x^2 + ax}}{a - 2x}$.

Thus far relates to Curves whoſe Ordinates are parallel to each other: We come now to Curves of the Spiral Kind, whoſe Ordinates all iſſue from a Point: Such as the Spiral BAG, whoſe Ordinates CB, CA, CG, are all referred to the Point C, called the Center of the Spiral.

ILLUSTRATION.



60. Let SAN be a Tangent to the Spiral at any Point A, alſo let CT be perpendicular thereto, and let the Arch CBA (conſidered as variable by the Motion of A towards G) be denoted by z, and the Ordinate CA by y. Then $\dot{z} : \dot{y} :: AC\ (y) : AT = \frac{y\dot{y}}{\dot{z}}$.* Hence, if upon CA, as a Diameter, a Semi-circle be deſcribed, and in it, from A, a Right-line AT equal to $\frac{y\dot{y}}{\dot{z}}$ be inſcribed, that Right-line will be a Tangent to the Spiral at the Point A.

* Art. 3 and 35.

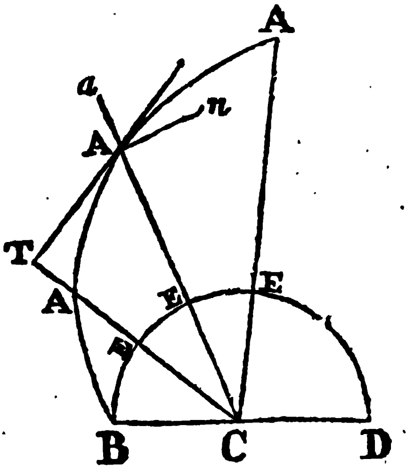
EXAMPLE I.

61. Let the Nature of the Curve CBA be ſuch that the Arch CBA may be, always, to its cor-

reſponding Ordinate CA in a conſtant Ratio; namely as a to b: Then, becauſe $z : y :: a : b$, we have $z = \frac{ay}{b}$, $\dot{z} = \frac{a\dot{y}}{b}$, and conſequently AT $\left(\frac{y\dot{y}}{\dot{z}}\right) = \frac{by}{a} = \frac{b}{a} \times$ AC: Therefore, AC and AT being in a conſtant Ratio, the Angle CAT muſt alſo be invariable. Which is a known Property of the logarithmic Spiral.

EXAMPLE II.

62. Let BAA be the Spiral of *Archimedes*; whoſe Nature is ſuch that the Part EA of the generating Ordinate, intercepted by the Spiral and a Circle BED deſcribed about the ſame Center C, is always in a conſtant Ratio to the correſponding Arch BE of that Circle.



Suppoſe A*n* perpendicular to AC, &c.

Put BC $= c$, CA $= y$, and let the given Ratio of AE to BE, be that of b to c: Then $b : c :: y - c$ (AE) : $\frac{cy - cc}{b} =$ BE: whoſe Fluxion therefore is $= \frac{c\dot{y}}{b}$. Now

If the Right-line CEA*a* be ſuppoſed to revolve about the Center C, the angular Celerity of the generating Point A, in the perpendicular Direction A*a*, will be to that of E as AC to EC; therefore as the latter of theſe Celerities is expreſſed by $\frac{c\dot{y}}{b}$[*], the former will be expreſſed by $\frac{y}{c} \times \frac{c\dot{y}}{b}$, or $\frac{y\dot{y}}{b}$: Which is to ($\dot{y}$) the Celerity of A, in the Direction A*a*, as $\frac{y}{b}$ to Unity, or as y to b. Therefore CT and AT are in the ſame Ratio, (by *Art.* 35) and conſequently AC : CT :: $\sqrt{yy + bb}$: y; and AC : AT :: $\sqrt{yy + bb} : b$; whence CT and AT are given equal to $\frac{y^2}{\sqrt{yy + bb}}$, and $\frac{by}{\sqrt{yy + bb}}$ reſpectively. From either of which (the Tangent AT) may be drawn by *Art.* 60. And, in the ſame manner may the Poſition of the Tangent of any other Spiral be determined.

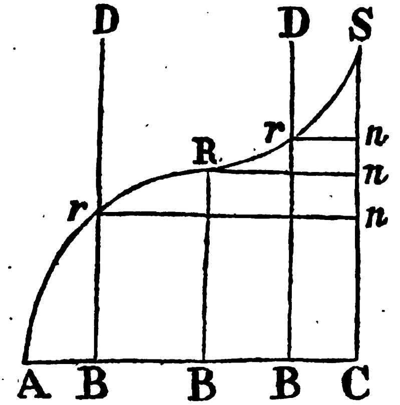
[*] Art. 5.

SECTION IV.

Of the Uſe of Fluxions in determining the Points of Retrogreſſion, or contrary Flexure in Curves.

63. WHEN a Curve ARS is, in one Part AR concave, and in the other Part RS convex, towards its Axis AC, the Point R limiting the two Parts is called a Point of Retrogreſſion, or contrary Flexure. The manner of determining which will appear from the following

ILLUSTRATION.



Suppoſe a Right-line BD to be carried along uniformly, parallel to itſelf, from A towards C; and let the Point *r* ſo move in that Line, at the ſame time, as to trace out, or deſcribe, the given Curve-line ARS.

Then (*by Art.* 48.) while the Celerity of the Point *r*, in the Line BD, decreaſes, the Curve will be concave to its Axis AC; but when it increaſes, convex to the ſame: Therefore, as any Quantity is a *Minimum* at the End of its Decreaſe and the Beginning of its Increaſe *, it follows that the ſaid Celerity, at the Point of Inflexion R, muſt be a *Minimum*: Whence, if the Fluxion of the Ordinate B*r*, expreſſing that Celerity †, be (as uſual) denoted by $\dot{y}$; then will $\ddot{y}$ (the Fluxion of $\dot{y}$) be equal to Nothing in that Circumſtance ‡.

*Art. 22.
†Art. 5.
‡Art. 22.

So far relates to Curves which are, in the former Part concave, and in the latter convex, to their Axes: But if (on the contrary) the Celerity of *r* firſt increaſes, and then decreaſes, that Celerity, at the required Point, between the Increaſe and Decreaſe, will be a *Maximum*; and therefore its Fluxion (or $\ddot{y}$) is *likewiſe* equal to Nothing in this Caſe §.

§Art. 22.

Furthermore, if CS (perpendicular to AC) be now conſidered as an Axis, and the Abſciſſa S*n* (or its Complement B*r* = y) be ſuppoſed to flow uniformly, (as AB was ſuppoſed before); then, by the ſame Argument, the ſecond Fluxion ($-\ddot{x}$) of the Ordinate *nr* (or its Complement AB $= x$) will be equal to Nothing. Hence it is evident that, at the Point of contrary Flexure, the ſecond Fluxion of the Ordinate will become equal to Nothing, if the Abſciſſa be made to flow uniformly; and *vice verſâ.*

EXAMPLE I.

64. Let the Nature of the Curve ARS (*ſee the preceding Figure*) be defined by the Equation $ay = a^{\frac{3}{2}}x^{\frac{1}{2}} + xx$ (the Abſciſſa AB and the Ordinate B*r* being, as uſual, repreſented by x and y reſpectively). Then $\dot{y}$, expreſſing the Celerity of the Point *r*, in the Line BD, will be equal to $\frac{\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x} + 2x\dot{x}}{a}$: Whoſe Fluxion, or that of $\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}} + 2x$ (becauſe a and $\dot{x}$ are conſtant) muſt be equal to Nothing*; that is, $-\frac{1}{4}a^{\frac{3}{2}}x^{-\frac{3}{2}}\dot{x} + 2\dot{x}$ *Art. 63. $= 0$: Whence $a^{\frac{3}{2}}x^{-\frac{3}{2}} = 8$, $a^{\frac{3}{2}} = 8x^{\frac{3}{2}}$, $64x^3 = a^3$, and $x = \frac{1}{4}a = AB$; therefore BR $\left(= \frac{a^{\frac{3}{2}}x^{\frac{1}{2}} + xx}{a}\right) = \frac{9}{16}a$: From which the Poſition of the Point R is given.

EXAMPLE II.

65. Let the Nature of the propoſed Curve be defined by the Equation $ayy - aax - x^3 = 0$.

Then, by taking the firſt and ſecond Fluxions thereof (ſuppoſing $\dot{x}$ conſtant) we ſhall alſo have $2ay\dot{y} - aa\dot{x} - 3x^2\dot{x} = 0$, and $2a\dot{y}^2 + 2ay\ddot{y} - 6x\dot{x}\dot{x} = 0$; whereof the latter, when $\ddot{y}$ is $= 0$, becomes $2a\dot{y}^2 - 6x\dot{x}^2 = 0$, and therefore $\dot{y}^2 = \frac{3x\dot{x}^2}{a}$: But, by the former $\dot{y} = \frac{a^2\dot{x} + 3x^2\dot{x}}{2ay}$; whence $\frac{3x\dot{x}^2}{a} = \frac{\overline{a^2\dot{x} + 3x\dot{x}}|^2}{\overline{2ay}|^2}$, and conſequently $12axy^2$

$= \overline{a^2 + 3x^2}|^2$; but, by the given Equation, $12axy^2 = 12a^2x^2 + 12x^4$, therefore $12a^2x^2 + 12x^4 = \overline{a^2 + 3x^2}|^2$, or $3x^4 + 6a^2x^2 - a^4 = 0$: Whence x will be found $= a\sqrt{:\sqrt{\tfrac{4}{3}} - 1}$.

Otherwiſe.

Since $ay^2 = a^2x + x^3$, we have $y = \frac{\overline{a^2x + x^3}|^{\frac{1}{2}}}{\sqrt{a}}$, and therefore $\dot{y} = \frac{\overline{\frac{1}{2}a^2\dot{x} + \frac{3}{2}x^2\dot{x}} \times \overline{a^2x + x^3}|^{-\frac{1}{2}}}{\sqrt{a}}$: Whoſe Fluxion, or that of $\overline{a^2 + 3x^2} \times \overline{a^2x + x^3}|^{-\frac{1}{2}}$ (becauſe $\dot{x}$ is conſtant) being put $= 0$, we get $6\dot{x} \times \overline{a^2x + x^3}|^{-\frac{1}{2}} + \overline{a^2 + 3x^2} \times \overline{-\frac{1}{2}a^2 - \frac{3}{2}x^2} \times \overline{a^2x + x^3}|^{-\frac{3}{2}} = 0$, or $6x \times \overline{a^2x + x^3} + \overline{a^2 + 3x^2} \times -\frac{\overline{a^2 + 3x^2}}{2}$: Whence $3x^4 + 6a^2x^2 - a^4 = 0$, and $x = a\sqrt{:\sqrt{\tfrac{4}{3}} - 1}$, the ſame as before.

EXAMPLE III.

66. Let the propoſed Curve be the Conchoid of *Nicomedes*, whereof the Equation is $x^2y^2 = \overline{a+y}|^2 \times$
Art. 57. $\overline{b^2 - y^2}$ *, or $x^2 = \frac{\overline{a+y}|^2 \times \overline{b^2 - y^2}}{y^2}$.

Here we have $x\dot{x} = \frac{\overline{\dot{y}\times\overline{a+y}+\overline{b^2-y^2}-y\dot{y}\times\overline{a+y}|^2}\times y^2}{y^4}$ $\frac{-y\dot{y}\times\overline{a+y}|^2\times\overline{b^2-y^2}}{y^4} = -\frac{\overline{a+y}\times\overline{ab^2+y^3}}{y^3}\times\dot{y} = \overline{\frac{-a^2b^2}{y^3}-\frac{ab^2}{y^2}-a-y}\times\dot{y}$: Whence, making $\dot{y}$ invariable, we alſo have $\dot{x}^2+x\ddot{x} = \overline{\frac{3a^2b^2}{y^4}+\frac{2ab^2}{y^3}-1}\times\dot{y}^2$: Which, becauſe $\ddot{x}$ is $= 0$ *, will be $\dot{x}^2 = \overline{\frac{3a^2b^2}{y^4}+\frac{2ab^2}{y^3}-1}\times\dot{y}^2 = \frac{3a^2b^2+2ab^2y-y^4}{y^4}\times\dot{y}^2$. But ſince, by the former Equation, $x\dot{x} = -\frac{\overline{a+y}\times\overline{ab^2+y^3}}{y^3}\times\dot{y}$, we likewiſe get $\dot{x}^2 = \frac{\overline{a+y}|^2\times\overline{ab^2+y^3}|^2}{x^2y^6}\times\dot{y}^2$, and conſequently $\overline{3a^2b^2+2ab^2y-y^4}\times x^2y^2 = \overline{a+y}|^2\times\overline{ab^2+y^3}|^2$: But, by the Equation of the Curve x^2y^2 is $= \overline{a+y}|^2\times\overline{b^2-y^2}$; therefore $\overline{3a^2b^2+2ab^2y-y^4}\times\overline{a+y}|^2\times\overline{b^2-y^2} = \overline{a+y}|^2\times\overline{ab^2+y^3}|^2$, and $\overline{3a^2b^2+2ab^2y-y^4}\times\overline{b^2-y^2} = \overline{ab^2+y^3}|^2$; whence $y^4+4ay^3+3a^2y^2-2ab^2y-2a^2b^2=0$; which divided by $y+a$, gives $y^3+3ay^2-2ab^2=0$; from whence y may be determined. But if $b=a$, the Equation will become more ſimple by dividing again by $y+a$; in which Caſe we get $y^2+2ay-2a^2=0$, and conſequently $y = a\sqrt{3}-a$.

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* Art. 63.

EXAMPLE IV.

67. Let $a^4y = 180a^3x^2-110a^2x^3+30ax^4-3x^5$. Then will $a^4\dot{y} = 360a^3x\dot{x}-330a^2x^2\dot{x}+120ax^3\dot{x}-15x^4\dot{x}$;

And $a^4\ddot{y} = 360a^3\dot{x}^2 - 660a^2x\dot{x}^2 + 360ax^2\dot{x}^2 - 60x^3\dot{x}^2$

* Art. 63. Therefore, $6a^3 - 11a^2x + 6ax^2 - x^3 \mp 0$ *:

Which being diviſible by any one of the three Quantities $a-x$, $2a-x$, or $3a-x$, the Root x muſt therefore have three Values, a, $2a$, and $3a$, and conſequently the Curve, defined by the given Equation, as many Points of contrary Flexure.

But, if you would know whether the Part of the Curve lying between any two adjacent Points, thus found, be convex or concave towards the Axis; ſee whether the Value of the Expreſſion for the ſecond Fluxion of the Ordinate, between the two correſponding Roots, be poſitive or negative: For, in the former Caſe, the Curve is convex, and in the latter concave †,
† Art. 5 and 48. (provided the *whole* Curve lies on the ſame Side the Axis). Thus, in the Example before us; becauſe the ſecond Fluxion of the Ordinate is always as $6a^3 - 11aax + 6axx - x^3$ ($= \overline{a-x} \times \overline{2a-x} \times \overline{3a-x}$) and it appears that the Value of this Expreſſion, while x is leſs than the firſt Root a, will be poſitive; the Curve, therefore, at the Beginning, will be convex to its Axis: But when x becomes greater than a, the ſaid Expreſſion being negative, the Curve will then be concave, and ſo continue 'till x is equal to the ſecond Root $2a$; after which the Fluxion again becoming affirmative, the Curve will accordingly be convex till $x = 3a$; beyond which Limit the Curvature continually tends the ſame Way.

But it will be proper to obſerve, that there are Caſes where the ſecond Fluxion of the Ordinate may become equal to Nothing, without either changing its Value from poſitive to negative, or the contrary, (ſimilar to thoſe already taken Notice of in *Sect.* II. *p.* 45 and 46.) which Caſes always happen when the Equation admits of an even Number of equal Roots: And then the Point found as above is not a Point of Inflexion, becauſe the Curvature on either Side of it tends the ſame Way.

SECT.

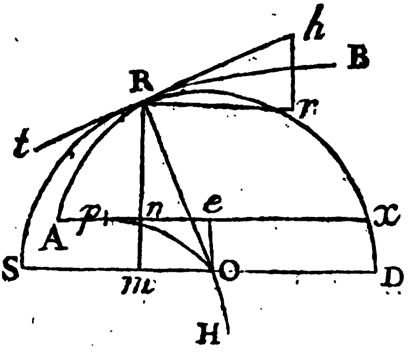
SECTION V.

The Uſe of Fluxions in determining the Radii of Curvature, and the Evolutes of Curves.

68. A Curve *p*OH is ſaid to be the Evolute of another Curve ARB, when it is of ſuch a Nature, that a Thread ROH, coinciding therewith (or wrapped upon the ſame) being unwound or diſengaged from it, by a Power acting at the End R, ſhall, by that End (the Thread continuing tight) deſcribe the given Curve ARB.

ILLUSTRATION.

From the Point O, where the Right-line RO (called the Radius of Curvature) touches the Evolute *p*OH,



let the Semi-circle SRD be deſcribed; which Semicircle, having the ſame Radius with the given Curve, at R, will conſequently have the ſame Degree of Curvature.——But the Curvature in two Curves is the ſame, when, the Fluxions of their Abſciſſas being the ſame, both the Firſt, and Second Fluxions of their

correſponding Ordinates R*n* and R*m* are reſpectively
equal to each other: For, the Firſt Fluxions being
equal, the two Curves will have, at the common Point
*Art.48. R, one and the ſame Tangent *t*R*b* *: And, if the Se-
cond Fluxions be likewiſe equal, the Curvature, or
Deflection from that Tangent, will alſo be the ſame in
both; becauſe theſe laſt expreſs the Increaſe or Decreaſe
†Art.19. of Motion in the Direction of the Ordinate †, upon
‡Art.48. which the Curvature intirely depends ‡.

This being premiſed, let the Abſciſſa S*m* of the Semi-
circle (conſidered as variable) be put $= w$, its Ordinate
R$m = v$, R$r = \dot{w}$, $rh = \dot{v}$, and R$h = \dot{z}$: Then, R*h* be-
‖Art.48. ing a Tangent to the Circle at R ‖, the Triangles R*hr*
and RO*m* will be equiangular, and therefore $\dot{w}$ (R*r*) :
$\dot{z}$ (RA) :: v (R*m*) : RO $= \frac{v\dot{z}}{\dot{w}}$; which, becauſe the
Radius of every Circle is a conſtant Quantity, muſt be invariable, and conſequently its Fluxion $\frac{\dot{v}\dot{z}+v\ddot{z}}{\dot{w}} = 0$: Whence v is found $= \frac{\dot{v}\dot{z}}{-\ddot{z}} = \frac{\dot{z}^2}{-\ddot{v}}$ (becauſe, $\dot{w}$ being conſtant, and $\dot{w}^2 + \dot{v}^2 = \dot{z}^2$, we have, in Fluxions $2\dot{v}\ddot{v} = 2\dot{z}\ddot{z}$, and ſo $\frac{\dot{v}\dot{z}}{-\ddot{z}} = \frac{\dot{z}^2}{-\ddot{v}}$). Therefore ſince v is $= \frac{\dot{z}^2}{-\ddot{v}}$, we alſo get SO$=$RO $\left(\frac{v\dot{z}}{\dot{w}}\right) = \frac{\dot{z}^3}{-\dot{w}\ddot{v}} = \frac{\overline{\dot{v}^2+\dot{w}^2}\rvert^{\frac{3}{2}}}{-\dot{w}\ddot{v}}$: Which laſt is a general Expreſſion for the Radius of any Circle, whatever, in Terms of the Fluxions of its Abſciſſa (*w*) and Ordinate (*v*). But, by what is premiſed above, theſe Fluxions are reſpectively equal to thoſe of the Abſciſſa A*n* (*x*) and Ordinate R*n* (*y*) of the propoſed Curve ARB. Therefore, by writing $\dot{x}$, $\dot{y}$, and $\ddot{y}$, inſtead of $\dot{w}$, $\dot{v}$, and $\ddot{v}$, we have $\frac{\overline{\dot{y}^2+\dot{x}^2}\rvert^{\frac{3}{2}}}{-\dot{x}\ddot{y}} \left(= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right)$ for the general Value of the Radius of Curvature, RO.

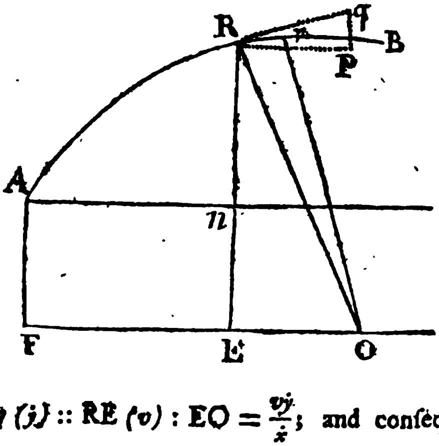
The ſame otherwiſe.

If the Radius of the Circle be put $= R$, and every Thing elſe be ſuppoſed as above; then (by the Property of the Circle) we ſhall have v^2 $(Rm^2) = 2\ Rw - w^2$ $(Sm \times Dm)$: Whence, in Fluxions (making $\dot{w}$ conſtant) we get $2v\dot{v} = 2R\dot{w} - 2w\dot{w}$, and $2\dot{v}^2 + 2v\ddot{v} = -2\dot{w}^2$: From the laſt of which Equations v is found $= \frac{\dot{v}^2 + \dot{w}^2}{-\ddot{v}}$ $= \frac{\dot{z}^2}{-\ddot{v}}$; and conſequently RO $\left(\frac{v\dot{z}}{\dot{w}}\right) = \frac{\dot{z}^3}{-\dot{w}\ddot{v}} = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, *the ſame as before.*

Otherwiſe without the Circle.

Let RO and rO be two Rays perpendicular to the Curve, indefinitely near to each other; and from their Interſection O, let OF be drawn parallel to An, cutting Rn and AF (parallel to Rn) in E and F.

Therefore, ſuppoſing RE$=v$, A$n=x$, R$n=y$, *&c.* (as before) we ſhall have, by ſimilar Triangles, as RP



$(\dot{x})$: Pq $(\dot{y})$ :: RE (v) : EO $= \frac{v\dot{y}}{\dot{x}}$; and conſequently FO (A$n$+EO) $= x + \frac{v\dot{y}}{\dot{x}}$: Which Value (as well as that of AF) continuing the ſame whether we regard the Radius RO, or the Radius *r*O, its Fluxion muſt therefore be equal to Nothing; that is, $\dot{x}+\frac{\overline{\dot{v}\dot{y}+v\ddot{y}}\times\dot{x}-v\dot{y}\ddot{x}}{\dot{x}^2}=0$; whence $v=\frac{\dot{x}^3+\dot{x}\dot{v}\dot{y}}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}$, and conſequently RO $\left(\frac{v\dot{z}}{\dot{x}}\right)=\frac{\dot{x}^2\dot{z}+\dot{v}\dot{y}\dot{z}}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}=\frac{\dot{x}^2\dot{z}+\dot{y}^2\dot{z}}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}=\frac{\dot{z}^3}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}$: Which, if $\dot{x}$ is ſuppoſed conſtant, or $\ddot{x}=0$, will become $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, *as above.*

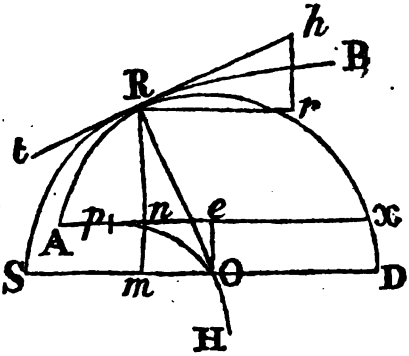
But if $\dot{y}$ be ſuppoſed conſtant, it will be $\frac{\dot{z}^3}{\dot{x}\ddot{y}}$. And, if $\dot{z}$ be conſtant, it will then be $\frac{\dot{z}\dot{y}}{\ddot{x}}$: For, ſince $\dot{x}^2+\dot{y}^2=\dot{z}^2$, by taking the Fluxion thereof, we have $2\dot{x}\ddot{x}+2\dot{y}\ddot{y}=0$; whence $\ddot{y}=-\frac{\dot{x}\ddot{x}}{\dot{y}}$; and therefore RO $\left(=\frac{\dot{z}^3}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}\right)=\frac{\dot{z}^3}{\dot{y}\ddot{x}+\frac{\dot{x}^2\ddot{x}}{\dot{y}}}=\frac{\dot{y}\dot{z}^3}{\overline{\dot{y}^2+\dot{x}^2}\times\ddot{x}}=\frac{\dot{y}\dot{z}}{\ddot{x}}$, *as before.*

Now from the ſeveral Values of the Radius of Curvature RO, found above, the correſponding Values of A*e* and *e*O will likewiſe be given.

Thus, if $\dot{x}$ be made conſtant; then, RO being $=\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, we ſhall have A*e* $\left(An+Om=An+\frac{\dot{y}}{\dot{z}}\times RO\right)=x+\frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}$, and *e*O $\left(Rm-nR=\frac{\dot{x}}{\dot{z}}\times RO-Rn\right)=\frac{\dot{z}^2}{-\ddot{y}}-y$.

But, if $\dot{y}$ be made conſtant, then, RO being $=\frac{\dot{z}^3}{\dot{y}\ddot{x}}$, we ſhall have AE $=x+\frac{\dot{z}^2}{\ddot{x}}$, and *e*O $=\frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x}}-y$.

Laſtly, if $\dot{z}$ be ſuppoſed conſtant; then RO being $= \frac{\dot{y}\dot{z}}{\ddot{x}}$, we ſhall have $Ae = x + \frac{\dot{y}^2}{\ddot{x}}$, and $eO = \frac{\dot{x}\dot{y}}{\ddot{x}} - y$.



Which ſeveral Expreſſions will ſerve as ſo many general Theorems for determining the Quantity of Curvature, and the Evolutes of given Curves: But, before we proceed to Examples, it will be proper to obſerve, that the Right-line Ap, denoting the Radius of Curvature at the Vertex A (to be found by making x, or y, $= 0$) muſt always be ſubtracted from RO and Ae, to have the true Length of the Arch pO, and its correſponding Abſciſſa pe.

EXAMPLE I.

69. Let the given Curve ARB be the common Parabola, whoſe Equation is $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$: Then will $\dot{y} = \frac{1}{2}a^{\frac{1}{2}}\dot{x}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}\dot{x}}{2x^{\frac{1}{2}}}$, and (making $\dot{x}$ conſtant) $\ddot{y} = -\frac{1}{2} \times \frac{1}{2}a^{\frac{1}{2}}\dot{x}^2x^{-\frac{3}{2}} = \frac{-a^{\frac{1}{2}}\dot{x}^2}{4x^{\frac{3}{2}}}$: Whence $\dot{z}\ (\sqrt{\dot{x}+\dot{y}^2} = \frac{\dot{x}}{2}\sqrt{\frac{4x+a}{x}}$,

and the Radius of Curvature RO $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \frac{\overline{a+4x}|^{\frac{3}{2}}}{2\sqrt{a}}$: Which at the Vertex A, where $x=0$, will be $= \frac{1}{2}a =$ Ap. Moreover Ae $\left(x + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}\right) = \frac{1}{2}a + 3x$, and therefore pe (Ae—Ap) $= 3x$, the Abſciſſa of the Evolute: Likewiſe Oe $\left(\frac{\dot{z}^2}{-\ddot{y}} - y\right) = \frac{4x^{\frac{3}{2}}}{\sqrt{a}}$ the Ordinate of the Evolute. Therefore, $\overline{Oe}|^2 \times a$ being in a conſtant Ratio to $\overline{pe}|^3$, namely as 16 to 27, the Curve is, in this Caſe, the Semi-cubical Parabola: Whoſe Arch pO (RO—Ap) is alſo given $= \frac{\overline{a+4x}|^{\frac{3}{2}}}{2\sqrt{a}} - \frac{1}{2}a$.

EXAMPLE II.

70. Let the Curve ARB denote a Parabola of any other Kind: Then, becauſe $y = ax^n$ is an Equation to all Kinds of Parabolas, we have $\dot{y} = nax^{n-1}\dot{x}$ and $\ddot{y} = n \times \overline{n-1} \times ax^{n-2}\dot{x}^2$: Therefore $\dot{z}$ $(\sqrt{\dot{x}^2+\dot{y}^2}) = \dot{x}\sqrt{1+n^2a^2x^{2n-2}}$, RO $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \frac{\overline{1+n^2a^2x^{2n-2}}|^{\frac{3}{2}}}{-n\times\overline{n-1}\times ax^{n-2}}$, A$e$ $\left(x + \frac{\dot{y}\dot{z}^2}{-\ddot{y}}\right) = x - \frac{x+n^2a^2x^{2n-1}}{n-1}$, O$e$ $\left(\frac{\dot{z}^2}{-\ddot{y}} - y\right) = \frac{1+\overline{2n-1}\times na^2x^{2n-2}}{-\overline{n-1}\times nax^{n-2}}$, and A$p$ $= -\frac{n^2a^2 0^{2n-1}}{n-1}$; Which, if $n=\frac{1}{2}$, will become $= \frac{a^2}{2}$; but, if n be greater than $\frac{1}{2}$, it will be $= 0$; and, if n be leſs than $\frac{1}{2}$,

it will be infinite: Whence it appears, that the Radius of Curvature at the Vertex will be a finite Quantity in Curves whoſe firſt (or leaſt) Ordinates are in the Subduplicate Ratio of their Abſciſſas, and in all other Caſes, either Nothing, or Infinite.

EXAMPLE III.

71. Suppoſe the given Curve to be an Ellipſis; whoſe Equation (putting a and c for the two principal Diameters) is $a^2y^2 = c^2 \times \overline{ax - x^2}$.

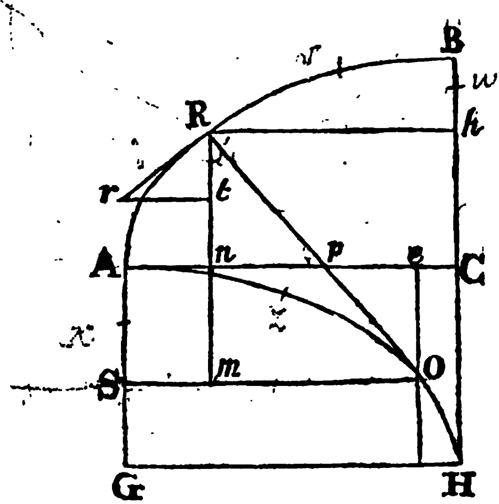
Here, by taking the Firſt and Second Fluxions of the given Equation, we have $2a^2y\dot{y} = c^2\dot{x} \times \overline{a - 2x}$, and $2a^2\dot{y}^2 + 2a^2y\ddot{y} = c^2\dot{x} \times -2\dot{x} = -2c^2\dot{x}^2$; whence $\dot{y} = \frac{c^2\dot{x} \times \overline{a - 2x}}{2a^2y}$, and $-\ddot{y} = \frac{a^2\dot{y}^2 + c^2\dot{x}^2}{a^2y}$: Which, by ſubſtituting the Values of y and $\dot{y}$, will become $\dot{y} = \frac{c\dot{x} \times \overline{a - 2x}}{2a\sqrt{ax - x^2}}$, and $-\ddot{y} = \frac{a^2c^2\dot{x}^2 \times \overline{a - 2x}|^2}{4a^2 \times \overline{ax - xx} \times ac\sqrt{ax - x^2}} + \frac{c\dot{x}^2}{a\sqrt{ax - x^2}} = \frac{c\dot{x}^2}{a} \times \frac{\overline{a - 2x}|^2 + 4 \times \overline{ax - x^2}}{4 \times \overline{ax - x^2}\sqrt{ax - x^2}} = \frac{ca\dot{x}^2}{4 \times \overline{ax - x^2}|^{\frac{3}{2}}}$:

Therefore $\dot{z}\ (\sqrt{\dot{y}^2 + \dot{x}^2}) = \sqrt{\frac{c^2\dot{x}^2 \times \overline{a - 2x}|^2}{4a^2 \times \overline{ax - x^2}} + \dot{x}^2}$ $= \frac{\dot{x}}{2a}\sqrt{\frac{c^2a^2 + \overline{a^2 - c^2} \times \overline{4ax - 4x^2}}{ax - x^2}}$, and the Radius of Curvature $\left(\frac{\dot{x}^3}{-\dot{x}\ddot{y}}\right) = \frac{\overline{a^2c^2 + \overline{a^2 - c^2} \times \overline{4ax - 4x^2}}|^{\frac{3}{2}}}{2a^4c}$: Which when the Diameters a and c are equal, or the Ellipſis degenerates to a Circle, will be every where equal to $\frac{\overline{a^2c^2}|^{\frac{3}{2}}}{2a^4c}$, or $\frac{1}{2}a$; agreeable to the Definition of a Circle.

EXAMPLE IV.

72. *To find the Radius of Curvature, and the Evolute of the common Cycloid.*

Let ARB be the given Curve, and AOH its Evolute; alſo let Rb and OS be parallel to AC, and eO and Rn



perpendicular to AC; and put ARB (=2BC) = a, AR = z, An = x, and Rn = y: Then BR = $a - z$, Bb = $\frac{1}{2}a - y$; and, by the Property of the Curve, a^2 (AB²) : $\overline{a-z}|^2$ (BR²) :: $\frac{1}{2}a$ (BC) : $\frac{1}{2}a - y$ (Bb) whence $y = \frac{2az - z^2}{2a}$; therefore $\dot{y} = \frac{a\dot{z} - z\dot{z}}{a}$, $\dot{z}^2 - \dot{y}^2$ $(\dot{x}^2) = \frac{\overline{2az - z^2} \times \dot{z}^2}{a^2}$, and $\dot{x} = \frac{\dot{z}\sqrt{2az - z^2}}{a}$. Whence (making $\dot{z}$ conſtant) $\ddot{x} = \frac{\dot{z}^2 \times \overline{a-z}}{a\sqrt{2az - z^2}}$; from which

we get RO, or AO $\left(= \frac{\dot{y}\dot{z}}{\ddot{x}}{}^{*}\right) = \sqrt{2az - z^2}$, and cO, *Art.68.
or AS $(= \frac{\dot{y}\dot{x}}{\ddot{x}} - y) = \frac{2az - z^2}{2a}$; which, when $z = a$, or ROH coincides with BH, become AOH (BH) $= a$, and CH (AG) $= \frac{1}{2}a$. Hence, becauſe it appears, that, $\widehat{\text{AH}}|^2$ (a^2) : AO^2 $(2az - z^2)$ :: AG $(\frac{1}{2}a)$: AS $\left(\frac{2az - z^2}{2a}\right)$ it follows that the Evolute AOH is alſo a Cycloid equal, and ſimilar, to the Involute ARB.

If the Evolute had been given, or ſuppoſed, a Cycloid, and the Involute required, the Proceſs would have been, more ſimple, as follows,

Let AH (2AG) $= a$, AO $(=$ RO$) = z$, AS $= x$, SO $= y$, BR $= v$, B$b = w$, R$r = \dot{v}$, R$t = \dot{w}$, &c. Then it will be †, †Art.48.

$\dot{y} : \dot{z}$ (:: Om : OR) :: Rt $(\dot{w})$: R$r = \frac{\dot{w}\dot{z}}{\dot{y}}$.

$\dot{z} : \dot{y}$ :: z (RO) : Om $= \frac{z\dot{y}}{\dot{z}}$,

$\dot{z} : \dot{x}$ :: z (RO) : Rm $= \frac{z\dot{x}}{\dot{z}}$,

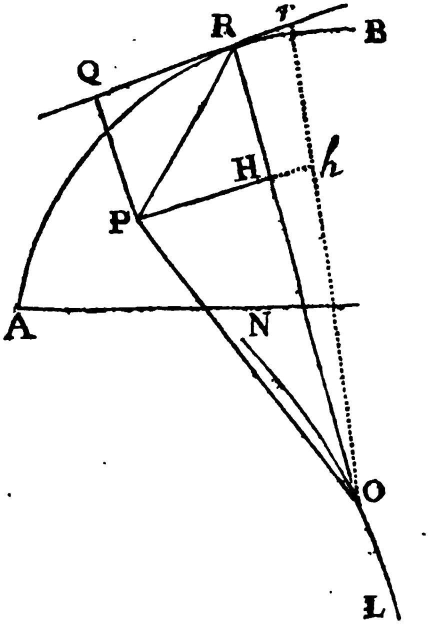
Whence we have $\dot{v} = \frac{\dot{w}\dot{z}}{\dot{y}}$, R$n$ (Rm — AS) $= \frac{z\dot{x}}{\dot{z}} - x$, and A$n$ (OS — Om) $= y - \frac{z\dot{y}}{\dot{z}}$; which Expreſſions anſwer to any Curve whatever.

But, in the Caſe above propoſed, AH^2 (a^2) : AO^2 (z^2) :: AG $(\frac{1}{2}a)$: AS (x); therefore $x = \frac{z^2}{2a}$, $\dot{x} = \frac{z\dot{z}}{a}$, and $\dot{y}$ $(\sqrt{\dot{z}^2 - \dot{x}^2}) = \frac{\dot{z}\sqrt{a^2 - z^2}}{a}$; and conſequently R$n$ $\left(\frac{z\ddot{x}}{\dot{z}} - x\right) = \frac{z^2}{a} - \frac{z^2}{2a} = \frac{z^2}{2a} = \frac{1}{2}a - w$ (or CB — Bb):

Whence alſo $w = \frac{a^2 - z^2}{2a}$, and $\dot{v}\left(\frac{\dot{w}\dot{z}}{y}\right) = \frac{a\dot{w}}{\sqrt{a^2 - z^2}} = \frac{a\dot{w}}{\sqrt{2aw}}$: Therefore it will be $\dot{v} : \dot{w}\ (:: a : \sqrt{2aw}) :: \sqrt{\frac{1}{2}a} : \sqrt{w}$; that is, as $Rr : Rt :: \sqrt{BC} : \sqrt{Bb}$: Which is a known Property of the Cycloid.

Hitherto regard has been had to Curves where the Ordinates are parallel to each other: But when the Ordinates are all referred to a given Point, as in Spirals, &c. other Theorems will become neceſſary; and may be thus derived.

73. Let ARB be the propoſed Curve, P the Point, or Center, to which its Ordinates are referred, NOL



the Evolute, and RO the Ray of Curvature at R: Moreover, let PH be perpendicular to RO; and, ſuppoſing the Ordinate PR (y) to become variable by the Motion of the Point R along the Curve, let the Fluxions of AR and PH (p), expreſſing the Celerities of the Points R and H in Directions perpendicular to RO*, be denoted by $\dot{z}$ and $\dot{p}$ reſpectively.

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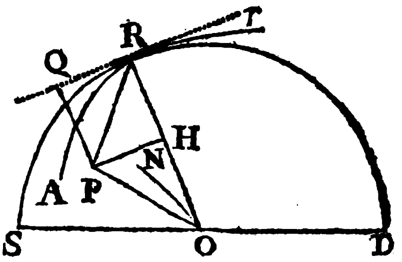
*Art. 5.

Therefore, the Celerities, of any two Points, in a Right-line revolving about a Center, being as the Diſtances from that Center, it follows that $\dot{p} : \dot{z} :: \mathrm{OH} : \mathrm{OR}$; whence by Diviſion (putting $\mathrm{RH} = v$) we have $\dot{z} - \dot{p} : \dot{z} :: v\ (\mathrm{RH}) : \mathrm{RO} = \frac{v\dot{z}}{\dot{z} - \dot{p}} = \frac{vp\dot{z}}{p\dot{z} - p\dot{p}}$: But $p\dot{z} = y\dot{y}$ (*by Art.* 60.) and therefore $\mathrm{RO} = \frac{vy\dot{y}}{y\dot{y} - p\dot{p}}$; which, becauſe $y^2 - p^2$ is $= v^2$ (and therefore $y\dot{y} - p\dot{p} = v\dot{v}$) will alſo be $= \frac{vy\dot{y}}{v\dot{v}} = \frac{y\dot{y}}{\dot{v}}$.

The ſame otherwiſe.

Let SRD be a Circle deſcribed about the Point O, as a Center, and ſuppoſe the Diſtance PR to be variable by the Motion of the Point R along the Arch of the Circle (inſtead of the Curve): Then, drawing OP, and putting $\mathrm{OR} = r$, $\mathrm{PR} = y$, &c. as before, we ſhall get OP^2 $(\mathrm{OR}^2 + \mathrm{PR}^2 - 2\mathrm{OR} \times \mathrm{RH}) = r^2 + y^2 - 2rv$; which (as well as r) being a conſtant Quantity, its Fluxion $2y\dot{y} - 2r\dot{v}$ muſt be equal to nothing; and therefore $r = \frac{y\dot{y}}{\dot{v}}$, the very ſame as above. Nor is it of any Conſequence whether $\dot{y}$ and $\dot{v}$ be here looked upon as reſpecting the Circle, or the Curve; ſince, at R, they muſt be the ſame in both Caſes; otherwiſe the Curvature could not be the ſame *. Now from the Value of RO thus found, which (corrected, when neceſſary) will alſo expreſs the Length of the Arch NO of the Evolute ‡, the Ordinate PO and the Tangent OH of the Evolute may



*Art. 63.

‡Art. 63.

may be eaſily deduced. For OH (RO—RH) $= \frac{y\dot{y}}{\dot{v}} - v = \frac{p\dot{p}}{\dot{v}}$, and PO ($= \sqrt{OH^2+PH^2} = \frac{p\sqrt{\dot{p}^2+\dot{v}^2}}{\dot{v}}$; whence the Nature of the Evolute is known.

EXAMPLE I.

74. Let the given Curve AR be the logarithmic Spiral, whoſe Nature is ſuch, that the Angle PRQ (or RPH) which the Ordinate makes with the Curve is every where the ſame.

Then (denoting the Sine of that Angle by b, and the Radius of the Tables by a) we have RH (v) $= \frac{by}{a}$, and therefore RO $\left(\frac{y\dot{y}}{\dot{v}}\right) = \frac{ay\dot{y}}{b\dot{y}} = \frac{ay}{b}$; which being to PR ($y$) in the conſtant Ratio of a to b, or of PR to RH, the Triangles ROP and RPH muſt therefore be ſimilar, and ſo the Angle POH, which the Ordinate PO makes with the Evolute, being every where equal to PRQ, will likewiſe be invariable. Whence it appears that the Evolute is alſo a logarithmic Spiral, ſimilar to the Involute; and that a Right-line drawn from the Center, perpendicular to the Ordinate, of any logarithmic Spiral, will paſs thro' the Center of Curvature.

EXAMPLE II.

75. Let the Curve propoſed be the Spiral of *Archimedes*; where we have $p = \frac{by}{\sqrt{y^2+b^2}}$, and $v = \frac{y^2}{\sqrt{y^2+b^2}}$ (ſee *Art.* 62.) Therefore $\dot{v} = 2y\dot{y} \times \overline{y^2+b^2}|^{-\frac{1}{2}} + yy \times$

$-\frac{1}{2}\times 2y\dot{y}\times\overline{y^2+b^2}|^{-\frac{3}{2}}\pm\frac{2y\dot{y}}{\overline{y^2+b^2}|^{\frac{1}{2}}}-\frac{y^3\dot{y}}{\overline{y^2+b^2}|^{\frac{3}{2}}}=$ $\frac{2y\dot{y}\times\overline{y^2+b^2}-y^3\dot{y}}{\overline{y^2+b^2}|^{\frac{3}{2}}}=\frac{y^3\dot{y}+2b^2y\dot{y}}{\overline{y^2+b^2}|^{\frac{3}{2}}}$; whence the Radius of

* Curvature $\frac{y\dot{y}}{\dot{v}}$ is here $=\frac{\overline{yy+bb}|^{\frac{3}{2}}}{y^2+2b^2}$; which being $=\frac{b}{2}$, *Art.73.
when $y=0$, the Arch of the Evolute †, reckoned from †Art.68.
the Vertex, is therefore $=\frac{\overline{yy+bb}|^{\frac{3}{2}}}{y^2+2b^2}-\frac{b}{2}$.

After the very ſame Manner you may proceed in other Caſes: But if the Value of $\dot{v}$ (or $\frac{y\dot{y}}{\dot{v}}$) changes, in any Caſe, from Poſitive to Negative, the Radius of Curvature (RO) after becoming infinite, will fall on the other Side of the Tangent, and the correſponding Point of the Curve, when $\dot{v}=0$, will be a Point of *Contrary-Flexure.* Whence it may be obſerved that the Point of Inflection, in a Curve whoſe Ordinates are referred to a Center, may be found by making the Fluxion of the Perpendicular, drawn from the Center to the Tangent, equal to Nothing, which Caſe is not taken Notice of in the preceding Section.

SECTION VI.

Of the Inverſe Method, or the Manner of determining the Fluents of given Fluxions.

76. IN the *Inverſe Method*, which teaches the Manner of finding the reſpective flowing Quantities of given Fluxions, there will be no great Difficulty in conceiving the Reaſons, if what is already delivered in *Sect.* 1. *on the direct Method*, has been duly conſidered: Though the Difficulties that occur in this Part, upon another Account, are indeed vaſtly ſuperior.

It is an eaſy Matter, or not impoſſible at moſt, to find the Fluxion of any flowing Quantity whatever; but in the *Inverſe Method* the Caſe is quite different: For, as there is no Method for deducing the Fluent from the Fluxion *a priori*, by a direct Inveſtigation, ſo it is impoſſible to lay down Rules for any other Forms of Fluxions, than thoſe particular ones which we know, from the direct Method, belong to ſuch and ſuch kinds of flowing Quantities. Thus, for Example, the Fluent of $2x\dot{x}$ is known to be x^2, becauſe it is found in *Art.* 6. and 14. that $2x\dot{x}$ is the Fluxion of x^2: But the Fluent of $y\dot{x}$ is unknown, ſince no Expreſſion has been diſcovered that produces $y\dot{x}$ for its Fluxion.

77. Now, as the principal Rule in the *direct Method* is that for the Fluxions of Powers, derived in *Art.* 8. (where it is proved that the Fluxion of x^n is, *univerſally*, expreſſed by $nx^{n-1}\dot{x}$); ſo the moſt general Rule, that can be given in the *Inverſe Method*, muſt be that ariſing from the converſe thereof; which *ſhews how to aſſign the Fluent of any Power of a variable Quantity drawn into the Fluxion of the Root*; and which, expreſſed in Words, will be as follows.

Divide by the Fluxion of the Root, add Unity to the Exponent of the Power, and divide by the Exponent ſo increaſed.

For, dividing the Fluxion $nx^{n-1}\dot{x}$ by $\dot{x}$ (the Fluxion of the Root x) it becomes nx^{n-1}; and, adding 1 to the Exponent $(n-1)$ we have nx^{n}; which, divided by n, gives x^{n}, the true Fluent of nx^{n-1}, by *Art.* 8.

Hence (by the ſame Rule) the
Fluent of $3x^2\dot{x}$ will be $= x^3$;

That of $8x^2\dot{x} = \frac{8x^3}{3}$;

That of $2x^5\dot{x} = \frac{x^6}{3}$

That of $y^{\frac{1}{2}}\dot{y} = \frac{2}{3}y^{\frac{3}{2}}$;

That of $ay^{\frac{5}{3}}\dot{y} = \frac{3ay^{\frac{8}{3}}}{8}$;

That of $y^{\frac{m}{n}}\dot{y} = \frac{y^{\frac{m}{n}+1}}{\frac{m}{n}+1} = \frac{ny^{\frac{m+n}{n}}}{m+n}$;

That of $\frac{a\dot{x}}{x^{n}}$, or $a\dot{x}x^{-n}$, $= \frac{ax^{1-n}}{1-n}$;

That of $\overline{a+z}|^{3} \times \dot{z} = \frac{\overline{a+z}|^{4}}{4}$;

And that of $\overline{a^{m}+z^{m}}|^{n} \times z^{m-1}\dot{z} = \frac{\overline{a^{m}+z^{m}}|^{n+1}}{m\times\overline{n+1}}$:

For *here* the Root, or the Quantity under the general Index n, being $a^{m}+z^{m}$, and its Fluxion $= mz^{m-1}\dot{z}$ (*Art.* 14.) we ſhall, by dividing by the laſt of theſe Quantities, have $\frac{\overline{a^{m}+z^{m}}|^{n}}{m}$; whence, increaſing the

Index by Unity, and dividing by $(n+1)$ the Index ſo increaſed, there comes out $\frac{\overline{a^m+z^m}|^{n+1}}{m\times\overline{n+1}}$.

After the very ſame Manner the Fluents of other Expreſſions may be deduced, when the Quantity, or Multiplicator, without the Vinculum is either equal, or in a conſtant Ratio, to the Fluxion of the Quantity, under the Vinculum: As in the Expreſſion $\overline{a+cz^n}|^m \times dz^{n-1}\dot{z}$; where the Number of Dimenſions of z under the *Vinculum* (or general Index) being equal to thoſe of z without the *Vinculum* $+ 1$, the Fluent may therefore be had, as in the preceding Examples; and will come out $\frac{\overline{a+cz^n}|^{m+1}\times d}{nc\times\overline{m+1}}$: And, that this (or any other Expreſſion derived in like Manner) is the true Fluent will evidently appear, by ſuppoſing x equal to $a+cz^n$ the Quantity under the Vinculum; for then (equal Quantities having equal Fluxions) $\dot{x}$ will be

* Art. 8. $= ncz^{n-1}\dot{z}$ *; and conſequently $\overline{a+cz^n}|^m \times dz^{n-1}\dot{z}$ $\left(=x^m \times \frac{d\dot{x}}{nc}\right) = \frac{dx^m\dot{x}}{nc}$; whoſe Fluent is therefore

‡ Art. 77. $\frac{dx^{m+1}}{nc\times\overline{m+1}}$ ‡ $= \frac{d\times\overline{a+cz^n}|^{m+1}}{nc\times\overline{m+1}}$, *as before.*

78. In aſſigning the Fluents of given Fluxions there is another Particular that ought to be attended to, not yet taken notice of; and that is, whether the flowing Quantity, found by the common Rule, above delivered, does not require the Addition or Subtraction of ſome conſtant Quantity to render it complete. This

indeed can, only, be known from the Nature of the Problem under Conſideration; but that ſuch an Addition or Subtraction may, in ſome Caſes, become neceſſary is evident from the Subject itſelf; ſince a flowing Quantity increaſed, or decreaſed, by a conſtant Quantity, has ſtill the ſame Fluxion; and therefore the Fluent of that Fluxion is as properly expreſſed by the whole compound Expreſſion, as by the variable Part of it, alone; Thus, for Inſtance, the Fluent of $nx^{n-1}\dot{x}$ may be either repreſented by x^n or by $x^n \pm a$, becauſe (a being conſtant) the Fluxion of $x^n \pm a$, as well as of x^n, is $nx^{n-1}\dot{x}$.

79. Hence it appears that it is the variable Part of a Fluent only which is aſſignable by the common Method; the conſtant Part (when ſuch becomes neceſſary) being to be aſcertained from the particular Nature of the Problem. Now to do this, the beſt Way is to conſider how much the variable Part of the Fluent, firſt found, differs from the Truth, in that particular Circumſtance when the required Quantity which the whole Fluent ought to expreſs, is equal to Nothing; then that Difference, added to, or ſubtracted from, the ſaid variable Part, as occaſion requires, will give the Fluent truly corrected: For, ſince the Difference of two Quantities flowing with the ſame Celerity (or having equal Fluxions) is either, Nothing at all, or *conſtantly* the *ſame*, the Difference in that Circumſtance will likewiſe be the Difference in all other Circumſtances: And therefore being added to the leſſer Quantity, or ſubtracted from the greater, both become equal.

80. To render what is above delivered as familiar as may be, I ſhall put down a few Examples; in which the variable Quantities repreſented by x and y are ſuppoſed to begin their Exiſtence together, or to be generated, at the ſame time.

1. Let $\dot{y} = a^2x\dot{x}$; then the Fluent, found as uſual, will be $y = \frac{a^2x^2}{2}$; where taking $y = 0$, $\frac{a^2x^2}{2}$ alſo vaniſhes, (becauſe then $x=0$ by Hypotheſis): Therefore the Fluent requires no Correction in this Caſe.

2. Let $\dot{y} = \overline{a+x}|^3 \times \dot{x}$: Here we firſt have $y = \frac{\overline{a+x}|^4}{4}$; but when $y=0$, then $\frac{\overline{a+x}|^4}{4}$ becomes $= \frac{a^4}{4}$ (ſince x by Hypotheſis is then $= 0$;) Therefore $\frac{\overline{a+x}|^4}{4}$ *always* exceeds y by $\frac{a^4}{4}$; and ſo the Fluent properly corrected will be $y = \frac{\overline{a+x}|^4 - a^4}{4} = a^3x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4}$.

But the very ſame Fluent may be otherwiſe found, without needing any Correction: For the given Equation ($\dot{y} = \overline{a+x}|^3 \times \dot{x}$), by expanding $\overline{a+x}|^3$, is transformed to $\dot{y} = a^3\dot{x} + 3a^2x\dot{x} + 3ax^2\dot{x} + x^3\dot{x}$; whence $y = a^3x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4}$; the ſame as *above*.

Hence it appears that the Fluent of an Expreſſion, found according to one Form, may require a very different Correction from the Fluent of the ſame Fluxion found according to another Form.

3. Let $\dot{y} = \overline{a^2-x^2}|^{\frac{1}{2}} \times x\dot{x}$; then, firſt, $y = -\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3}$; where taking $y=0$, $-\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3}$ becomes =

$= -\frac{a^3}{3}$; therefore $-\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3}$ is too little by $\frac{a^3}{3}$; and ſo the Fluent corrected will be $y = \frac{a^3}{3} - \frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3}$.

4. Let $\dot{y} = \overline{a^m+x^m}|^n \times x^{m-1}\dot{x}$: Here we firſt have $y = \frac{\overline{a^m+x^m}|^{n+1}}{m\times\overline{n+1}}$; and making $y=0$, the latter Part of the Equation becomes $\frac{\overline{a^m}|^{n+1}}{m\times\overline{n+1}} = \frac{a^{mn+m}}{m\times\overline{n+1}}$; whence the Equation, or Fluent, truly corrected is $y = \frac{\overline{a^m+x^m}|^{n+1} - a^{mn+m}}{m\times\overline{n+1}}$,

5. Laſtly, let $\dot{y} = \overline{a+bx^m+cx^n}|^p \times \overline{mbx^{m-1}\dot{x}+ncx^{n-1}\dot{x}}$; then, in the firſt Place, we have $y = \frac{\overline{a+bx^m+cx^n}|^{p+1}}{p+1}$; which corrected, as above, becomes $y = \frac{\overline{a+bx^m+cx^n}|^{p+1} - a^{p+1}}{p+1}$.

81. Hitherto x and y are both ſuppoſed equal to Nothing at the ſame time; but that will not always be the Caſe in the Solution of Problems. Thus, for Inſtance, though the Sine and Tangent of an Arch are both equal to Nothing when the Arch itſelf is equal to Nothing, yet

the Secant is then equal to the Radius: It will be proper therefore to add an Example or two wherein the Value of y is equal to Nothing, when that of x is equal to any given Quantity a.

Let, then, the Equation $\dot{y} = x^2\dot{x}$ be firſt propoſed; whereof the Fluent (firſt taken) is $y = \frac{x^3}{3}$; but when $y = 0$, then $\frac{x^3}{3} = \frac{a^3}{3}$, by Hypotheſis; therefore the Fluent, corrected, is $y = \frac{x^3 - a^3}{3}$.

Again, let the propoſed Equation be $\dot{y} = -x^n\dot{x}$; then will $y = -\frac{x^{n+1}}{n+1}$; which corrected becomes $y = \frac{a^{n+1} - x^{n+1}}{n+1}$.

Laſtly, let $\dot{y} = \overline{c^3 + bx^2}|^{\frac{1}{2}} \times x\dot{x}$; then, firſt, $y = \frac{\overline{c^3+bx^2}|^{\frac{3}{2}}}{3b}$; and, when $y = 0$ and $x = a$, $\frac{\overline{c^3+bx^2}|^{\frac{3}{2}}}{3b}$ becomes $= \frac{\overline{c^3+ba^2}|^{\frac{3}{2}}}{3b}$: therefore the Fluent corrected is $y = \frac{\overline{c^3+bx^2}|^{\frac{3}{2}} - \overline{c^3+ba^2}|^{\frac{3}{2}}}{3b}$.

82. All the Examples hitherto given relate to ſuch Fluxions as involve one variable Quantity only in each Term, whoſe Fluents are aſſignable from the Converſe of the firſt General Rule, *in Section* I. But, beſides theſe, various other Forms of Fluxions may be propoſed, involving two or more variable Quantities, whoſe Fluents may alſo be found by Help of the other two General Rules delivered in the ſame Section.

Thus the Fluent of $y\dot{x}+x\dot{y}$ is expreſſed by xy*; that of $\frac{y\dot{x}-x\dot{y}}{y^2}$ by $\frac{x}{y}$†; that of $a\dot{x}+x\dot{y}+y\dot{x}$ by $ax+xy$‡; and that of $\overline{nx\dot{y}y^{n-1}+y^n\dot{x}-nax^{n-1}\dot{x}}\times\overline{y^nx-ax^n}\big|^{\frac{p}{m}}$ by $\frac{m\times\overline{y^nx-ax^n}\big|^{\frac{p+m}{m}}}{p+m}$: For, dividing (in the laſt Caſe) by the Fluxion of the Root $\overline{y^nx-ax^n}$*, which (by *Art.* 14 and 15) is $nxy^{n-1}\dot{y}+y^n\dot{x}-nax^{n-1}\dot{x}$, we firſt have $\overline{y^nx-ax^n}\big|^{\frac{p}{m}}$; whence, adding Unity to the Exponent $\frac{p}{m}$, and dividing by the Exponent ſo increaſed, we get $\frac{\overline{y^nx-ax^n}\big|^{\frac{p}{m}+1}}{\frac{p}{m}+1}=\frac{m\times\overline{y^nx-ax^n}\big|^{\frac{p+m}{m}}}{p+m}$ for the true Fluent of the Quantity propoſed. But it ſeldom happens that theſe Kinds of Fluxions which involve two different variable Quantities in one Term, and yet admit of known, or perfect, Fluents, are to be met with in Practice: I ſhall therefore take no further Notice of them in this Place (but refer the Reader to the ſecond Part of the Work) my Deſign here being to inſiſt only upon what is moſt general and uſeful in the Subject; which brings me to further conſider thoſe Forms of Fluxions, involving one variable Quantity only, that frequently occur in the Solution of Problems, whoſe Fluents may (after proper Transformation) be found, by the Rule already delivered in *Art.* 77.

* Art. 10.
† Art. 13.
‡ Art. 10.
* Art. 77.

83. It has been already hinted, that if a Fluxion of the Binomial Kind, as $\overline{a+cz^{n}}^{m} \times dz^{n-1}\dot{z}$, has the Index $(n-1)$ of the variable Quantity (z) without the *Vinculum* + 1, equal to (n) the Index of the ſame Quantity under the *Vinculum*, the Fluent thereof may be then truly found by the forementioned Rule. But the ſame Obſervation may be farther extended to *thoſe Caſes where the Index without the Vinculum increaſed by Unity is equal to any Multiple of that under the Vinculum*; as in the Expreſſions, $\overline{a+cz^{n}}^{m} \times dz^{2n-1}\dot{z}$, $\overline{a+cz^{n}}^{m} \times dz^{3n-1}\dot{z}$, $\overline{a+cz^{n}}^{m} \times dz^{4n-1}\dot{z}$, &c. Whoſe Fluents are thus determined.

Put $a+cz^{n} = x$, then will $z^{n} = \frac{x-a}{c}$, and $nz^{n-1}\dot{z} = \frac{\dot{x}}{c}$*; and therefore $z^{2n-1}\dot{z} = \frac{x-a}{c} \times \frac{\dot{x}}{nc} = \frac{x\dot{x}-a\dot{x}}{ncc}$; whence by Subſtitution we get $\overline{a+cz^{n}}^{m} \times dz^{2n-1}\dot{z} = \frac{x^{m} \times d \times \overline{x\dot{x}-a\dot{x}}}{nc^{2}} = d \times \frac{x^{m+1}\dot{x}-ax^{m}\dot{x}}{nc^{2}}$: Whoſe Fluent (by *Art.* 77.) is therefore $= \frac{d}{nc^{2}} \times \overline{\frac{x^{m+2}}{m+2} - \frac{ax^{m+1}}{m+1}}$; which, by reſtoring the Value of x, becomes $\frac{d}{nc^{2}} \times \overline{\frac{\overline{a+cz^{n}}^{m+2}}{m+2} - \frac{a \times \overline{a+cz^{n}}^{m+1}}{m+1}} =$

* Art. 8.

$d \times$

$\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2}\times\overline{\frac{a+cz^n}{m+2}-\frac{a}{m+1}}=\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2}\times\overline{\frac{cz^n}{m+2}-\frac{a}{\overline{m+2}\times\overline{m+1}}}$; the true Fluent of $\overline{a+cz^n}|^m\times dz^{2n-1}\dot{z}$.

Again; for the Fluent of $\overline{a+cz^n}|^m\times dz^{3n-1}\dot{z}$, because $z^{n-1}\dot{z}=\frac{\dot{x}}{nc}$, and $z^n=\frac{x-a}{c}$, we have $z^{3n-1}\dot{z}$ $\left(=z^{2n}\times z^{n-1}\dot{z}\right)=\frac{\overline{x-a}|^2}{c^2}\times\frac{\dot{x}}{nc}=\frac{x^2\dot{x}-2ax\dot{x}+a^2\dot{x}}{nc^3}$:

Whence, $\overline{a+cz^n}|^m$ being $=x^m$, we get $\overline{a+cz^n}|^m\times dz^{3n-1}\dot{z}=dx^m\times\frac{x^2\dot{x}-2ax\dot{x}+a^2\dot{x}}{nc^3}=\frac{d}{nc^3}\times\overline{x^{m+2}\dot{x}-2ax^{m+1}\dot{x}+a^2x^m\dot{x}}$; whose Fluent is therefore $=\frac{d}{nc^3}\times\overline{\frac{x^{m+3}}{m+3}-\frac{2ax^{m+2}}{m+2}+\frac{a^2x^{m+1}}{m+1}}=\frac{dx^{m+1}}{nc^3}\times\overline{\frac{x^2}{m+3}-\frac{2ax}{m+2}+\frac{a^2}{m+1}}=\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^3}\times\overline{\frac{\overline{a+cz^n}|^2}{m+3}-\frac{2aa+2acz^n}{m+2}+\frac{a^2}{m+1}}=\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^3}\times\overline{\frac{c^2z^{2n}}{m+3}-\frac{2acz^n}{\overline{m+3}\times\overline{m+2}}+\frac{2a^2}{\overline{m+3}\times\overline{m+2}\times\overline{m+1}}}$.

Univerſally, let r denote any whole poſitive Number whatever, and let the Fluent of $\overline{a+cz^n}^{m} \times dz^{rn-1}\dot{z}$ be required; then, by putting $a+cz^n = x$, and proceeding as above, our propoſed Fluxion is transformed to $\frac{dx^m\dot{x}}{nc^r} \times \overline{x-a}^{r-1}$; which, expanding $\overline{x-a}^{r-1}$ (by the Binomial Theorem) becomes

$$\frac{d}{nc^r} \times \overline{x^{m+r-1}\dot{x} - \overline{r-1} \times ax^{m+r-2}\dot{x} + \overline{r-1} \times \frac{r-2}{2} \times a^2x^{m+r-3}\dot{x}}$$

&c. whoſe Fluent is therefore

$$= \frac{d}{nc^r} \times \overline{\frac{x^{m+r}}{m+r} - \frac{\overline{r-1} \times ax^{m+r-1}}{m+r-1} + \frac{\overline{r-1} \times \overline{r-2} \times a^2x^{m+r-2}}{2 \times \overline{m+r-2}}} \text{ \&c.} =$$

$$\frac{dx^{m+1}}{nc^r} \times \overline{\frac{x^{r-1}}{m+r} - \frac{\overline{r-1} \times ax^{r-2}}{m+r-1} + \frac{\overline{r-1} \times \overline{r-2} \times a^2x^{r-3}}{2 \times \overline{m+r-2}}}$$

&c.

Where, r being a whole poſitive Number, the Multiplicators $1, r-1, \overline{r-1} \times \overline{r-2}, \overline{r-1} \times \overline{r-2} \times \overline{r-3}$, &c. will therefore become equal to Nothing, after the r firſt terms; and ſo, the Series terminating, the Fluent itſelf will be truly exhibited in that Number of Terms: Except when $m+r$ is likewiſe a whole poſitive Number, leſs than r; in which Circumſtance the Diviſors $m+r$, $m+r-1$, $m+r-2$, &c. becoming equal to Nothing, before the Multiplicators, the correſponding Terms of the Series will be infinite. And in that Caſe the Fluent is ſaid to fail, ſince Nothing can then be determined from it.

84. Beſides the foregoing, there is another Way of deriving the Fluent of $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, in Terms of the original flowing Quantity z; which will afford a Theorem more commodious for Practice than that above given: The Method of Inveſtigation is thus.

Let $d\times\overline{a+cz^n}|^{m+1} \times \overline{Az^p+Bz^{p-v}+Cz^{p-2v}+Dz^{p-3v}}$ &c. (where p, v, A, B, C, &c. denote unknown, but determinate, Quantities) be aſſumed for the Fluent ſought: Then by taking the Fluxion of the Quantity ſo aſſumed we ſhall have

$$dcn\times\overline{m+1}\times z^{n-1}\dot{z}\times\overline{a+cz^n}|^m\times\overline{Az^p+Bz^{p-v}+Cz^{p-2v}+Dz^{p-3v}\ \&c.}+d\times\overline{a+cz^n}|^{m+1}\times\overline{pAz^{p-1}\dot{z}+\overline{p-v}\times Bz^{p-v-1}\dot{z}+\overline{p-2v}\times Cz^{p-v-1}\dot{z}\ \&c.}$$

which being put* equal to the given Fluxion, $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, and the whole Equation divided by $\overline{a+cz^n}|^m \times dz^{-1}\dot{z}$, there comes out

$$\left.\begin{array}{l} +cn\times\overline{m+1}\times z^n\times\overline{Az^p+Bz^{p-v}+Cz^{p-2v}+Dz^{p-3v}\ \&c.}\\ +\overline{a+cz^n}\times\overline{pAz^p+\overline{p-v}\times Bz^{p-v}+\overline{p-2v}\times Cz^{p-2v}\ \&c.}\end{array}\right\}=z^{rn}$$

Whence, by collecting the Coefficients of the like Powers of z, we have

$$\left.\begin{array}{llll} \left.\begin{array}{l}n\times\overline{m+1}\\ +p\end{array}\right\}\times cAz^{p+n} & +\left.\begin{array}{l}n\times\overline{m+1}\\ +p-v\end{array}\right\}\times cBz^{p+n-v} & +\left.\begin{array}{l}n\times\overline{m+1}\\ +p-2v\end{array}\right\}\times cCz^{p+n-2v} & \&c.\\ -z^{rn} & +paAz^p & +\overline{p-v}\times aBz^{p-v} & \&c.\end{array}\right\}=0$$

Where, comparing $p+n$ and rn, the two greateſt Exponents of z, we find $p=rn-n=\overline{r-1}\times n$; and by comparing the two next inferior Exponents $p+n-v$, and p, we likewiſe

*Art. 8. 10.

likewiſe get $v=n$; which Values being ſubſtituted above, our Equation is reduced to

$$\left.\begin{array}{l}\overline{m+r}\times ncAz^{rn}+\overline{m+r-1}\times ncBz^{rn-n}+\overline{m+r-2}\times ncCz^{rn-2n}\ \text{\&c.}\\ -z^{rn}+\quad\overline{r-1}\times naAz^{rn-n}+\overline{r-2}\times naBz^{rn-2n}\ \text{\&c.}\end{array}\right\}=0$$

Where, putting $m+r=s$, and comparing the Coefficients of the homologous Terms *, we have $A=\frac{1}{snc}$, $B=-\frac{\overline{r-1}\times aA}{\overline{s-1}\times c}=-\frac{\overline{r-1}\times a}{s\times\overline{s-1}\times nc^2}$, $C=-\frac{\overline{r-2}\times aB}{\overline{s-2}\times c}=\frac{\overline{r-1}\times\overline{r-2}\times a^2}{s\times\overline{s-1}\times\overline{s-2}\times nc^3}$, $D=-\frac{\overline{r-3}\times aC}{\overline{s-3}\times c}=-\frac{\overline{r-1}\times\overline{r-2}\times\overline{r-3}\times a^3}{s\times\overline{s-1}\times\overline{s-2}\times\overline{s-3}\times nc^4}$, &c. &c. which Values, with thoſe of p and v, being ſubſtituted in the aſſumed Fluent, it becomes $d\times\overline{a+cz^n}\big|^{m+1}\times$

$$\overline{\frac{z^{rn-n}}{snc}-\frac{\overline{r-1}\times az^{rn-2n}}{s\times\overline{s-1}\times nc^2}+\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{s\times\overline{s-1}\times\overline{s-2}\times nc^3}\ \text{\&c.}}=\frac{d\times\overline{a+cz^n}\big|^{m+1}}{snc}\times\overline{\frac{z^{rn-n}}{1}-\frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c}+\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{\overline{s-1}\times\overline{s-2}\times c^2}\ \text{\&c.}}$$

the true Fluent of $\overline{a+cz^n}\big|^{m+1}\times dz^{rn-1}\dot{z}$, which was to be determined: Which Fluent therefore, when r is a whole poſitive Number, will always terminate in as many Terms as are expreſſed by that Number; except in that particular Caſe, ſpecified in the laſt Article. Thus, if $r=2$, or

the

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* *Vid. p.* 181. *of my Treatiſe of Algebra.*

the given Fluxion be $\overline{a+cz^n}|^m \times dz^{2n-1}\dot{z}$; then, s $(m+r)$ being $=m+2$, the Fluent itſelf will become $\frac{d\times\overline{a+cz^n}|^{m+1}}{nc\times\overline{m+2}} \times \overline{\frac{z^n}{1} - \frac{a}{\overline{m+1}\times c}} = \frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2} \times \frac{cz^n}{m+2} - \frac{a}{\overline{m+2}\times\overline{m+1}}$; which is exactly the ſame with the firſt of thoſe found in *Art.* 83. by a different Method.

The like Agreement will likewiſe be found, when r is $= 3$: But when r, either denotes a broken, or a negative, Number, the Series for the Fluent will then run on to Infinity; becauſe no one of the Multiplicators $r-1$, $r-2$, $r-3$, $r-4$, &c. can in that Caſe be equal to Nothing.

85. The foregoing Fluent, it may be obſerved, was found by aſſuming $d\times\overline{a+cz^n}|^{m+1} \times Az^p + Bz^{p-v} + Cz^{p-2v}$ &c. and comparing the two greateſt Exponents, of the Equation thence reſulting: But if, inſtead of $Az^p + Bz^{p-v} + Cz^{p-2v}$ &c. an aſcending Series, as $Az^p + Bz^{p+v} + Cz^{p+2v}$ &c. (where the Exponents of z continually increaſe) be taken, and the two leaſt Indices of z in the Equation (in like Manner reſulting) be compared together, the ſame Fluent will be had according to a different Form, which will be of good Uſe in many Caſes, when the foregoing fails, or runs out into an Infinite Series.

Thus, if $p+v$, $p+2v$, &c. be wrote in the Room of $p-v$, $p-2v$, &c. reſpectively, in the firſt Equation of the laſt Article, it will appear that

$$\left.\begin{array}{l} +cn\times\overline{m+1}\times z^{n}\times\overline{Az^{p}+Bz^{p+v}+Cz^{p+2v}}\ \&c. \\ +\overline{a+cz^{n}}\times pAz^{p}+\overline{p+v}\times Bz^{p+v}+\overline{p+2v}\times Cz^{p+2v}\ \&c. \end{array}\right\}=z^{rn}$$

Which Equation may be reduced to

$$\left.\begin{array}{l} paAz^{p}+\overline{p+v}\times aBz^{p+v}+\overline{p+2v}\times aCz^{p+2v}\ \&c. \\ -z^{rn}+\left.\begin{array}{l}n\times\overline{m+1}\\ +p\end{array}\right\}\times cAz^{p+n}+\left.\begin{array}{l}n\times\overline{m+1}\\ +p+v\end{array}\right\}\times cBz^{p+n+v}\ \&c. \end{array}\right\}=0$$

Where, by comparing the two leaſt Exponents, &c. p will be found $=rn$, $v=n$; $A=\frac{1}{pa}=\frac{1}{rna}$; $B=-\frac{\overline{p+n\times\overline{m+1}}\times cA}{\overline{p+v}\times a}=-\frac{\overline{r+m+1}\times ncA}{\overline{r+1}\times na}=-\frac{\overline{r+m+1}\times c}{r\times\overline{r+1}\times na^{2}}$; $C=-\frac{\overline{p+v+n\times\overline{m+1}}\times cB}{\overline{p+2v}\times a}=-\frac{\overline{n+m+2}\times ncB}{\overline{r+2}\times na}=\frac{\overline{r+m+1}\times\overline{r+m+2}\times c^{2}}{r\times\overline{r+1}\times\overline{r+2}\times na^{3}}$ &c. &c. Therefore, denoting $r+m$ by s (as above) the Fluent of $\overline{a+cz^{n}}|^{m}\times dz^{rn-1}\dot{z}$, will (*alſo*) be truly repreſented by

$$d\times\overline{a+cz^{n}}|^{m+1}\times\overline{\frac{z^{rn}}{rna}-\frac{\overline{s+1}\times cz^{rn+n}}{r\times\overline{r+1}\times na^{2}}+\frac{\overline{s+1}\times\overline{s+2}\times c^{2}z^{rn+2n}}{r\times\overline{r+1}\times\overline{r+2}\times na^{3}}\ \&c.}$$

or its Equal

$$\frac{\overline{a+cz^{n}}|^{m+1}\times dz^{rn}}{rna}\times\overline{1-\frac{\overline{s+1}\times cz^{n}}{\overline{r+1}\times a}+\frac{\overline{s+1}\times\overline{s+2}\times c^{2}z^{2n}}{\overline{r+1}\times\overline{r+2}\times a^{2}}\ \&c.}$$

Which Series will terminate when s (or $r+m$) is a whole negative Number; and therefore in all ſuch Caſes the Fluent is exactly determined; provided r be not alſo a negative Integer leſs than s; for in this particular Circumſtance the Fluent fails, the Diviſor firſt becoming equal to Nothing. *Vid. Art.* 83.

The Uſe of the two foregoing general Expreſſions, for the Fluent of $\overline{a+cz^{n}}\big|^{m} \times dz^{rn-1}\dot{z}$, will appear from the following Examples.

EXAMPLE I.

86. *Let it be required to find the Fluent of* $\frac{bx\dot{x}}{\overline{a+x}|^{\frac{1}{2}}}$, *or*

$$\overline{a+x}\big|^{-\frac{1}{2}}\times bx\dot{x}.$$

By comparing the Fluxion here propoſed with $\overline{a+cz^{n}}\big|^{m}\times dz^{rn-1}\dot{z}$, we have $a=a$, $c=1$, $z=x$, $n=1$, $m=-\frac{1}{2}$, $d=b$, $rn-1$ (or $r-1$) $=1$; whence $r=2$, and s $(r+m)=\frac{3}{2}$; whereof the former being a whole poſitive Number, let theſe Values be therefore ſubſtituted in

$$\left(\frac{d\times\overline{a+cz^{n}}\big|^{m+1}}{snc}\times\overline{\frac{z^{rn-n}}{1}-\frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c}+\frac{\overline{r-1}\times\overline{r-2}\times a^{2}z^{rn-3n}}{\overline{s-1}\times\overline{s-2}\times c^{2}}}, \text{\&c.}\right)$$

the firſt of the two general Expreſſions for the Fluent, and it will become $\frac{b\times\overline{a+x}|^{\frac{1}{2}}}{\frac{3}{2}}\times\overline{x-\frac{a}{\frac{1}{2}}}=\frac{b\times\overline{a+x}|^{\frac{1}{2}}\times\overline{2x-4a}}{3}$, the Quantity ſought in this Caſe.

EXAMPLE II.

87. *Let the Fluxion propoſed be* $\frac{b\dot{x}x^{3n-1}}{\sqrt{a+fx^n}}$, or

$$\overline{a+fx^n}\big|^{-\frac{1}{2}}\times bx^{3n-1}\dot{x}.$$

Here, by proceeding as above, we have $a=a$, $c=f$, $z=x$, $n=n$, $m=-\frac{1}{2}$, $d=b$, $r=3$, and $s\ (r+m)=\frac{5}{2}$: Whence, by ſubſtituting theſe ſeveral Values in the ſame general Expreſſion, we get $\frac{b\times\overline{a+fx^n}\big|^{\frac{1}{2}}}{\frac{5}{2}nf}\times$

$$\overline{x^{2n}-\frac{2ax^n}{\frac{3}{2}f}+\frac{2a^2}{\frac{3}{2}\times\frac{1}{2}f^2}} = \frac{b\times\overline{a+fx^n}\big|^{\frac{1}{2}}}{nf^3}\times\frac{\overline{6f^2x^{2n}-8afx^n+16a^2}}{15}.$$

EXAMPLE III.

88. *Wherein the Quantity propoſed is* $\frac{\dot{y}\sqrt{g^2+y^2}}{y^6}$, or

$$\overline{g^2+y^2}\big|^{\frac{1}{2}}+y^{-6}\dot{y}.$$

Here we have $a=g^2$, $c=1$, $z=y$, $n=2$, $m=\frac{1}{2}$, $d=1$, $rn-1$ (or $2r-1$)$=-6$; whence $r\ (=\frac{-6+1}{2})$ $=-\frac{5}{2}$, and $s\ (r+m)=-2$; whereof the latter being a whole Negative Number, let the ſeveral Values here exhibited be therefore ſubſtituted in

$a+$

$\left(\frac{\overline{a+cz^n}|^{m+1} \times dz^{rn}}{rna} \times \overline{1 - \frac{\overline{s+1} \times cz^n}{\overline{r+1} \times a} + \frac{\overline{s+1} \times \overline{s+2} \times c^2 z^{2n}}{\overline{r+1} \times \overline{r+2} \times a^2}}\right.$ &c.) the latter of the two general Expreſſions above derived, and it will become $\frac{\overline{g^2+y^2}|^{\frac{3}{2}} \times y^{-5}}{-5g^2} \times \overline{1 - \frac{-1 \times y^2}{-\frac{3}{2} \times g^2}} = \frac{\overline{g^2+y^2}|^{\frac{3}{2}} \times \overline{2y^2 - 3g^2}}{15g^4y^5}$; the true Fluent required.

EXAMPLE IV.

89. *Laſtly, let the given Fluxion be* $\overline{a-fz^n}|^{\frac{1}{2}} \times z^{-\frac{7}{2}n-1}\dot{z}$.

Then, a being $= a$, $c = -f$, $m = \frac{1}{2}$, $d = 1$, $r = -\frac{7}{2}$, and the reſt as in the general Fluxion $\overline{a+cx^n}|^m \times dz^{rn-1}\dot{z}$; we ſhall, by ſubſtituting in the ſecond Form (becauſe s is here equal to (-3) a whole negative Number) have $\frac{\overline{a-fz^n}|^{\frac{3}{2}} \times z^{-\frac{7}{2}n}}{-\frac{7}{2}na} \times \overline{1 - \frac{-2 \times -fz^n}{-\frac{5}{2}a} \frac{-2 \times -1 \times f^2z^{2n}}{-\frac{5}{2} \times -\frac{3}{2}a^2}} = \frac{\overline{a-fz^n}|^{\frac{3}{2}}}{-\frac{7}{2}naz^{\frac{7}{2}n}} \times \overline{1 + \frac{4fz^n}{5a} + \frac{8f^2z^{2n}}{15a^2}}$
$= -\frac{\overline{a-fz^n}|^{\frac{3}{2}} \times \overline{30a^2 + 24afz^n + 16f^2z^{2n}}}{105na^3z^{\frac{7}{2}n}}$.

90. Having inſiſted largely on the Manner of finding ſuch Fluents as can be truly exhibited in Algebraic Terms; it remains now to ſay ſomething with regard

to thoſe other Forms of Expreſſions, involving one variable Quantity only, which, *yet*, are ſo affected by compound Diviſors and radical Quantities, that their Fluents cannot be *accurately* determined by any Method whatſoever; of which there are innumerable Kinds: But there is one general Method whereby the Fluents of ſuch Expreſſions are approximated, to any aſſigned Degree of Exactneſs; namely, the Method of *Infinite Series*; which it will, therefore, be neceſſary to explain; ſo far as relates to the Manner of expounding the Value of any compound Fraction, or ſurd Quantity, by Help of ſuch a Series.

EXAMPLE I.

91. *Let, then, the Fraction* $\frac{ax}{a-x}$ *be, firſt, given; to be converted into an Infinite Series.*

Divide the Numerator ax by the Denominator $a-x$, as is taught in Compound Diviſion of common Algebra; then the Operation will ſtand as follows;

$$\begin{array}{l}
a-x)\,ax \quad \left(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+\text{\&c.}\right. \\
\underline{ax-xx} \\
\quad +xx \\
\quad \underline{+xx-\frac{x^3}{a}} \\
\qquad\quad +\frac{x^3}{a} \\
\qquad\quad \underline{+\frac{x^3}{a}-\frac{x^4}{a^2}} \\
\qquad\qquad\quad +\frac{x^4}{a^2} \\
\qquad\qquad\qquad \text{\&c.}
\end{array}$$

Where the Quotient, or Series $x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} + \frac{x^5}{a^4} + \frac{x^6}{a^5}$ &c. infinitely continued, is taken to expound the Value of the propoſed Fraction $\frac{ax}{a-x}$:

92. But, though the Series thus ariſing ought to be carried on to an Infinity of Terms, to have the true Value of the Quantity firſt propoſed; or, though the Quotient, continued to ever ſo great a Number of Terms, will be *ſtill* ſomething defective of the Truth; yet, if the Value of the Quantity (x) in the Numerator be but ſmall in Compariſon of the Quantity (a) in the Denominator, the Remainder, after a few Terms in the Quotient, will become ſo exceeding ſmall, as to be neglected without any conſiderable Error; and then the Value of the *Whole*, or of the Quantity firſt propoſed, will be, very nearly, exhibited, by taking a ſmall Number of the leading Terms only.

Thus, for Inſtance, let the Value of a be expounded by 10, and that of x by Unity; then the Remainder $\left(\frac{x^3}{a}\right)$ after the two firſt Terms of the Quotient, being $= \frac{1}{10}$, this Value, divided by the given Diviſor ($a-x=$) 9, will therefore give $\frac{1}{90} = 0{,}01111111$, &c. for the Defect, by taking the two firſt Terms only: But, if the three firſt Terms be taken, the Defect will be *ſtill* leſs conſiderable; amounting to no more than $\frac{1}{900}$, or 0,00111111, &c.

This may likewiſe be made to appear, without any regard to the Remainder, by collecting into one Sum, the Values of all the Terms to be taken: For, if only the firſt two $\left(x + \frac{x^2}{a}\right)$ be propoſed, their Sum will be

= 1, 1; which, deducted from the true Value of the given Fraction $\frac{ax}{a-x}\left(=\frac{10}{9}\right) = 1,1111111$ &c. the Difference will come out 0,01111, *the very ſame as before.*

Thus, alſo, by collecting the Sum of the three, four and five, &c. firſt Terms of the Series, you will have 1,11; 1,111; and 1,1111 &c. which, being ſucceſſively deducted from 1,111111111 &c. (as above) there will remain 0,001111 &c. 0,0001111 &c. 0,00001111 &c. for the Errors or Defects in thoſe Caſes reſpectively.

93. From what has been ſaid in the preceding Article it appears, that Infinite Serieſes, in Algebra (according to a common Obſervation) are ſimilar to, or correſpond with, Decimal Fractions in common Arithmetick: For, as a Decimal Fraction may be carry'd on to any propoſed Number of Places, however great, and yet never amount to a Quantity, which but a very little exceeds the Value of the three or four firſt Places; ſo a Series may be infinite with regard to the Number of its Terms, and yet a few of the leading Terms only, may be ſufficient to expreſs the Value of the *Whole*, very nearly: Provided, always, that the Series has a ſufficient Rate of Convergency, or that its Terms decreaſe in a pretty large Proportion: For, otherwiſe, *even*, a great Number of Terms may be uſed to little Purpoſe: Thus, in the foregoing Series, $x + \frac{x^2}{a} + \frac{x^3}{a^2}$ &c. if x be taken $= a$, no Number of Terms will be ſufficient to exhibit the Value of the correſponding Fraction $\frac{ax}{a-x}$, it being infinite in that Circumſtance.

94. Having endeavoured to ſhew, that the true Value of an infinite Series may be nearly obtained by adding together a few of the firſt Terms only, I ſhall now proceed to give other Examples of the Manner of

converting fractional, and ſurd, Quantities into ſuch Kinds of Serieſes, in order to the Approximation of the Fluents of Expreſſions affected by them.

EXAMPLE II.

Let the Quantity propoſed be the Fraction $\frac{c^2}{c^2 + 2cy + y^2}$; then, by proceeding as in the firſt Example, you will have

$$c^2 + 2cy + y^2)\; c^2 \ldots\ldots \left(1 - \frac{2y}{c} + \frac{3y^2}{c^2} - \frac{4y^3}{c^3}\right. \text{ \&c.}$$

$$\begin{array}{l}
\underline{c^2 + 2cy + y^2} \\
-2cy - y^2 \\
\underline{-2cy - 4y^2 - \frac{2y^3}{c}} \\
\quad +3y^2 + \frac{2y^3}{c} \text{ \&c.}
\end{array}$$

Where, from a few of the firſt Terms of the Quotient, the Law of Continuation is manifeſt; the Numerators being in Arithmetical Progreſſion; and the Signs, + and —, alternately.

EXAMPLE III.

95. *Let the Quantity given be* $\frac{1 + x^2 - 2x^4}{1 - x - x^2}$.

Then the Quotient will be $1 + x + 3x^2 + 4x^3 + 5x^4 + 9x^5 + 14x^6$ &c. where the *Law of Continuation* is manifeſt; being ſuch that the Coefficient of each ſucceeding Term is equal to the Sum of thoſe of the two Terms immediately preceding it.

EXAMPLE IV.

96. *Let the Radical Quantity* $\sqrt{a^2+x^2}$ *be propoſed.*

Here, according to the common Method of extracting the Square Root, the Proceſs will ſtand as follows:

$$
\begin{array}{l}
2a+\frac{x^2}{a}-\frac{x^4}{4a^3}\Big)\, aa+xx\,\Big(\,a+\frac{x^2}{2a}-\frac{x^4}{8a^3}\ \&c.\\
\qquad\qquad\qquad aa\\
\qquad\qquad\qquad \overline{\qquad\quad +xx}\\
\qquad\qquad\qquad\quad +xx+\frac{x^4}{4a^2}\\
\qquad\qquad\qquad\quad \overline{\qquad\quad -\frac{x^4}{4a^2}}\\
\qquad\qquad\qquad\qquad -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}\\
\qquad\qquad\qquad\qquad \overline{\qquad\qquad +\frac{x^6}{8a^4}-\frac{x^8}{64a^6}\ \&c.\ \&c.}
\end{array}
$$

97. The Law of Continuation in Series, thus ariſing, from radical Quantities, is not eaſily diſcovered: But, if you would carry on the Series to any propoſed Number of Terms, the Work will be a good deal ſhortned, by dividing the Remainder by the Diviſor, when half that Number of Terms is found (as in common Diviſion) and obſerving, at the ſame time, to neglect all ſuch Terms whoſe Indices would exceed the greateſt, or the greateſt Plus the common Difference, in the ſaid Remainder, according as the whole Number of Terms propoſed to be found is odd, or even.

Thus, if it were propoſed to continue the foregoing Series $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}$ to 6 Terms, then the Diviſor (or double Quotient) being $2a+\frac{x^2}{a}-\frac{x^4}{4a^3}$, and the Remainder $\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$ (as appears from the laſt Article) the reſt of the Operation will ſtand thus:

$$
\begin{array}{l}
2a+\frac{x^2}{a}-\frac{x^4}{4a^3}\Big)\,\frac{x^6}{8a^4}-\frac{x^8}{64a^6}+0\Big(\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9}\\
\qquad\qquad\qquad\;\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}\\
\qquad\qquad\qquad\;\overline{\qquad\quad -\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}}\\
\qquad\qquad\qquad\qquad\quad -\frac{5x^8}{64a^6}-\frac{5x^{10}}{128a^8}\\
\qquad\qquad\qquad\qquad\quad \overline{\qquad\qquad\quad +\frac{7x^{10}}{128a^8}}
\end{array}
$$

Which three Terms thus found being added to thoſe found above, we have $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9}$, for the 6 firſt Terms of an infinite Series exhibiting the Value of $\sqrt{a^2+x^2}$.

98. Another Way of reſolving any radical Quantity, is to aſſume a Series (with unknown Coefficients) for the Value thereof; and then the Series ſo aſſumed being raiſed to the ſecond, third, or fourth Power, &c. according as the Root to be extracted is a ſquare, cubic, or biquadratic one, &c. an Equation will be obtained (free from Surds) from whence, by comparing the homologous Terms, the aſſumed Coefficients, and conſequently the Series ſought, will be determined; as in

EXAMPLE V.

Where it is propoſed to extract the Square Root of $a^{2n}+x^{2n}$ in an Infinite Series.

In which Caſe, aſſuming $A+Bx^{2n}+Cx^{4n}+Dx^{6n}+Ex^{8n}$ &c. for the required Series, and taking the Square thereof, we have

$$\left.\begin{array}{l} A^2+2ABx^{2n}+2ACx^{4n}+2ADx^{6n}+2AEx^{8n}\text{ \&c.} \\ \qquad\qquad\qquad +B^2x^{4n}\quad +2BCx^{6n}+2BDx^{8n}\text{ \&c.} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad +C^2x^{8n}\text{ \&c.} \end{array}\right\} = a^{2n}+x^{2n};$$

and conſequently

$$\left.\begin{array}{l} A^2\quad +2ABx^{2n}+2ACx^{4n}+2ADx^{6n}+2AEx^{8n}\text{ \&c.} \\ -a^{2n}-\quad x^{2n}+\quad B^2x^{4n}+2BCx^{6n}+2BDx^{8n}\text{ \&c.} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad +\ C^2x^{8n}\text{ \&c.} \end{array}\right\} = 0$$

Therefore $A^2-a^{2n}=0$, $2AB-1=0$, $2AC+B^2=0$, $2AD+2BC=0$, $2AE+2BD+C^2=0$, * &c. From which we get $A=a^n$; $B\left(=\frac{1}{2A}\right)=\frac{1}{2a^n}$; $C\left(=-\frac{B^2}{2A}\right)=-\frac{1}{8a^{3n}}$; $D\left(=-\frac{BC}{A}\right)=\frac{1}{16a^{5n}}$; $E\left(=-\frac{2BD+C^2}{2A}\right)=-\frac{5}{128a^{7n}}$ &c. whence we have

$A+Bx^{2n}+Cx^{4n}+Dx^{6n}$ &c. $\left(=\sqrt{a^{2n}+x^{2n}}\right)=a^n$

---

* *Vid. p.* 181 *of my Treatiſe of Algebra.*

+

$+ \frac{x^{2n}}{2a^{n}} - \frac{x^{4n}}{8a^{3n}} + \frac{x^{6n}}{16a^{5n}} - \frac{5x^{8n}}{128a^{7n}}$ &c. Which Series, if *n* be expounded by Unity, will become $a + \frac{x^2}{2a} - \frac{x^4}{8a^3}$ &c. the very ſame with that in the preceding Article found by the common Method.

EXAMPLE VI.

99. *Let it be required to reſolve* $\overline{a+bx^n}|^{\frac{1}{3}}$ *into an Infinite Series.*

Here, by aſſuming $A + Bx^n + Cx^{2n} + Dx^{3n}$ &c. and cubing the ſame, &c. we have

$$\left.\begin{array}{l} A^3 + 3A^2Bx^n + 3A^2Cx^{2n} + 3A^2Dx^{3n} + \&c. \\ -a - bx^n \quad + 3AB^2x^{2n} + 6ABCx^{3n} + \&c. \\ \qquad\qquad\qquad\qquad + \quad B^3x^{3n} + \&c. \end{array}\right\} = 0$$

Therefore $A = a^{\frac{1}{3}}$; $B\left(= \frac{b}{3A^2}\right) = \frac{b}{3a^{\frac{2}{3}}}$; $C\left(= -\frac{B^2}{A}\right) = -\frac{b^2}{9a^{\frac{5}{3}}}$; $D\left(= -\frac{6ABC + B^3}{3A^2}\right) = \frac{5b^3}{81a^{\frac{8}{3}}}$ &c.

and conſequently, $\overline{a+bx^n}|^{\frac{1}{3}}$ $(= A + Bx^n + Cx^{2n} + \&c.)$ $= a^{\frac{1}{3}} + \frac{bx^n}{3a^{\frac{2}{3}}} - \frac{b^2x^{2n}}{9a^{\frac{5}{3}}} + \frac{5b^3x^{3n}}{81a^{\frac{8}{3}}} + \&c.$

And, in the ſame Manner, may the Root of any other Quantity be extracted: But as the celebrated Binomial Theorem, diſcovered by the illuſtrious Sir *Iſaac Newton*, is vaſtly more eaſy and expeditious, in raiſing Powers and extracting Roots than that, or any other, Method, I ſhall now explain the Uſes thereof; but, firſt of all, it may not be amiſs to ſhew how the Theorem itſelf, from the Principles of Fluxions, may be derived.

Let, then, $1+y$ be a Binomial whoſe firſt Term is Unity, and its ſecond Term any propoſed Quantity y; and let the Quantity to be expanded or thrown into a Series be $\overline{1+y}|^{v}$; where the Exponent v is ſuppoſed to denote any Number whatever, whole or broken, poſitive or negative.

Now it is evident that the firſt Term of the required Series muſt be Unity; becauſe when y is $=0$, the other Terms all vaniſh; and, in that Caſe, $\overline{1+y}|^{v}$ is equal to Unity. Let, therefore, $1+Ay^{m}+By^{n}+Cy^{p}+Dy^{q}$ &c. be aſſumed to expreſs the true Value of the ſaid Series; or, which is the ſame, let

$\overline{1+y}|^{v}=1+Ay^{m}+By^{n}+Cy^{p}+Dy^{q}$ &c. where A, B, C, D, &c. m, n, p, q, &c. denote unknown, but determinate Quantities:

Then, by taking the Fluxion of the whole Equation, (ſuppoſing y variable) we ſhall have $v\dot{y}\times\overline{1+y}|^{v-1}=m\dot{y}Ay^{m-1}+n\dot{y}By^{n-1}+p\dot{y}Cy^{p-1}+q\dot{y}Dy^{q-1}$ &c. Whence, multiplying the Sides of the two Equations, croſs-wiſe, and dividing by $\dot{y}\times\overline{1+y}|^{v-1}$, there comes out $\overline{1+y}\times\overline{mAy^{m-1}+nBy^{n-1}+pCy^{p-1}+qDy^{q-1}\text{ \&c.}}$ $=v+vAy^{m}+vBy^{n}+vCy^{p}+vDy^{q}$ &c. which, by Reduction, is

$$\left.\begin{array}{lllll} mAy^{m-1} & +nBy^{n-1} & +pCy^{p-1} & +qDy^{q-1} & \text{\&c.} \\ * & +mAy^{m} & +nBy^{n} & +pCy^{p} & \text{\&c.} \\ -v & -vAy^{m} & -vBy^{n} & -vCy^{p} & \text{\&c.} \end{array}\right\}=0$$

Now, ſince we are at Liberty to take the Exponents of y what we will, ſo as to anſwer the Conditions of the Equation, or ſo that all the Terms here put down may mutually deſtroy each other; let them, therefore, be ſo taken that the Terms themſelves may be homologous, that is, let $m-1=0$, $n-1=m$, $p-1=n$, $q-1=p$, *&c.* Then, m being $=1$, $n=2$, $p=3$, $q=4$, *&c.* if theſe ſeveral Values be ſubſtituted above, the Equation itſelf will become

$$\left.\begin{array}{l} A+2By+3Cy^2+4Dy^3+ \&c. \\ {*}+Ay+2By^2+3Cy^3 \quad \&c. \\ -v-vAy-vBy^2-vCy^3 \quad \&c. \end{array}\right\}=0$$

Where, taking $A-v=0$, $2B+A-vA=0$, $3C+2B-vB=0$, $4D+3C-vC=0$, *&c.* ſo that every Column of homologous Terms (and, conſequently, the whole Expreſſion) may vaniſh, we alſo get $A=v$; $B\ (=\frac{vA-A}{2}=\frac{A\times\overline{v-1}}{2})=\frac{v\times\overline{v-1}}{2}$; $C\ (=\frac{vB-2B}{3}=B\times\frac{\overline{v-2}}{3})=v\times\frac{v-1}{2}\times\frac{v-2}{3}$; $D\ (=\frac{vC-3C}{4}=C\times\frac{\overline{v-3}}{4})=v\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}$, *&c. &c.*

Whence, by writing theſe Values, with thoſe of m, n, p, q, *&c.* in the Series $1+Ay^m+By^n+Cy^p$ *&c.* firſt aſſumed, we, at length, find $\overline{1+y}^{\,v}=1+vy+\frac{v}{1}\times\frac{v-1}{2}\times y^2+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times y^3+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}\times y^4+$ *&c.* which was to be inveſtigated.

From the Series here brought out, any Power or Root, of any other compound Quantity, whether Binomial, Trinomial, *&c.* is eaſily deduced: For, if $p\times$ be put to repreſent the firſt Term of any ſuch Quantity, and Q the Quotient of the reſt of the Terms divided by the firſt; then the Quantity itſelf will be expreſſed by $P+PQ$ or $P\times\overline{1+Q}$, and the v Power thereof by $P^{v}\times\overline{1+Q}|^{v}$ which therefore is equal to

$$P^{v}\times\overline{1+vQ+\frac{v}{1}\times\frac{v-1}{2}\times Q^{2}+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times Q^{3}+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}\times Q^{4}+\&c.},$$

by what is juſt now determined.

But when v is a Fraction, as in the Notation of Roots, the Theorem here given will be render'd ſomewhat more commodious for Practice, if, inſtead of v, a Fraction as $\frac{m}{n}$ be ſubſtituted; by which means it will become

$$P^{\frac{m}{n}}\times\overline{1+Q}|^{\frac{m}{n}}=P^{\frac{m}{n}}\times\overline{1+\frac{m}{n}Q+\frac{m}{n}\times\frac{m-n}{2n}Q^{2}+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}Q^{3}+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}\times\frac{m-3n}{4n}Q^{4}+\&c.}$$

whoſe Uſe, in converting radical Quantities into Infinite Serieſes will appear from the following Examples.

EXAMPLE VII.

100. *Wherein it is propoſed to extract the Square Root of $a^{2}+x^{2}$, in an Infinite Series.*

Here the Quantity to be expanded being $\overline{a^{2}+x^{2}}|^{\frac{1}{2}}$, or $\overline{aa}|^{\frac{1}{2}}\times\overline{1+\frac{xx}{aa}}|^{\frac{1}{2}}$, by comparing it with the general Form, $P^{\frac{m}{n}}\times\overline{1+Q}|^{\frac{m}{n}}$, we have $P=a^{2}$, $Q=\frac{x^{2}}{a^{2}}$, $m=1$, and $n=2$: Whence, by ſubſtituting theſe Values in the laſt general Equation, we get

$$\overline{a^2+x^2}|^{\frac{1}{2}} = a \times \overline{1 + \frac{1}{2} \times \frac{x^2}{a^2} + \frac{1}{2} \times -\frac{1}{4} \times \frac{x^4}{a^4} + \frac{1}{2} \times -\frac{1}{4} \times -\frac{3}{6} \times \frac{x^6}{a^6} + \frac{1}{2} \times -\frac{1}{4} \times -\frac{3}{6} \times -\frac{5}{8} \times \frac{x^8}{a^8} + \&c.} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}\ \&c.$$

Which Series agrees exactly with thoſe found in *Art.* 97. and 98. by different Methods.

EXAMPLE VIII.

101. *Let it be required to extract the Cube-Root of b^3-y^3, in an Infinite Series.*

Here by comparing $\overline{b^3}|^{\frac{1}{3}} \times \overline{1-\frac{y^3}{b^3}}|^{\frac{1}{3}}$ $\left(= \overline{b^3-y^3}|^{\frac{1}{3}}\right)$ with $P^{\frac{m}{n}} \times \overline{1+Q}|^{\frac{m}{n}}$, it will be $P=b^3$, $Q=-\frac{y^3}{b^3}$, $m=1$, and $n=3$: Therefore, by Subſtitution, we get

$$\overline{b^3-y^3}|^{\frac{1}{3}} \left(= b \times \overline{1-\frac{y^3}{b^3}}|^{\frac{1}{3}}\right) = b \times \overline{1 + \frac{1}{3} \times -\frac{y^3}{b^3} + \frac{1}{3} \times -\frac{2}{6} \times \frac{y^6}{b^6} + \frac{1}{3} \times -\frac{2}{6} \times -\frac{5}{9} \times -\frac{y^9}{b^9} + \frac{1}{3} \times -\frac{2}{6} \times -\frac{5}{9} \times -\frac{8}{12} \times \frac{y^{12}}{b^{12}} + \&c.} = b - \frac{y^3}{3b^2} - \frac{y^6}{9b^5} - \frac{5y^9}{81b^8} - \frac{10y^{12}}{243b^{11}}$$

&c.

EXAMPLE IX.

102. *Let the Quantity to be converted into an Infinite Series be* $\frac{a}{\sqrt{ax-xx}}$.

In this Caſe the given Quantity being firſt transformed to $\sqrt{\frac{a}{x}} \times \overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$ and $\overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$ afterwards compared with $\overline{1+Q}|^{\frac{m}{n}}$, we have $Q = -\frac{x}{a}$, $m = -1$, and $n = 2$; and therefore $\overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$ $(= \overline{1+Q}|^{\frac{m}{n}} = 1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-2n}{2n}Q^2 + \text{\&c.})$ $1 + {-\frac{1}{2}} \times \frac{-x}{a} + {-\frac{1}{2}} \times {-\frac{3}{4}} \times \frac{x^2}{a^2} + {-\frac{1}{2}} \times {-\frac{3}{4}} \times {-\frac{5}{6}} \times \frac{-x^3}{a^3}$ &c. $= 1 + \frac{x}{2a} + \frac{3x^2}{8a^2} + \frac{5x^3}{16a^3} + \frac{35x^4}{128a^4} +$ &c. Which therefore, multiplied by $\sqrt{\frac{a}{x}}$, gives $\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}}}{2a^{\frac{1}{2}}} + \frac{3x^{\frac{3}{2}}}{8a^{\frac{3}{2}}} + \frac{5x^{\frac{5}{2}}}{16a^{\frac{5}{2}}} + \frac{35x^{\frac{7}{2}}}{128a^{\frac{7}{2}}} +$ &c. $= \frac{a}{\sqrt{ax-xx}}$, the Quantity propoſed.

103. It may not be improper to obſerve here, that, when both the Terms of the propoſed Quantity are affirmative, and its Exponent alſo affirmative and leſs than Unity, the two firſt Terms of the equal Series will be poſitive, and the reſt negative and poſitive, alternately; but if only the firſt Term of the Binomial be affirmative, all the Terms of the Series, after the firſt, will be negative: Moreover, if the Exponent of

the given Quantity be negative, and both the Terms affirmative, the Signs will change alternately; but if only the firſt be affirmative, all the Terms of the equal Series will be poſitive.

EXAMPLE X.

104. *Let the Quantity propoſed be the Trinomial*

$$\overline{x^3+2x^4+3x^5}|^{\frac{1}{3}}.$$

Here, by dividing the reſt of the Terms by the firſt, &c. our given Quantity is reduced to $\overline{x^3}|^{\frac{1}{3}} \times \overline{1+2x+3x^2}|^{\frac{1}{3}}$. Therefore, in this Caſe $P=x^3$, $Q=2x+3x^2$, $m=1$, and $n=3$: Whence (by Subſtitution) $\overline{x^3+2x^4+3x^5}|^{\frac{1}{3}} = x \times 1 + \frac{1}{3} \times \overline{2x+3x^2}| + \frac{1}{3} \times -\frac{2}{6} \times \overline{2x+3x^2}|^2 + \frac{1}{3} \times -\frac{2}{6} \times -\frac{5}{9} \times \overline{2x+3x^2}|^3$ &c. $= x \times 1 + \frac{2x+3x^2}{3} - \frac{\overline{2x+3x^2}|^2}{9} + \frac{5 \times \overline{2x+3x}|^3}{81}$ &c.

Which, reduced to ſimple Terms, is $= x + \frac{2x^2}{3} + \frac{5x^3}{9} - \frac{68x^4}{81}$ &c.

105. When the propoſed Expreſſion conſiſts of a rational, multiply'd by an irrational, Quantity, the Series anſwering to the irrational one muſt be firſt found, and afterwards multiply'd by the rational Quantity: But, if two, or more, compound irrational Quantities are to be drawn into each other, then take the Series anſwering to each Quantity, ſeparately, and multiply them together; obſerving, always, to neglect all ſuch Terms whoſe Indices would exceed that of the laſt, or higheſt,

Term, which the Series ſought is propoſed to be continued to.

EXAMPLE XI.

106. *Let the Quantity propoſed be* $\overline{1+x} \times \overline{1-x}|^{\frac{1}{10}}$

Firſt we have $\overline{1-x}|^{\frac{1}{10}} = 1 - \frac{x}{10} - \frac{9x^2}{10 \times 20} - \frac{9 \times 19x^3}{10 \times 20 \times 30} - \frac{9 \times 19 \times 29x^4}{10 \times 20 \times 30 \times 40}$ — &c. Which, multiply'd by $1+x$, produces $\overline{1+x} \times \overline{1-x}|^{\frac{1}{10}} = 1 + \frac{9x}{10} - \frac{29x^2}{10.20} - \frac{9.49x^3}{10.20.30} - \frac{9.19.69x^4}{10.20.30.40}$ &c. $= 1 + \frac{9x}{10} - \frac{29x^2}{200} - \frac{147x^3}{2000} - \frac{3933x^4}{80000}$ — &c.

EXAMPLE XII.

107. *Where the Quantity to be expreſſed in an Infinite Series is* $\frac{\overline{a^2-x^2}|^{\frac{1}{2}}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$, or $\overline{a^2-x^2}|^{\frac{1}{2}} \times \overline{c^2-x^2}|^{-\frac{1}{2}}$.

Here we have, $\overline{a^2-x^2}|^{\frac{1}{2}}$ $(a \times \overline{1-\frac{xx}{aa}}|^{\frac{1}{2}} = a \times$ $\overline{1 + \frac{1}{2} \times -\frac{x^2}{a^2} + \frac{1}{2} \times -\frac{1}{4} \times \frac{x^4}{a^4} + \frac{1}{2} \times -\frac{1}{4} \times -\frac{3}{6} \times -\frac{x^6}{a^6}}$ $+$ &c. $= a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$ &c.

And $\overline{c^2 - x^2}|^{-\frac{1}{2}}$ $(= c^{-1} \times \overline{1 - \frac{xx}{cc}}|^{-\frac{1}{2}} = c^{-1} \times$ $\overline{1 + -\frac{1}{2} \times -\frac{x^2}{c^2} + -\frac{1}{2} \times -\frac{3}{4} \times \frac{x^4}{c^4} + \&c.} = \frac{1}{c} +$ $\frac{x^2}{2c^3} + \frac{3x^4}{8c^5} + \frac{5x^6}{16c^7}$ &c. Whence, multiplying theſe two Values, one by the other, we get $\frac{a}{c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times x^2 + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times x^4 +$ $\overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times x^6 + \&c.$ for the four firſt Terms of the Series ſought.

EXAMPLE XIII.

108. *Let the Quantity to be expanded be the Multinomial, or infinite Series,* $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}|^v$; *whoſe Exponent v denotes any Number whatever, whole or broken, poſitive or negative.*

Here, dividing by the firſt Term, the given Quantity is transformed to $x^{pv} \times \overline{1 + ax^n + bx^{2n} + cx^{3n} + dx^{4n} + \&c.}|^v$; which, if $ax^n + bx^{2n} + cx^{3n}$ &c. be put $=y$, will become $x^{pv} \times \overline{1+y}|^v$; which laſt Expreſſion (by *Art.* 99.) is $=$ $x^{pv} \times \overline{1 + vy + \frac{v}{1} \times \frac{v-1}{2} \times y^2 + \frac{v}{1} \times \frac{v-1}{2} \times \frac{v-2}{3} \times y^3 + \&c.}$ Whence (for Brevity ſake) putting $A = v$, $B = \frac{v}{1} \times \frac{v-1}{2}$, $C = \frac{v}{1} \times \frac{v-1}{2} \times \frac{v-2}{3}$, $D = \frac{v}{1} \times$

$\frac{v-1}{2} \times \frac{v-2}{3} \times \frac{v-3}{4}$, &c. and ſubſtituting for y, there comes out $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}\Big|^{v} =$ $x^{pv} + Aax^{pv+n} + \overline{Ab + Ba^2} \times x^{pv+2n} +$ $\overline{Ac + 2Bab + Ca^3} \times x^{pv+3n} + \overline{Ad + 2Bac + Bb^2 + 3Ca^2b + Da^4}$ $\times x^{pv+4n} + \overline{Ae + 2Bad + 2Bbc + 3Ca^2c + 3Cab^2 + 4Da^3b}$ $\overline{+ Ea^5} \times x^{pv+5n} + \&c.$

EXAMPLE XIV.

109. *To extract the Square Root of $a^2 - x^2$, and from thence to determine the Fluent of $\dot{x}\sqrt{a^2 - x^2}$, in an Infinite Series.*

By proceeding as in the foregoing Examples, the Value of $\sqrt{a^2 - x^2}$ in an Infinite Series will be found to be $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \&c$, Which multiplied by $\dot{x}$ gives $\dot{x}\sqrt{a^2 - x^2} = a\dot{x} - \frac{x^2\dot{x}}{2a} - \frac{x^4\dot{x}}{8a^3} - \frac{x^6\dot{x}}{16a^5} - \frac{5x^8\dot{x}}{128a^7}$ &c. Whoſe Fluent therefore (by *Art.* 77.) is $= ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5}\ \frac{5x}{1152a^7} - \&c.$ Which was to be determined.

EXAMPLE XV.

110. *Let it be required to approximate the Fluent of*

$$\frac{\overline{a^2-x^2}|^{\frac{1}{2}}\times x^n\dot{x}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$$ *in an Infinite Series.*

It appears, from Example 12, that the Value of $\frac{\overline{a^2-x^2}|^{\frac{1}{2}}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$, expreſſed in a Series, is $\frac{a}{c}+\overline{\frac{a}{2c^3}-\frac{1}{2ac}}\times x^2+\overline{\frac{3a}{8c^5}-\frac{1}{4ac^3}-\frac{1}{8a^3c}}\times x^4+\overline{\frac{5a}{16c^7}-\frac{3}{16ac^5}-\frac{1}{16a^3c^3}-\frac{1}{16a^5c}}\times x^6+\&c.$ Which Value being therefore multiplied by $x^n\dot{x}$, and the Fluent taken (by the common Method) we get $\frac{ax^{n+1}}{\overline{n+1}\times c}+\overline{\frac{a}{2c^3}-\frac{1}{2ac}}\times\frac{x^{n+3}}{n+3}+\overline{\frac{3a}{8c^5}-\frac{1}{4ac^3}-\frac{1}{8a^3c}}\times\frac{x^{n+5}}{n+5}+\overline{\frac{5a}{16c^7}-\frac{3}{16ac^5}-\frac{1}{16a^3c^3}-\frac{1}{16a^5c}}\times\frac{x^{n+7}}{n+7}+\&c.$

EXAMPLE XVI.

111. *Wherein it is propoſed to approximate the Fluent of* $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}|^v \times x^{m-1}\dot{x}$ *in a Series.*

Here, if A be put $= v$, $B = v \times \frac{v-1}{2}$, $C = v \times \frac{v-1}{2} \times \frac{v-2}{3}$, $D = v \times \frac{v-1}{2} \times \frac{v-2}{3} \times \frac{v-3}{4}$, &c. the Quantity $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n}\ \&c.}|^v$ expanded, will be $= x^{pv} + Aax^{pv+n} + \overline{Ab + Ba^2} + x^{pv+2n} + \overline{Ac + 2Bab + Ca^3} \times x^{pv+3n} + \overline{Ad + 2Bac + Bb^2 + 3Ca^2b + Da^4} \times x^{pv+4n} + \&c.$ as appears from *Art.* 108. Therefore this Expreſſion being multiplied by $x^{m-1}\dot{x}$, and the Fluent taken (as uſual) we ſhall have

$$\frac{x^{pv+m}}{pv+m} + \frac{Aax^{pv+m+n}}{pv+m+n} + \frac{\overline{Ab+Ba^2} \times x^{pv+m+2n}}{pv+m+2n} + \frac{\overline{Ac + 2Bab + Ca^3} \times x^{pv+m+3n}}{pv+m+3n} + \frac{\overline{Ad + 2Bac + Bb^2 + 3Ca^2b + Da^4} \times x^{pv+m+4n}}{pv+m+4n} + \&c.$$

for the Quantity propoſed to be found.

SEC.

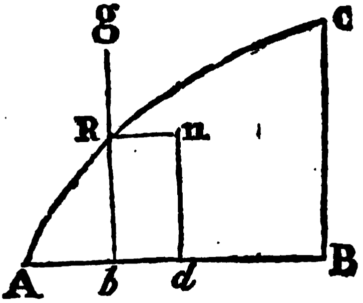
SECTION VII.

Of the Uſe of Fluxions in finding the Areas of Curves.

CASE I.

112. L*ET* ARC *be a Curve of any Kind whoſe Ordinates are perpendicular to an Axis* AB.

Imagine a Right-line bRg (perpendicular to AB) to move parallel to itſelf from A towards B; and let the Celerity thereof, or the Fluxion of the Abſciſſa Ab, in any propoſed Poſition of that Line, be denoted by bd: Then it will appear, *from Art.* 4. that the Rectangle (bn) under bd and the Ordinate bR, will expreſs the correſponding Fluxion of the generated Area abR: Which Fluxion, if $Ab=x$, and $bR=y$, will therefore be $=y\dot{x}$: From whence, by ſubſtituting for y or $\dot{x}$ (according to the Equation of the Curve) and taking the Fluent, the Area itſelf will become known.

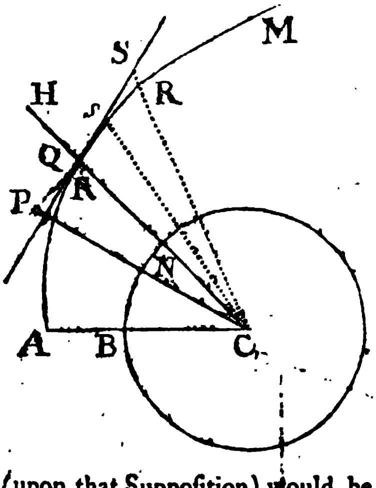


CASE II.

113. *Let* ARM *be any Curve whoſe Ordinates* CR, CR *are all referred to a Point or Center.*

Conceive a Right-line CRH to revolve about the given Center C, and let a Point R move along the ſaid Line, ſo as to trace out, or deſcribe the propoſed Curve Line ARM.

Now it is evident, that, if the Point R was to move from any Poſition Q, without changing its Direction and Velocity, it would proceed along the Tangent QS (inſtead of the Curve) and deſcribe Areas Q*s*C, QSC about the Center C, proportional to the Times of their Deſcription; becauſe thoſe Areas, or Triangles, having the ſame Altitude (CP), are as the Baſes Q*s* and QS, and theſe are as the Times, becauſe the Motion in the Tangent (upon that Suppoſition) would be uniform.



Hence, if RS be taken to denote the Value of ($\dot{x}$) the Fluxion of the Curve Line AR, the correſponding Fluxion of the Area ARC, will be truly repreſented by the, uniformly generated, Triangle QCS *: Which, putting the Perpendicular (CP) drawn from the Center to the Tangent, $= s$, will therefore be $\left(= \frac{QS \times CP}{2}\right) = \frac{s\dot{x}}{2}$; from whence the Area itſelf may be determined.

But, ſince in many Caſes, the Value of $\dot{x}$ cannot be computed (from the Property of the Curve) without ſome Trouble, the two following Expreſſions, for the Fluxion of the Area, will commonly be found more commodious, *viz.* $\frac{sy\dot{y}}{2t}$ and $\frac{y^2\dot{x}}{2a}$; where $t = RP$ and $x =$ the Arch BN of a Circle, deſcribed about the Center C, at any

* Art. 2 and 5.

any Diſtance a ($=$ CB). Theſe Expreſſions are derived from that above, in the following Manner; *viz.* $\dot z : \dot y :: y$ (CR) $: t$ (RP) *; therefore $\dot z = \frac{y\dot y}{t}$; and *Art. 35.
conſequently $\frac{s\dot z}{2} = \frac{sy\dot y}{2t}$; which is the firſt Expreſſion.

Again, becauſe the Celerity of R in the Direction of the Tangent is denoted by $\dot z$, that in a Direction perpendicular to CQ (whereby the Point R revolves about the Center C) will therefore be ($= \frac{\mathrm{CP}}{\mathrm{CR}} \times \dot z$) * $=$ *Art. 35.
$\frac{s\dot z}{y}$; which being to ($\dot x$) the Celerity of the Point N (about the ſame Center) as the Diſtance (or Radius) CR (y) to the Radius CN (a) we ſhall, by multiplying Extremes and Means, have $\frac{as\dot z}{y} = y\dot x$; and conſequently $\frac{s\dot z}{2} = \frac{y^2\dot x}{2a}$; which is the other Expreſſion.

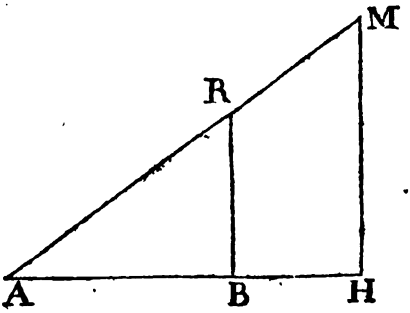
The Method of applying this, together with the preceding Forms, will appear at large from the following Examples: Wherein x, y, z, and u are all along put to denote the Abſciſſa, Ordinate, Curve-line, and the Area reſpectively, unleſs where the contrary is expreſsly ſpecified.

EXAMPLE I.

114. *Let it be propoſed to determine the Area of a right-angled Triangle* AHM.

Put the Baſe AH $= a$, the Perpendicular HM $= b$; and let AB (x) be any Portion of the Baſe, conſidered as a flowing Quantity, and let BR (y) be the Ordinate, or Perpendicular, correſponding:

Then, becauſe of the ſimilar Triangles AHM and ABR, it will be, $a : b :: x : y = \frac{bx}{a}$: Whence $y\dot{x}$



Art.112. (the Fluxion of the Area ABR) is, in this Caſe, $= \frac{bx\dot{x}}{a}$; and conſequently the Fluent thereof, or the Area

†Art.77. itſelf $= \frac{bx^2}{2a}$ †: Which therefore, when $x=a$, and BR coincides with HM, will become $\frac{ab}{2} = \frac{\text{AH}\times\text{HM}}{2} =$ the Area of the whole Triangle AHM; which we alſo know from other Principles.

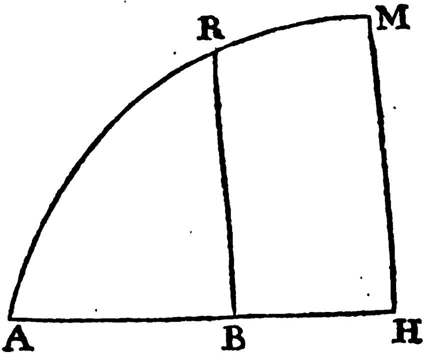
EXAMPLE II.

115. *Let the Curve* ARMH, *whoſe Area you would find, be the common Parabola.*

In which Caſe the Relation of AB (x) and BR (y) being expreſſed by $y^2=ax$ (where a is the Parameter)

‡Art.112. we thence get $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$; and therefore $\dot{u}$ $(= y\dot{x}$ ‡ $)$ $= a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$: Whence $u = \frac{2}{3} \times a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}\, a^{\frac{1}{2}}x^{\frac{1}{2}} \times x = \frac{2}{3}yx$

(becauſe $a^{\frac{1}{2}}x^{\frac{1}{2}} = y$) $= \frac{2}{3} \times AB \times BR$: Hence *a Parabola is* $\frac{2}{3}$ *of a Rectangle of the ſame Baſe and Altitude.*



The Area is here found in Terms of x; but it will, many times, be more eaſily brought out in Terms of y (without radical Quantities) as in the very Caſe laſt propoſed: Where x being $= \frac{y^2}{a}$, we therefore have $\dot{x} = \frac{2y\dot{y}}{a}$; and conſequently $\dot{z}\ (y\dot{x}) = \frac{2y^2\dot{y}}{a}$: Whence $z = \frac{2y^3}{3a} = \frac{2y}{3} \times \frac{y^2}{a} = \frac{2y}{3} \times x = \frac{2}{3} \times AB \times BR$; *the ſame as before.*

EXAMPLE III.

116. *Let* ARM *(ſee the preceding Figure) be a Parabola of any Kind; whereof the general Equation is* $y^{m+n} = a^m x^n$.

Therefore, by extracting the Root, or dividing each Exponent by $m+n$, we have $y = a^{\frac{m}{m+n}} \times x^{\frac{n}{m+n}}$; whence

$\dot{u}\,(y\dot{x}) = a^{\frac{m}{m+n}} \times \dot{x}x^{\frac{n}{m+n}}$; and conſequently u (the true Fluent, or Area) $= a^{\frac{m}{m+n}} \times \dfrac{x^{\frac{n}{m+n}+1}}{\frac{n}{m+n}+1} =$ $\dfrac{a^{\frac{m}{m+n}} \times x^{\frac{n}{m+n}} \times x \times \overline{m+n}}{m+2n} = \dfrac{m+n}{m+2n} \times yx = \dfrac{m+n}{m+2n} \times$ AB x BR.

No Notice has been yet taken of any conſtant Quantity to be added to, or ſubtracted from, the variable One, firſt found, in order to render it complete, agreeable to the Obſervation in *Art.* 78.

But that no ſuch Correction is required in any of the preceding Examples, is evident from the Nature of the Figure; becauſe, when x and y are nothing, the Area (u) ought alſo to be nothing, which it actually is according to the Equations above exhibited.

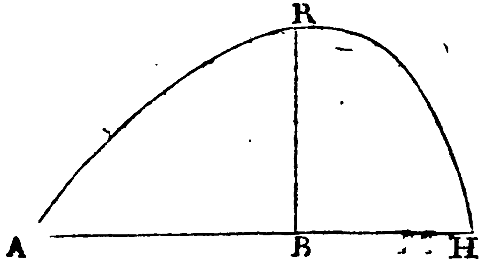
The Fluent found in the ſucceeding Example, will, however, ſtand in need of a Correction.

EXAMPLE IV.

117. *Where it is propoſed to find the Area of the Curve* ARH, *whoſe Equation is* $x^4 - a^2x^2 + a^2y^2 = 0$.

Here, the given Equation is reduced to $y = \dfrac{x \times \overline{a^2 - x^2}|^{\frac{1}{2}}}{a}$; whence $\dot{u}\ (= y\dot{x}) = \dfrac{\overline{a^2 - x^2}|^{\frac{1}{2}} \times x\dot{x}}{a}$:

*Art. 77. Whereof the Fluent (by the common Rule *) is —



$\frac{\overline{a^2-x^2}\big|^{\frac{3}{2}}}{3a}$: Which, when $x=0$ and $u=0$, becomes $-\frac{a^2}{3}$; this therefore ſubtracted from $-\frac{\overline{a^2-x^2}\big|^{\frac{3}{2}}}{3a}$, leaves $\frac{a^2}{3}-\frac{\overline{a^2-x^2}\big|^{\frac{3}{2}}}{3a}$ for the Fluent corrected, or the true Value of the Area ABR *.

When the Ordinate BR $\left(\frac{x\sqrt{a^2-x^2}}{a}\right)$ becomes equal to Nothing, and B coincides with H, then x will become $=a=$ AH; and therefore the Area of the whole Curve ARH will be barely $=\frac{a^2}{3}=\frac{1}{3}\text{AH}^2$

EXAMPLE V.

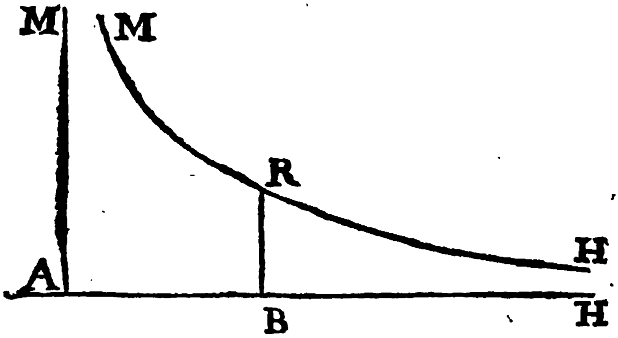
118. *Let it be required to determine the Area of the hyperbolical Curve whoſe Equation is* $x^m y^n = a^{m+n}$.

In this Caſe we have $y=\frac{a^{\frac{m+n}{n}}}{x^{\frac{m}{n}}}=a^{\frac{m+n}{n}}\times x^{\frac{-m}{n}}$;

---

* Art. 78.

and therefore $\dot{u}\ (=y\dot{x}) = a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}}\dot{x}$: Whoſe Fluent is $\frac{a^{\frac{m+n}{n}} \times x^{1-\frac{m}{n}}}{1-\frac{m}{n}} = \frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{n-m}$; which, when x is



$=0$, will alſo be $=0$, if n be greater than m: Therefore, the Fluent requires no Correction in this Caſe; the Area AMRB, included between the Aſymptote AM and the Ordinate BR, being truly defined by $\left(\frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{n-m}\right)$ the Quantity above determined. But, if n be leſs than m, then the Fluent, when $x=0$, will be infinite (becauſe the Index $\frac{n-m}{n}$ being negative, o becomes a Diviſor to na^{m+n}:) Whence the Area AMRB will alſo be infinite.

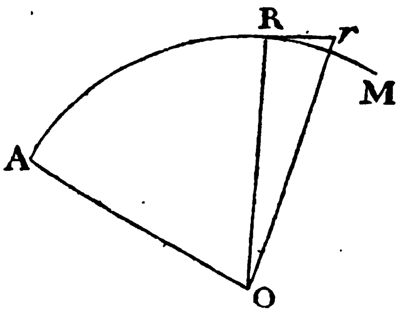
But, here, the Area BRH comprehended between the Ordinate, the Curve, and the Part BH of the other Aſymptote, is finite, and will be truly expounded by $\frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{m-n}$ the ſame Quantity with its Signs changed. For the

Fluxion of the Part AMRB being $a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}} \dot{x}$, that of its Supplement BRH muſt conſequently be $-a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}} \dot{x}$: Whereof the Fluent is $-\frac{a^{\frac{m+n}{n}} \times x^{1-\frac{m}{n}}}{1-\frac{m}{n}}$ $= \frac{a^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{m-n}$ = the Area BRH: Which wants no Correction; becauſe, when x is infinite, and the Area BRH $= 0$, the ſaid Fluent will alſo intirely vaniſh, ſeeing the Value of $x^{\frac{m-n}{n}}$ (which is a Diviſor to $a^{\frac{m+n}{n}}$) is then infinite.

EXAMPLE VI.

119. *Where let it be required to determine the Area of the circular Sector* AOR.

Then, putting the Radius AO (or OR) $= a$, the



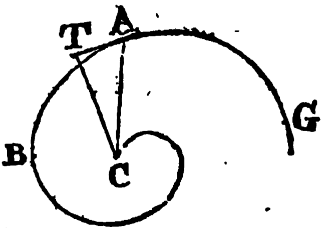
Arch AR (conſidered as variable by the Motion of R) $= x$, and Rr $= \dot{x}$, the Fluxion of the Area will here

*Art. 113. be expreſſed by $\frac{a\dot{z}}{2}$ (= the Triangle ORr *:) Whence the Area itſelf is $= \frac{az}{2} = AO \times \frac{1}{2}AR$: From which it appears that the Area of any Circle is expreſſed by a Rectangle under half the Circumference and half the Diameter.

EXAMPLE VII.

120. *Wherein it is propoſed to determine the Area* CBAG *of the logarithmic Spiral.*

Let the Right-line AT touch the Curve at A; upon which, from the Center C, let fall the Perpendicular CT: Then, ſince by the Nature of the Curve the



Angle TAC is every where the ſame, the Ratio of AT (t) to CT (s) will here be conſtant: And therefore the

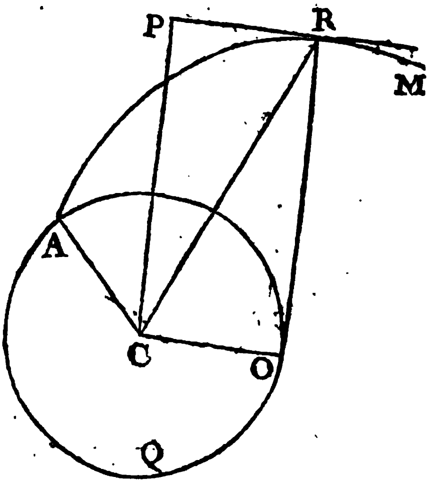
*Art. 113. Fluent of $\frac{s}{t} \times \frac{y\dot{y}}{2}$ * $= \frac{s}{t} \times \frac{y^2}{4}$ = the Area which was to be found.

EX.

EXAMPLE VIII.

121. *Let the Curve* ARM *be the Involute of a given Circle* AOQ.

In which Caſe the intercepted Part of the Tangent RP (t) being every where equal to the Radius CO (a)



of the generating Circle, we therefore have CP (s) $=$ $\sqrt{CR^2 - RP^2}) = \sqrt{y^2 - a^2}$: Whence $\dot{u}$ $\left(= \frac{sy\dot{y}}{2t}\right.$ * $\left.\right)$ *Art. 113.

$= \frac{\sqrt{y^2 - a^2} \times y\dot{y}}{2a}$; and conſequently $u = \frac{\overline{y^2 - a^2}|^{\frac{3}{2}}}{6a} =$ $\frac{CP^3}{6CA} =$ the required Area ACR:

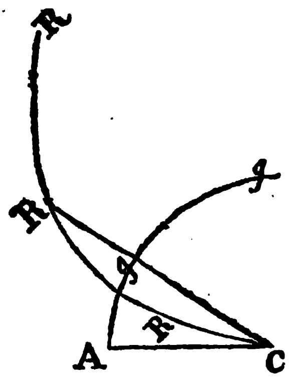
Which will alſo expreſs the Area ARO generated by the Radius of Evolution RO; becauſe, RO being = the

*Art.119. the Arch AO, the Sector ACO ($\frac{1}{2}$AO × OC *) is equal to the Triangle CRO ($\frac{1}{2}$RO × OC) which equal Quantities being ſucceſſively ſubtracted from CARO, there remains AOR=ACR.

EXAMPLE IX.

122. *Let the Curve* CRR, *whoſe Area* CRgC *you would find, be the Spiral of* Archimedes.

Let AC be a Tangent to the Curve at the Center



C, about which Center, with any Radius AC (=a) ſuppoſe a Circle Agg to be deſcribed; then the Arch (or Abſciſſa) Ag correſponding to any propoſed Ordinate CR, being to that Ordinate in a given, or conſtant, Ratio (ſuppoſe as m to n) we have x (Ag) = $\frac{my}{n}$; therefore $\dot{u} = \frac{y^2\dot{x}}{2a}$* $= \frac{my^2\dot{y}}{2an}$, and conſequently u
*Art.113. $= \frac{my^3}{6an}$ = the Area CRRgC.

EX.

EXAMPLE X.

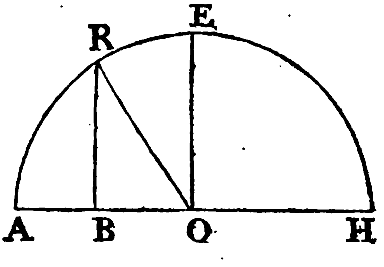
123. *Let the Equation of the Spiral* CRR (*ſee the laſt Figure*) *be* $x = by + cy^2 + dy^3 + ey^4 + fy^5 + \&c.$

Then, $\dot{x}$ being $= b\dot{y} + 2cy\dot{y} + 3dy^2\dot{y} + 4ey^3\dot{y} + \&c.$ we ſhall have $\dot{u}\ \left(= \frac{y^2\dot{x}}{2a}\right) = \frac{by^2\dot{y}}{2a} + \frac{2cy^3\dot{y}}{2a} + \frac{3dy^4\dot{y}}{2a} + \frac{4ey^5\dot{y}}{2a} + \&c.$ and therefore $u = \frac{by^3}{6a} + \frac{2cy^4}{8a} + \frac{3dy^5}{10a} + \frac{4ey^6}{12a}\ \&c.$ = the true Value of the Area in this Caſe.

EXAMPLE XI.

124. *Let it be propoſed to find the Area of a Semicircle* AREH.

Here, putting the Diameter $AH = a$, $AB = x$, and $BR = y$ &c. (*as uſual*) we have y^2 (BR^2) $= ax - xx$



($AB \times BH$), and conſequently $\dot{u}\ (y\dot{x}) = \dot{x}\sqrt{ax - xx} = a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} \times \overline{1 - \frac{x}{a}}\Big|^{\frac{1}{2}}$: Which Expreſſion not being of the Kind deſcribed in *Art.* 83 and 85. that admit of Fluents in finite Terms, let it therefore be reſolved into an Infinite Series * and you will have $\dot{u} = a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x} \times \overline{1 - \frac{x}{2a} - \frac{x^2}{8a^2} - \frac{x^3}{16a^3} - \frac{5x^4}{128a^4} \&c.} = a^{\frac{1}{2}} \times \overline{x^{\frac{1}{2}} \dot{x} - \frac{x^{\frac{3}{2}} \dot{x}}{2a} - \frac{x^{\frac{5}{2}} \dot{x}}{8a^2} - \frac{x^{\frac{7}{2}} \dot{x}}{16a^3} \&c.}$ From whence, the Fluent of every Term being taken, according to the common Method, there will come out $u = a^{\frac{1}{2}} \times \overline{\frac{2x^{\frac{3}{2}}}{3} - \frac{x^{\frac{5}{2}}}{5a} - \frac{x^{\frac{7}{2}}}{28a^2} - \frac{x^{\frac{9}{2}}}{72a^3} - \frac{5x^{\frac{11}{2}}}{704a^4} \&c.} = x\sqrt{ax} \times \overline{\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2} - \frac{x^3}{72a^3} - \frac{5x^4}{704a^4} - \&c.} =$ the Area ABR. Now, when, $x = \frac{1}{2} a$, the Ordinate BR will coincide with the Radius OE; in which Caſe the Area becomes $= \frac{1}{2} a\sqrt{\frac{1}{2}aa} \times \overline{\frac{2}{3} - \frac{1}{10} - \frac{1}{112} - \frac{1}{576} - \frac{1}{11264} \&c.} = \frac{a^2\sqrt{\frac{1}{2}}}{2} \times \overline{0{,}6666 - 0{,}1 - 0{,}0089 - 0{,}0017 - 0{,}0004 \&c.} = 0{,}1964a^2$; which, multiply'd by 2, gives $0{,}3928a^2$ for the Area of the Semi-circle AEH, nearly.

As the foregoing Series, in finding the Area of the whole Quadrant AOE, converges but ſlowly, a conſiderable Number of Terms ought therefore to be taken to have the Concluſion but tolerably exact, the five firſt Terms above collected being ſufficient to bring out no more than three Places of Figures that can be depended on. For which Reaſon it may be of Uſe to conſider, whether, by computing the Area of ſome particular Portion (ABR) of the ſaid Quadrant, that of the whole may not be deduced; where x being ſmall in com-

* Art. 90 and 99.

compariſon of *a*, the Series may have ſuch a Rate of Convergency, that a ſmaller Number of Terms will be ſufficient *.

Now, in order to this, it is well known that, if the Arch AR be taken $= \frac{1}{3}$ AE (or 30 Degrees) the Sine BR will be $= \frac{1}{2}$ AO; and conſequently AB $(x) = \text{AO} - \text{OB} = \text{AO} - \sqrt{\text{OR}^2 - \text{BR}^2}$; which, if the Radius AO be expounded by Unity, (to facilitate the Operation) will be $= 0{,}1339746$ very nearly: This therefore, with the Value of *a*, being ſubſtituted in the forementioned Series, $\sqrt{ax^3} \times \overline{\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2} - \text{\&c.}}$ we have $\overline{0{,}0693505} \times \overline{0{,}6666666 - 0{,}0133975 - 0{,}0001603 - 0{,}0000042 - \text{\&c.}} = 0{,}0693505 \times 0{,}6531046 = 0{,}0452931 =$ the Area ABR: Which added to the Area OBR $(= \text{OB} \times \frac{1}{2}\text{BR} = \sqrt{\frac{3}{4}} \times \frac{1}{4} = 0{,}2165063)$ gives 0,2617994, for the Area of the Sector AOR; the treble whereof, or 0,7853982 (becauſe AR $= \frac{1}{3}$AE) will therefore be the Content of the whole Quadrant AOE: Which Number, found by taking four Terms of the Series only, is true to the laſt Decimal Place.

This Concluſion may be otherwiſe brought out, by finding a Series for the other Part of the Area, included between the Radius OE and the Ordinate BR; wherein the Co-ſine OB (inſtead of the verſed Sine AB) will be the converging (or variable) Quantity.

For, putting OB $= x$, and OR (OA) $= b$, we have y $(\text{BR} = \sqrt{\text{OR}^2 - \text{OB}^2} = \overline{b^2 - x^2}|^{\frac{1}{2}}$; and conſequently $(y\dot{x})$ the Fluxion of the Area OBRE * $= \dot{x} \times \overline{b^2 - x^2}|^{\frac{1}{2}} = b\dot{x} - \frac{x^2\dot{x}}{2b} - \frac{x^4\dot{x}}{8b^3} - \frac{x^6\dot{x}}{16b^5} - \frac{5x^8\dot{x}}{128b^7} - \frac{7x^{10}\dot{x}}{256b^9}$ &c. Whence the Area itſelf is $= bx - \frac{x^3}{6b} - \frac{x^5}{40b^3} - \frac{x^7}{112b^5} - \frac{5x^9}{1152b^7} - \frac{7x^{11}}{2816b^9}$ &c.

* Art. 92.

* Art. 112.

Now, if x (OB) be aſſumed $= \frac{1}{2}$ AO (ſo that the Arch ER may be $= \frac{1}{3}$ AE) and the Value of b (AO) be expounded by Unity, we ſhall have

$$x^3 \quad \left(= x \times x^2 = ,5 \times \tfrac{1}{4} = \frac{,5}{4}\right) = ,125$$

$$x^5 \quad \left(= x^3 \times x^2 = \frac{,125}{4}\right) = ,03125$$

$$x^7 \quad \left(= x^5 \times x^2 = \frac{,03125}{4}\right) = ,0078125$$

$$x^9 \quad \left(= x^7 \times x^2 = \frac{x^7}{4}\right) = ,0019531 +$$

$$x^{11} \quad \left(= x^9 \times x^2 = \frac{x^9}{4}\right) = ,0004883 -$$

&c.

Which Values of the Powers of x being reſpectively divided by 6, 40, 112, 1152, 2816, &c. there will reſult 0,5000000 — 0,0208333 — 0,0007812 — 0,0000698 — 0,0000085 — 0,0000012 — 0,0000002 &c. = 0,4783057, for the Area OBRE in the forementioned Circumſtance, when OB $= \frac{1}{2}$ OA: From which, deducting the Triangle OBR $\left(= \sqrt{\tfrac{3}{4}} \times \tfrac{1}{4} = 0,2165063\right)$ the Remainder ,2617994 will conſequently be the Area of the Sector EOR; the treble whereof (becauſe ER is, *here*, $= \frac{1}{3}$ AE) will give the Area of the whole Quadrant, 0,7853982; *as before.*

EXAMPLE XII.

125. *Let the Curve, whoſe Area you would find, be the Ciſſoid of Diocles; whereof the Equation is* $y^2 = \frac{x^3}{a - x}$.

*Art. 112. Here we have $\dot{u}$ ($y\dot{x}$ *) $= \frac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a - x}} = \frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}} \times \overline{1 - \frac{x}{a}}\big|^{\frac{1}{2}}}$

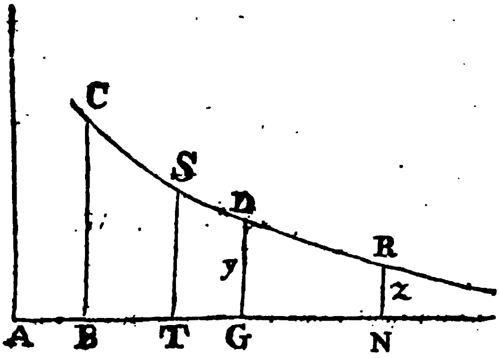
$= \frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}}} \times \overline{1-\frac{x}{a}}\Big|^{-\frac{1}{2}}$: Which being none of the Kind that admit of Fluents in finite Terms *, let it therefore be reſolved into an Infinite Series, and you will have $\dot{u} = \frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}}} \times \overline{1 + \frac{x}{2a} + \frac{3x^2}{8a^2} + \frac{5x^3}{16a^3} + \frac{35x^4}{128a^4} + \&c.} = \frac{1}{a^{\frac{1}{2}}} \times \overline{x^{\frac{3}{2}}\dot{x} + \frac{x^{\frac{5}{2}}\dot{x}}{2a} + \frac{3x^{\frac{7}{2}}\dot{x}}{8a^2} + \frac{5x^{\frac{9}{2}}\dot{x}}{16a^3} + \&c.}$ Whence u (the Area itſelf) will come out $= \frac{1}{a^{\frac{1}{2}}} \times \overline{\frac{2x^{\frac{5}{2}}}{5} + \frac{x^{\frac{7}{2}}}{7a} + \frac{x^{\frac{9}{2}}}{12a^2} + \frac{5x^{\frac{11}{2}}}{88a^3} + \&c.} = x^2\sqrt{\frac{x}{a}} \times \overline{\frac{2}{5} + \frac{x}{7a} + \frac{x^2}{12a^2} + \frac{5x^3}{88a^3} + \&c.}$

*Art. 83. and 85.

EXAMPLE XIII.

126. *Let the propoſed Curve* CSDR *be of ſuch a Nature, that (ſuppoſing* AB *Unity) the Sum of the Areas* CSTBC *and* CDGBC *anſwering to any two propoſed Abſciſſas* AT *and* AG, *ſhall be equal to the Area* CRNBC *whoſe correſponding Abſciſſa* AN *drawn into* AB *is equal to,* AT × AG, *the Product of the Meaſures of the two former Abſciſſas.*

Firſt, in order to determine the Equation of the Curve, (which muſt be known before the Area can be found) let the Ordinates GD and NR move parallel to themſelves towards HF; and, then, having put GD $= y$, NR $= z$, NR = z, AT = a, AG = s, and AN = u, the Fluxion of the Area CDGB will be repreſented by $y\dot{s}$, and that



of the Area CRNB by $z\dot{u}$ *: Which two Expreſſions muſt, by the Nature of the Problem, be equal to each other; becauſe the latter Area CRNB exceeds the former CDGB by the Area CSTB, which is here conſidered as a conſtant Quantity; and it is evident that two Expreſſions, that differ only by a conſtant Quantity, muſt always have equal Fluxions.

Since, therefore $y\dot{s}$ is $= z\dot{u}$, and $u = as$, *by Hypotheſis*, it follows that $\dot{u} = a\dot{s}$, and that the firſt Equation (by ſubſtituting for $\dot{u}$) will become $y\dot{s} = az\dot{s}$, or $y = az$, or laſtly $ys = zas$, that is, GD × AG = NR × AN: Therefore GD : NR :: AN : AG; whence it appears that every Ordinate of the Curve is reciprocally as its correſponding Abſciſſa.

Now, to find the Area of the Curve ſo determined, put BC = b, and BG = x; Then, ſince AG $(1 + x)$: AB (1) :: BC (b) : GD (y) we have $y = \frac{b}{1+x}$, and conſequently $\dot{u}\ (= y\dot{x}) = \frac{b\dot{x}}{1+x} = b \times \overline{\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} + x^4\dot{x}} -$ &c. Whence, BGDC, the Area itſelf

* Art. 113.

ſelf will be $= b \times x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ &c. Which was to be found.

It may here be obſerved that the Areas of the Spaces above mentioned, are analogous to, and have the very ſame Properties as *Logarithms*; and that thoſe Spaces, or Logarithms, may be of different Forms or Values, according as you take the Value of the firſt Ordinate BC, which may be aſſumed at Pleaſure: Thus, if BC be taken $=$ AB $=$ Unity, the Curve will become an equilateral Hyperbola whoſe Center is A (becauſe then AG $\times$ GD $=$ AB^2) and in that Caſe they are called hyperbolical Logarithms: But, if BC be taken $=0{,}43429448$ (ſo that the Logarithm, or the Area of the Space CDGB, anſwering to the Abſciſſa AG, when expreſſed by the Number 10, may be expounded by Unity, or AB^2) we ſhall then have the common, or *Brigean* Form of Logarithms.

From theſe Logarithms (given by the Tables) the Buſineſs of finding Fluents, is in many Caſes, very much facilitated: For, if the Fluxion given appears to agree with the Fluxion of any known Logarithmic Expreſſion, its Fluent may, it is evident, be had by the Tables, ready calculated, without the Trouble of an Infinite Series.

But, now to know what Kinds of Fluents are explicable by Means of Logarithms, it will be neceſſary to obſerve that, *the Fluxion of any hyperbolic Logarithm is always expreſſed by the Fluxion of the correſponding Number divided by that Number:* This appears from above, where $(y\dot{x})$ the Fluxion of the Area (or Logarithm) BGDC, when BC $=$ AB $= 1$, is truly repreſented by $\frac{\dot{x}}{1+x}$; where $1 + x$ $(=$ AG$)$ may ſtand for any Number whatever; and $\dot{x}$ for its Fluxion.

Hence the Fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ will be expreſſed by the hyperbolical Logarithm of $x + \sqrt{x^2 \pm a^2}$: For the Fluxion of $(x + \sqrt{x^2 \pm a^2})$ the Number itſelf, being $\dot{x} + \frac{x\dot{x}}{\sqrt{x^2 \pm a^2}} = \frac{\dot{x}\sqrt{x^2 \pm a^2} + x\dot{x}}{\sqrt{x^2 \pm a^2}} = \frac{\dot{x}}{\sqrt{x^2 \pm a^2}} \times \overline{\sqrt{x^2 \pm a^2} + x}$, this laſt Quantity, divided by that Number, gives $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$, the very Fluxion firſt propoſed.

It alſo appears that the Fluent of $\frac{\dot{x}}{\sqrt{2ax + x^2}}$ will be truly expounded by the hyperbolical Logarithm of $a + x + \sqrt{2ax + x^2}$: Becauſe the Fluxion of the Number $(a + x + \sqrt{2ax + x^2})$ is here $= \dot{x} + \frac{a\dot{x} + x\dot{x}}{\sqrt{2ax + xx}} = \frac{\dot{x}}{\sqrt{2ax + xx}} \times \overline{\sqrt{2ax + xx} + a + x}$; which divided by that Number produces $\frac{\dot{x}}{\sqrt{2ax + xx}}$.

Likewiſe the Fluent of $\frac{2a\dot{x}}{a^2 - x^2}$ will be repreſented by the hyperbolical Logarithm of $\frac{a + x}{a - x}$: Beeauſe, the Fluxion of $\frac{a + x}{a - x}$, being $\frac{\dot{x} \times \overline{a - x} + \dot{x} \times \overline{a + x}}{\overline{a - x}|^2} = \frac{2a\dot{x}}{\overline{a - x}|^2}$, if the ſame be therefore divided by $\frac{a + x}{a - x}$, we ſhall have $\frac{2a\dot{x}}{\overline{a - x}|^2} \times \frac{a - x}{a + x} = \frac{2a\dot{x}}{\overline{a - x} \times \overline{a + x}} = \frac{2a\dot{x}}{a^2 - x^2}$.

Laſtly, the Fluent of $\frac{2a\dot{x}}{x\sqrt{a^2\pm x^2}}$ will be denoted by the hyperbolical Logarithm of $\frac{a-\sqrt{a^2\pm x^2}}{a+\sqrt{a^2\pm x^2}}$; for here the Fluxion of the Number is $\frac{\mp x\dot{x}}{\sqrt{a^2\pm x^2}} \times \frac{\overline{a+\sqrt{a^2\pm x^2}}}{\overline{a+\sqrt{a^2\pm x^2}}|^2} \mp \frac{x\dot{x}}{\sqrt{a^2\pm x^2}} \times \frac{\overline{a-\sqrt{a^2\pm x^2}}}{\overline{a+\sqrt{a^2\pm x^2}}|^2} = \frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}|^2}$; which divided by $\frac{a-\sqrt{a^2\pm x^2}}{a+\sqrt{a^2\pm x^2}}$ gives $\frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}|^2} \times \frac{a+\sqrt{a^2\pm x^2}}{a-\sqrt{a^2\pm x^2}} = \frac{\pm 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}\times\overline{a-\sqrt{a^2\pm x^2}}} = \frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\mp x^2} = \frac{2a\dot{x}}{x\sqrt{a^2\pm x^2}}$, the Fluxion propoſed.

Theſe four are the principal Forms of Fluxions; whoſe Fluents may be found from a Table of Logarithms of the hyperbolic Kind: Which Table, upon Occaſion, may be eaſily ſupply'd by a Table of the common Form: For, ſince the hyperbolical Logarithm of any Number is to the common Logarithm of the ſame Number, in the conſtant Ratio of Unity to 0,43429448 (as appears from above) it follows that if any common Logarithm be, either, divided by 0,43429448, or multiply'd by its Reciprocal 2,30258509, you will thence obtain the hyperbolical Logarithm correſponding.

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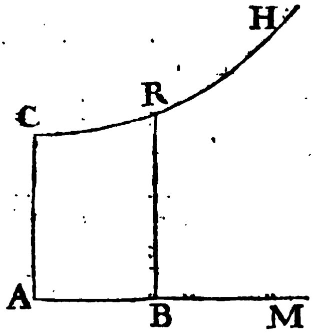
EXAMPLE XIV.

127. *Let it be required to determine the Area of the Curve; whoſe Equation is* $a^2y - x^2y - a^3 = 0$.

*Art. 112. In which Caſe y being $= \frac{a^3}{a^2 - x^2}$, we have $\dot{u}$ $(= y\dot{x})$ * $= \frac{a^3\dot{x}}{a^2 - x^2} = a\dot{x} + \frac{x^2\dot{x}}{a} + \frac{x^4\dot{x}}{a^3} + \frac{x^6\dot{x}}{a^5} + \frac{x^8\dot{x}}{a^7} +$ &c.



Whence $u = ax + \frac{x^3}{3a} + \frac{x^5}{5a^3} + \frac{x^7}{7a^5} + \frac{x^9}{9a^7} +$ &c. $=$ the Area ſought.

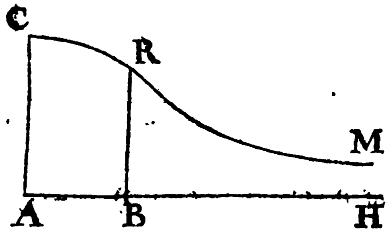
But the ſame Area (or Fluent) may be found without an Infinite Series, by Means of a Table of Logarithms, agreeable to the Obſervations in the laſt Article: For, ſince it there appears that the Fluent of $\frac{2a\dot{x}}{a^2 - x^2}$ is truly expreſſed by the hyperbolic Logarithm of $\frac{a+x}{a-x}$, it follows that that of $\frac{a^3\dot{x}}{a^2 - x^2}$ $\left(= \frac{2a\dot{x}}{a^2 - x^2} \times \frac{1}{2}a^2\right)$ will be expreſſed by the ſame Logarithm multiply'd by $\frac{1}{2}a^2$. Thus, for Example ſake, let a (=AC) be taken = 10, and x (= AB) = 5; then will $\frac{a+x}{a-x} = 3$; whoſe Logarithm taken from the common Tables is 0,47712135 which multiply'd by the *Modulus* 2,30258509 (ſee the laſt Article) gives 1,09861228 for the hyperbolical Logarithm of $\frac{a+x}{a-x}$; and this again multiply'd by 50 ($\frac{1}{2}a^2$) produces 54,930614 for the true Value of the Area ABRC, in the aforeſaid Circumſtance, when AC = 10, and AB = 5.

EXAMPLE XV.

128. *Where the propoſed Curve is that whoſe Equation is* $a^2y^2 + x^2y^2 = a^4$.

Here, by reducing the given Equation, we get $y = \frac{a^2}{\sqrt{a^2+x^2}}$: Therefore $y\dot{x} = \frac{a^2\dot{x}}{\sqrt{a^2+x^2}} =$ *. *Art. 11.

Whence, the Fluent of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ being = hyperb.



Log. of $x + \sqrt{a^2+x^2}$ (by *Art.* 126.) that of $\frac{a^2\dot{x}}{\sqrt{a^2+x^2}}$ will conſequently be = the ſame Logarithm multiply'd by a^2.

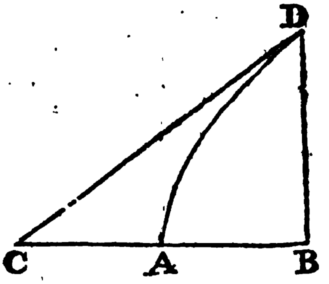
But to find whether the Fluent thus determined does not need a Correction ‡, let x be taken = 0; then the Fluent ‡Art. 72.

Fluent will become = hyp. Log. $a : \times a^2$: Which, therefore, muſt be ſubtracted, to have the true Value of the
* Art. 78. Area ACRB *; and then there reſults $a^2 \times$ hyp. Log. $x + \sqrt{a^2+x^2} - a^2 \times$ hyp. Log. $a = a^2 \times$ hyp. Log. $\frac{x+\sqrt{a^2+x^2}}{a} = u$.

EXAMPLE XVI.

129. *Let it be propoſed to find the Area of the Hyperbola* ABD, *and alſo the Area of the hyperbolical Sector* CAD; *ſuppoſing* C *to be the Center, and* A *the principal Vertex of the Curve.*

Here, putting the Semi-tranſverſe Axis CA $= a$, the Semi-conjugate $= c$, and CB $= x$; we have, by the



Property of the Curve, y (=BD) $= \frac{c}{a}\sqrt{xx - aa}$; and therefore $\dot{u} = y\dot{x} = \frac{c\dot{x}}{a}\sqrt{x^2-a^2}$ = the Fluxion
‡ Art. 112. of the Area ABD ‡

But to find the Fluxion of the Sector CAD, it is to be obſerved, that as the ſaid Sector is = CBD − ABD $= \frac{xy}{2} - u$, its Fluxion will therefore be =

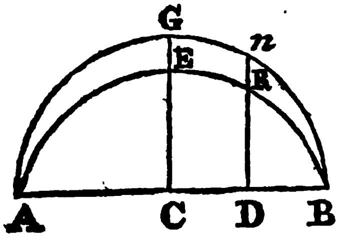
$\frac{x\dot y}{2}+\frac{y\dot x}{2}-\dot u=\frac{x\dot y}{2}-\frac{y\dot x}{2}$ (becauſe $\dot u = y\dot x$*) which, *Art. 112. by ſubſtituting for y and $\dot y$, their Equals $\frac{c}{a}\sqrt{x^2-a^2}$ and $\frac{cx\dot x}{a\sqrt{x^2-a^2}}$, is at length reduced to $\frac{ac}{2}\times\frac{\dot x}{\sqrt{x^2-a^2}}$: Whereof the Fluent (by Art. 126.) is $\frac{ac}{2}\times$ hyp. Log. $x+\sqrt{x^2-a^2}$; which corrected (by making $x=a$) will become $\frac{ac}{2}\times$ hyp. Log. $x+\sqrt{x^2-a^2}-\frac{ac}{2}\times$ hyp. Log. $a=\frac{ac}{2}\times$ hyp. Log. $\frac{x+\sqrt{x^2-a^2}}{a}$ = the Sector ADC: Which, ſubtracted from $\frac{cx\sqrt{x^2-a^2}}{2a}$ ($=\frac{BC\times BD}{2}$ = the Triangle ABD) leaves $\frac{cx\sqrt{x^2-a^2}}{2a}-\frac{ac}{2}\times$ hyp. Log. $\frac{x+\sqrt{x^2-a^2}}{a}$ for the required Area of the Hyperbola ABD.

EXAMPLE XVII.

130. *Let the Curve propoſed be the Ellipſis* AEB.

Then, putting the tranſverſe Axis AB = a, and the Conjugate (2CE) = c; we ſhall, by the Property of the Curve, have y (DR) $=\frac{c}{a}\sqrt{ax-xx}$, and therefore $\dot u\ (y\dot x)=\frac{c}{a}\times\dot x\sqrt{ax-xx}$ = the Fluxion of the Area ARD.

But $\dot{x}\sqrt{ax-xx}$ is known to expreſs the Fluxion of the correſponding Segment AD*n* of the circumſcribing

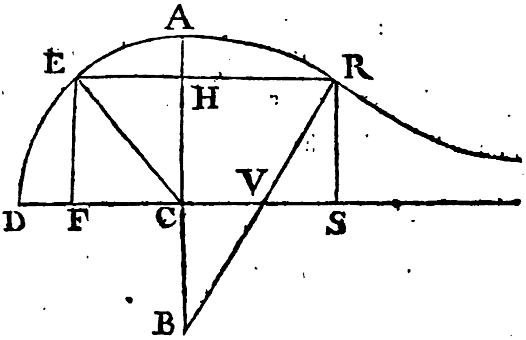


Semi-circle; whoſe Fluent is, therefore, given, by *Art.* 124; which being denoted by A, that of $\frac{c}{a}\times\dot{x}\sqrt{ax-x^2}$ will, conſequently, be $=\frac{c}{a}\times$ A. Hence, the Area of the Segment of an Ellipſis, is to the Area of the correſponding Segment of its circumſcribing Circle, as the leſſer Axis of the Ellipſis is to the greater; whence, it follows that the whole Ellipſis muſt be to the whole Circle in the ſame Ratio.

EXAMPLE XVIII.

131. *Let the Curve* AR &c. *whoſe Area* CARS *you would find, be the Conchoid of* Nicomedes.

Whereof the Equation (putting BC $= a$, and RV (= AC) $= b$) is $x^2y^2=\overline{a+y}|^2\times\overline{b^2-y^2}$ (*Vid. Art.* 57.) Which, by Reduction, becomes $x=\frac{a\sqrt{b^2-y^2}}{y}+$



$\sqrt{b^2 - y^2}$: But, to bring it down to a, *ſtill*, more ſimple Form, make $\sqrt{b^2 - y^2}$ (= SV) $= z$; then $y = \sqrt{b^2 - z^2}$; whence, by Subſtitution, $x = \frac{az}{\sqrt{b^2 - z^2}} + z$; and conſequently $\dot{x} =$

$$\frac{a\dot{z}\sqrt{b^2 - z^2} + \frac{z\dot{z}}{\sqrt{b^2 - z^2}} \times az}{b^2 - z^2} + \dot{z} =$$

$$\frac{a\dot{z} \times \overline{b^2 - z^2} + az^2\dot{z}}{b^2 - z^2 \times \sqrt{b^2 - z^2}} + \dot{z} = \frac{ab^2\dot{z}}{b^2 - z^2 \times \sqrt{b^2 - z^2}} + \dot{z};$$

and therefore $\dot{u}\ (y\dot{x}) = \sqrt{b^2 - z^2} \times \overline{\frac{ab^2\dot{z}}{b^2 - z^2 \times \sqrt{b^2 - z^2}} + \dot{z}} = \frac{ab^2\dot{z}}{b^2 - z^2} + \dot{z}\sqrt{b^2 - z^2}$.

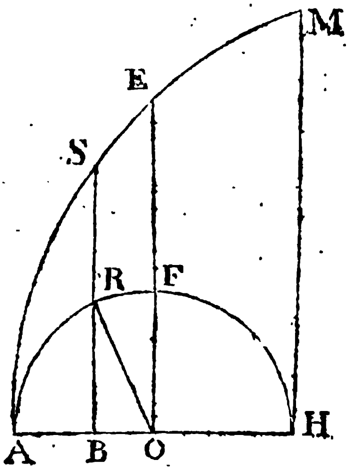
But now, to exhibit the Fluent hereof; upon C, as a Center, with the Radius AC (*b*) let a Quadrant of a Circle AED be deſcribed, and let RH, produced, meet the Periphery thereof in E, alſo let EF be parallel to AC, and let CE be drawn: It is evident (becauſe CE (CA) = VR and EF = RS) that CF is alſo = VS $= z$; and therefore, EF being $(= \sqrt{CE^2 - CF^2}) = \sqrt{b^2 - z^2}$, it appears that $\dot{z}\ \sqrt{b^2 - z^2}$ (the ſecond

Term of our given Quantity) expreſſes the Fluxion of the Area AEFC: Whence, if to this Area (found *by the Table of Segments*) the Fluent of the firſt Term
* Art. 126. $\frac{ab^2\dot{z}}{b^2 - z^2}$, or the hyp. Log. of $\frac{b+z}{b-z}$, × $\frac{1}{2}ab$ *, be added, the Sum will be the whole Area ARCS, that was to be determined.

EXAMPLE XIX.

132. *Let it be required to determine the Area* ASRA *included by the common Cycloid* ASM *and its generating Semi-circle* ARH.

Put the Radius AO (or RO) $= a$, the Sine BR $= y$, the Co-ſine OB $= x$, and the Arch AR ($=$ RS, by the Property of the Cycloid) $= z$: Then AB being $= a$



$- x$, its Fluxion will be $-\dot{x}$; whence $(\dot{u})$ that of the
* Art. 112. Area ARS is $= -z\dot{x}$ *. Now to find the Fluent thereof, make $w = -zx$ ($=$ the Fluent, if z was conſtant)

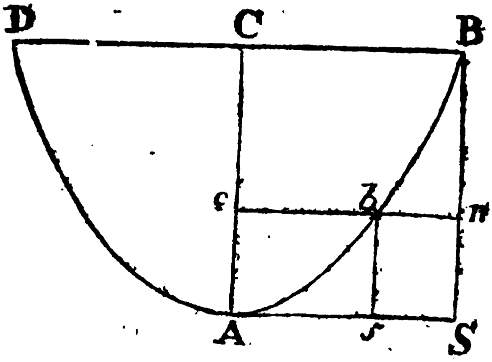
ſtant) then $\dot{w}$ being $= -z\dot{x} - x\dot{z}$ *, we ſhall have *Art. [illegible] $\dot{u}$ $(= -z\dot{x}) = \dot{w} + x\dot{z}$. But (*by Art.* 35.) $\dot{z}$ (AR Fluxion) : $\dot{y}$ (BR Fluxion) :: Radius : Co-ſine of the Angle ARB, or its Equal ROB :: OR (a) : OB (x): Therefore, by multiplying Extremes and Means, we get $x\dot{z} = a\dot{y}$: Whence, by Subſtitution $\dot{u}$ $(= \dot{w} + x\dot{z}) = \dot{w} + a\dot{y}$; and conſequently, by taking the Fluent, $u = w + ay = -zx + ay =$ AO × BR − BO × AR = the Area ARS.

Hence it follows that the Area (AEFA) when RB coincides with the Radius FO, is barely = AO × FO = AO²: And that the whole Area AMHFA is truly defined by −ARH × −OH, or by ARH × OH; that is by four times the Area of the generating Semi-circle.

EXAMPLE XX.

133. *Let the Curve propoſed be the* Catenaria DAB.

Then, drawing BS and *bs* parallel to the Axis AC, and AS and *cbn* perpendicular to the ſame; and making (as uſual) $Ac = x$, $cb = y$ and $Ab = z$, we ſhall have, by



the Property of the Curve, $2ax + x^2 = zz$: Whence $x = \sqrt{a^2 + z^2} - a$, and $\dot{x} = \frac{z\dot{z}}{\sqrt{a^2 + z^2}}$: From which the

Value

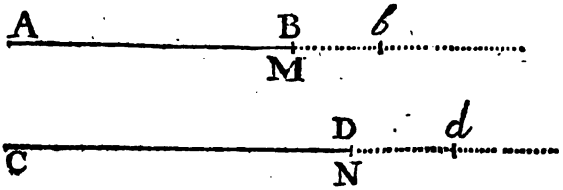
* Art. 135. Value of $\dot{y}$ (which in all Curves is $= \sqrt{\dot{z}^2 - \dot{x}^2}$ *) will here be found $= \sqrt{\dot{z}^2 - \frac{z^2\dot{z}^2}{a^2+z^2}} = \sqrt{\frac{a^2\dot{z}^2}{a^2+z^2}} = \frac{a\dot{z}}{\sqrt{a^2+z^2}}$; and this multiplied by $\sqrt{a^2+z^2} - a$ $(= bs)$ gives $a\dot{z} - \frac{a^2\dot{z}}{\sqrt{a^2+z^2}}$ (= the Rectangle Sb)
† Art. 112. = the Fluxion of the Area Asb †. From whence, by taking the Fluent, the Area itſelf is found $= az, - a^2$
‡ Art. 126. $\times$ *hyp. Log.* $\frac{z + \sqrt{a^2+z^2}}{a}$ ‡: Which therefore deducted from the Rectangle sc $(= yx = y\sqrt{a^2+z^2} - ay)$ leaves $y\sqrt{a^2+z^2} - ay - az, + a^2 \times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}$ for the required Area Abc. But, ſince $\dot{y} = \frac{a\dot{z}}{\sqrt{a^2+z^2}}$ we have $y = a \times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}$; whence, by Subſtitution, the Area, at laſt comes out $= y\sqrt{a^2+z^2} - az$, or $= a\sqrt{a^2+z^2} \times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}, - az$.

SCHOLIUM.

134. At the Beginning of this, and in the preceding Sections, we have ſeen how the Fluxions of Quantities are determined, *by conceiving the generating Motion to become uniform at the propoſed Poſition*; according to the
§ Art. 2. true Definition of a Fluxion §: But hitherto no particular Notice has been taken of *the Method of Increments, or indefinitely little Parts*, uſed (and miſtaken) by *many* for *that* of Fluxions: In which the Operations are, for the general Part, exactly the ſame; and which (tho' leſs accurate) may be applied to good Purpoſe in finding the Fluxions themſelves, in many Caſes. For which Reaſons it may not be improper to add here a

a few Lines on that Head, to ſhew the Beginner how the two Methods differ from each other; eſpecially as we ſhall be enabled, from thence, to draw out ſome Concluſions that will be of Uſe in the enſuing Part of the Work.

It hath been frequently inculcated in the foregoing Pages, that *the Fluxions of Quantities are always meaſured by how much the Quantities themſelves* would be *uniformly augmented in a given Time.* Therefore, if two



Quantities or Lines, AB and CD be generated together, by the uniform (or equable) Motion of two Points B and D, it follows, that any two Spaces B*b* and D*d* *actually* gone over (whereby AB and CD are augmented) in the ſame time, will truly expreſs the Fluxions of the generated Lines AB and CD: Whence it appears that the Increments (or Spaces actually gone over) and the Fluxions are the ſame in this Caſe, where the generating Velocities are equable.

But if, on the contrary, the Velocities of the two Points, in generating the Increments M*b* and N*d*, be ſuppoſed either to increaſe, or to decreaſe, the Lines or Increments ſo generated will, it is plain, no longer expreſs the Fluxions of AB and CD; being greater, or leſs than the Spaces that *might be uniformly* deſcribed, in the ſame Time, with the Velocities at M and N.

If, indeed, thoſe Increments, and the Time of their Deſcription, be taken ſo exceeding ſmall that the Motion of the Points during that Time may be conſidered as equable, the Ratio of the ſaid Increments, will then expreſs that of the Fluxions, or be as the Velocity at M to that at N, indefinitely near; but cannot be conceived to be *ſtrictly ſo*; unleſs, perhaps, in certain particular Caſes.

Hence we ſee that the *Differential Method*, which proceeds upon theſe indefinitely little Increments (actually generated) as we do upon Fluxions (or the Spaces that *might be uniformly* generated) differs little, or nothing, from the Method of Fluxions, except in the Manner of Conception, and in Point of Accuracy, wherein it appears defective: And yet it is very certain the Concluſions this Way derived are *mathematically* true; which has afforded Matter of Wonder to *ſome*: But the Reaſon why they are ſo is very eaſily explained. For, although the *whole complete* Increment is actually underſtood by the Notation and firſt Definition (of this Method) yet in the Solution of Problems the exact Meaſure thereof is not taken, but only that Part of it which would ariſe from an uniform Increaſe, agreeable to the Notion of a Fluxion; which admits of a ſtrict Demonſtration; But, after all, the *Differential Method* has one Advantage above that of Fluxions, which is, we are not there obliged to introduce the Properties of Motion. Since we reaſon upon the Increments themſelves, and not upon the Manner in which they may be generated.

It has been hinted above, that, though the Increments of Quantities are not, *ſtrictly*, as the Fluxions, yet from them the Ratio of the Fluxions may be deduced; and it appears that the ſmaller thoſe Increments are taken, the nearer their Ratio will approach to that of the Fluxions. Therefore, if we can, by any Means, find the Ratio to which the ſaid Increments, by conceiving them leſs and leſs, do perpetually converge, and which they may approach, before they vaniſh, nearer than any aſſignable Difference, that Ratio (called hereafter, for Diſtinction Sake, *the Ratio limiting that of the Increments*) will be, *ſtrictly*, that of the Fluxions.

This will more particularly appear from the following Inſtances; wherein the Manner of deriving the Ratio of the Fluxions, from that of the Increments, is ſhewn.

1°.

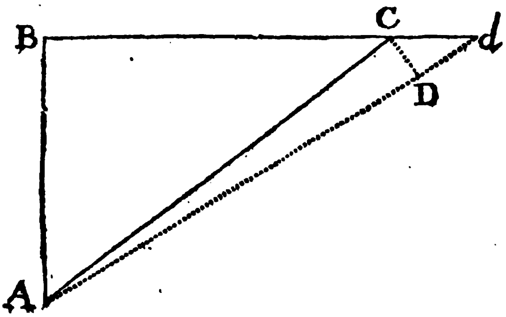
1°. *Let it be propoſed to determine the Ratio of the Fluxions of x and x^2.*

Now, if x be ſuppoſed to be augmented by any (ſmall) Quantity $\acute{x}$, ſo as to become $x + \acute{x}$, its Square (x^2) will be augmented to $\overline{x+\acute{x}}|^2 = x^2 + 2x\acute{x} + \acute{x}\acute{x}$; whence the Increment of x^2 will be $2x\acute{x} + \acute{x}\acute{x}$; which therefore is to ($\acute{x}$) the Increment of x, as $2x+\acute{x}$ to 1. Hence, becauſe the leſſer $\acute{x}$ is taken, the nearer this Ratio approaches to that of $2x$ to 1, which is its *Limit*, the Ratio of the Fluxions will therefore be expreſſed by that of $2x$ to 1, or, which is the ſame, by that of $2x\dot{x}$ to $\dot{x}$ (*as in Art.* 6.)

2°. *Let the Ratio of the Fluxions of x and x^n be required.*

Then, if x be augmented to $x+\acute{x}$, x^n will be augmented to $\overline{x+\acute{x}}|^n = x^n + nx^{n-1}\acute{x} + \frac{n}{1} \times \frac{n-1}{2} x^{n-2}\acute{x}^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}\acute{x}^3$ &c. (*Vid. Art.* 99. Whence the Increments of x and x^n will be to each other as 1 to $nx^{n-1} + \frac{n}{1} \times \frac{n-1}{2} x^{n-2}\acute{x} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}\acute{x}\acute{x}$ &c. Where the ſmaller $\acute{x}$ is taken, the nearer the Ratio will approach to that of 1 to nx^{n-1}; which appears to be its Limit: Therefore this laſt Ratio, or that of $\dot{x}$ to $nx^{n-1}\dot{x}$, is the Ratio of the Fluxions required, (*Vid. Art.* 8.)

3°. *Let it be propoſed to determine the Proportion of the Fluxions of the Sides* AC *and* BC, *of a right-angled, plane Triangle* ABC; *ſuppoſing the Perpendicular* AB *to remain invariable.*



If C*d* be aſſumed to repreſent any Increment of BC and D*d*, the correſponding Increment of AC (=AD) the Ratio of thoſe Increments will be, univerſally, expreſſed by that of the Sine of the Angle CD*d* to the Sine of the Angle DC*d* (*by plane Trigonometry*) and the leſs the Increments are ſuppoſed to be, the nearer will the Angle CD*d* approach to a right one, or to an Equality with B; which is its Limit: And the nearer will DC*d* approach, at the ſame time, to an Equality with BAC. Therefore *the Ratio here limiting that of the Increments* is that of the Sine of B (or Radius) to the Sine of BAC: Which alſo expreſſes that of the required Fluxions. (*Vid. Art.* 35.)

In the ſame way the Proportion of the Fluxions of other Kinds of algebraical and geometrical Quantities may be inveſtigated; but it will be unneceſſary to dwell longer upon this Head: I ſhall therefore only add one other Obſervation from hence (which will be of uſe hereafter) relating to the Value of an algebraic Fraction, in that particular Circumſtance when both its Numerator and Denominator become equal to Nothing, or vaniſh, at the ſame time. Which Value (it follows from above) will *be found by dividing the Fluxion of the Numerator by that of the Denominator.*

For, ſince the Value of any Fraction, in that Circumſtance, is to be looked on as *the limiting Ratio* towards which its two Terms converge, before they vaniſh, and ſeeing the Fluxions are, always, expreſſed by that Ratio, the Truth of the Rule, or Poſition, is manifeſt.

An Example, however, may not be improper:

Let therefore the Fraction $\frac{x^2-a^2}{x-a}$ be propounded, to find the Value thereof when $x=a$. In which Caſe, the true Value ſought, or the Fluxion of the Numerator divided by that of the Denominator, is $=\frac{2x\dot{x}}{\dot{x}}$ $=2x=2a$. And that this is the true Value, may be confirmed by common Diviſion, whereby the Fraction propoſed is reduced to $x+a$; whoſe Value when $x=a$, is therefore $=2a$, *the very ſame as before.*

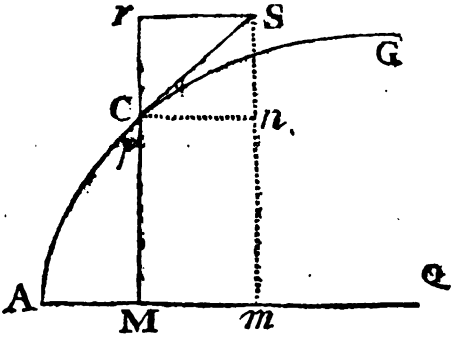
SECTION VIII.

The Uſe of Fluxions in the Rectification, or finding the Lengths, of Curves.

CASE I.

135. LET ACG *be a Curve of any Kind whoſe Ordinates are parallel to themſelves and perpendicular to the Axis* AQ.

If the Fluxion of the Abſciſſa AM be denoted by M*m*, or by C*n* (equal and parallel to M*m*) and *n*S,



equal and parallel to C*r*, be taken to repreſent the correſponding Fluxion of the Ordinate MC; then will the Diagonal CS (touching the Curve in C *) be the Line which the generating point (*p*) would deſcribe, was its Motion to become uniform at C (*Vid. Art.* 48 and 49.) which Line, therefore, truly expreſſes the Fluxion of the Space AC gone over, according to *the Definition* †.

* Art. 48 and 49.

† Art. 2.

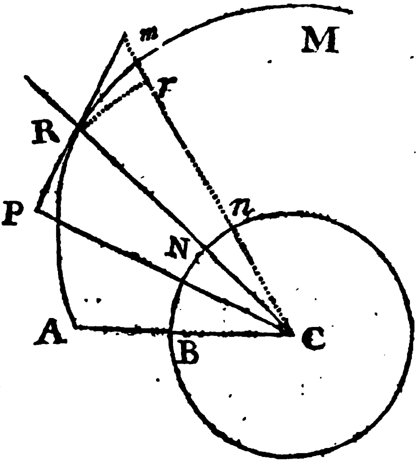
Hence, putting AM$=x$, CM$=y$, and AC$=z$, we have $\dot{z}$ ($=$CS$=\sqrt{Cn^2+Sn^2}$) $=\sqrt{\dot{x}^2+\dot{y}^2}$; from which, and the Equation of the Curve, the Value of z may be determined.

CASE

CASE II.

136. *Let all the Ordinates of the propoſed Curve* ARM *be referred to a Center* C.

Then, putting the Tangent RP (intercepted by the Perpendicular CP) $= t$, the Arch BN, of a Circle deſcribed about the Center C $= x$; the Radius CN (or CB) $= a$, &c. (*Vid. Art.* 113.) we have $\dot{z} : \dot{y} :: y$ (CR)



$: t$ (RP*) and conſequently $\dot{z} = \frac{y\dot{y}}{t}$: From whence the Value of z will be found, if the Relation of y and t is given. *Art. 35.

But in other Caſes it will be better to work from the following Equation, *viz.* $\dot{z} = \sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{a^2}}$. Which is thus derived.

Let the Right Line, CR, be conceived to revolve about the Center C; then ſince the Celerity of the generating Point R in a Direction perpendicular to CR is to ($\dot{x}$) the Celerity of the Point N, as CR (y) to CN (a) it will therefore be truly repreſented by $\frac{y\dot{x}}{a}$: Which being to ($\dot{y}$) the Celerity in the Direction of CR, produced, as CP (s) : RP (t) * it follows that $\frac{y^2\dot{x}^2}{a^2} : \dot{y}^2 :: s^2 : t^2$: Whence, by Compoſition, $\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2 : \dot{y}^2 :: s^2 + t^2\,(y^2) : t^2$; therefore $\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2 = \frac{y^2\dot{y}^2}{t^2}$, and conſequently $\sqrt{\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2}\left(= \frac{y\dot{y}}{t}\right) = \dot{z}$; as was to be ſhewn.

* Art. 35.

But the ſame Concluſion may be more eaſily deduced from the Increments of the flowing Quantities, according to the preceding Scholium.

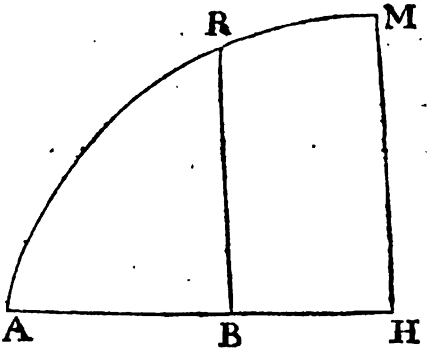
For, if Rm, *rm* and N*n* be aſſumed to repreſent ($\acute{z}$, $\acute{y}$ and $\acute{x}$) any very ſmall correſponding Increments of AR, CR and BN, it will be as CN (a) : CR (y) :: $\acute{x}$ (the Arch N*n*) : the ſimilar Arch R*r* $= \frac{y\acute{x}}{a}$. And, if the Triangle R*rm* (which, while the Point *m* is returning back to R, approaches continually nearer and nearer to a Similitude with CRP) be conſidered as *rectilineal*, we ſhall alſo obtain $\acute{z}^2$ ($= Rm^2 = Rr^2 + rm^2$) $= \frac{y^2\acute{x}^2}{a^2} + \acute{y}^2$: Whence, by writing $\dot{z}$, $\dot{x}$ and $\dot{y}$ for $\acute{z}$, $\acute{x}$ and $\acute{y}$ (according to the Scholium) there comes out $\dot{z}^2 = \frac{y^2\dot{x}^2}{a^2} + \dot{y}^2$, *as before.* –

EX-

EXAMPLE I.

137. *Let the Curve* ARM *whoſe Length is ſought, be the Semi-cubical Parabola.*

Whereof the Equation being $ax^2 = y^3$, or $x = \frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, we thence have $\dot{x} = \frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$: Whence $\dot{z}$ $(= \sqrt{\dot{y}^2 + \dot{x}^2}$*) *Art. 135.



$= \sqrt{\dot{y}^2 + \frac{9y\dot{y}^2}{4a}} = \frac{\dot{y} \times \overline{4a+9y}^{\frac{1}{2}}}{2a^{\frac{1}{2}}}$. Whoſe Fluent (found by the common Rule) is $\frac{\overline{4a+9y}^{\frac{3}{2}}}{27a^{\frac{1}{2}}}$; which, corrected (by making $y = 0$) becomes $\frac{\overline{4a+9y}^{\frac{3}{2}}}{27a^{\frac{1}{2}}} - \frac{8a}{27} = z.$

EX-

EXAMPLE II.

138. *Let the Curve propoſed be a Parabola of any (other) Kind.*

Then $x = \frac{y^n}{a^{n-1}}$ being a general Equation to all Kinds of Parabolas, we here have $\dot{x} = \frac{ny^{n-1}\dot{y}}{a^{n-1}}$, and therefore $\dot{z}$ $(=\sqrt{\dot{y}^2+\dot{x}^2}) = \sqrt{\dot{y}^2 + \frac{n^2y^{2n-2}\dot{y}^2}{a^{2n-2}}} = \dot{y} \times \overline{1 + \frac{n^2y^{2n-2}}{a^{2n-2}}}\Big|^{\frac{1}{2}}$: Whoſe Fluent, univerſally expreſſed in an Infinite Series, is $y + \frac{n^2y^{2n-1}}{\overline{2n-1} \times 2a^{2n-2}} - \frac{n^4y^{4n-3}}{\overline{4n-3} \times 8a^{4n-4}} + \frac{n^6y^{6n-5}}{\overline{6n-5} \times 16a^{6n-6}}$, &c. $= z$.

But, when $2n-2$, the Index of y, in the given Fluxion, is either equal to Unity, or to any *aliquot Part* of it, the Fluent may be accurately had in finite Terms, *by Article* 84.

For, by putting $\frac{1}{2n-2} = v$, and $\frac{n^2}{a^{2n-2}} = c$, our Fluxion $\left(\overline{1 + \frac{n^2y^{2n-2}}{a^{2n-2}}}\Big|^{\frac{1}{2}} \times \dot{y}\right)$ is, in the firſt place, reduced to $\overline{1+cy^{\frac{1}{v}}}\Big|^{\frac{1}{2}} \times \dot{y}$: Which being compared with $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, the general Expreſſion in the foreſaid *Article*, we have $a = 1$, $z = y$, $n = \frac{1}{v}$, $m = \frac{1}{2}$, $d = 1$, $\dot{x} = \dot{y}$, $rn - 1 = 0$, or $\frac{r}{v} - 1 = 0$; whence $r = v$, $s\ (r + m) = v + \frac{1}{2}$; and conſequently

$$\frac{d \times \overline{a+cz^n}|^{m+1}}{snc} \times \overline{\frac{z^{rn-n}}{1} - \frac{\overline{r-1} \times az^{rn-2n}}{\overline{s-1} \times c} + \&c.}\ ^{*}$$

$$\frac{\overline{1+cy^{\frac{1}{v}}}|^{\frac{3}{2}}}{c+\frac{c}{2v}} \times \overline{y^{\frac{v-1}{v}} - \frac{\overline{v-1} \times y^{\frac{v-2}{v}}}{\overline{v-\frac{1}{2}} \times c} + \frac{\overline{v-1} \times \overline{v-2} \times y^{\frac{v-3}{v}}}{\overline{v-\frac{1}{2}} \times \overline{v-\frac{3}{2}} \times c^2} - \&c.}$$ = the Fluent of $\overline{1+cy^{\frac{1}{v}}}|^{\frac{1}{2}} \times \dot{y}$; which was to be determined, and which will (it is plain) always terminate in v Terms, when v, or its Equal $\frac{1}{2n-2}$, is a whole poſitive Number.

If $\frac{2v+1}{2v}$ (derived from $v = \frac{1}{2n-2}$) be ſubſtituted for its Equal n, the Equation of the Curve, will be changed to $ax^{2v} = y^{2v+1}$; which, if v be expounded by 1, 2, 3, 4, &c. ſucceſſively, will become $ax^2 = y^3$, $ax^4 = y^5$, $ax^6 = y^7$, $ax^8 = y^9$ &c. reſpectively: In all which Caſes the Length of the Curve may therefore be accurately had from the Fluent above exhibited.

* Art. 84.

Moreover, if n be aſſumed $= 2$ (or $v = \frac{1}{2}$) the general Equation, $x = \frac{y^n}{a^{n-1}}$, will then become $x = \frac{y^2}{a}$; anſwering to the common (or conical) Parabola.

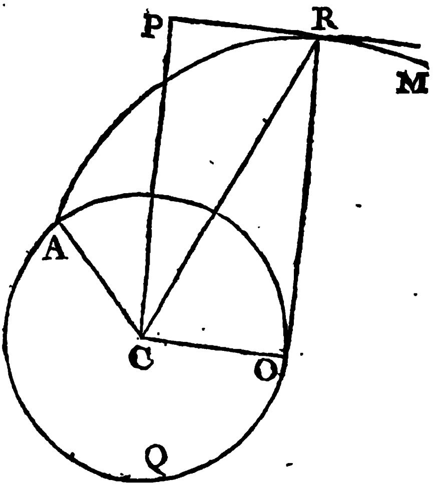
And therefore in that Caſe $\dot{x}$ $\left(= \overline{1 + \frac{n^2 y^{2n-2}}{a^{2n-2}}}\right|^{\frac{1}{2}} \times \dot{y})$ is $= \dot{y}\sqrt{1 + \frac{4y^2}{a^2}} = \frac{\dot{y}\sqrt{\frac{1}{4}a^2 + y^2}}{\frac{1}{2}a} = \frac{\dot{y}\sqrt{b^2 + y^2}}{b}$ (by putting $b = \frac{1}{2}a$) $= \frac{\dot{y} \times \overline{b^2 + y^2}}{b\sqrt{b^2 + y^2}} = \frac{1}{b} \times \frac{b^2\dot{y} + y^2\dot{y}}{\sqrt{b^2 + y^2}} = \frac{1}{b} \times \frac{b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} = \frac{1}{b}$ into $\frac{\frac{1}{2}b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} + \frac{\frac{1}{2}b^2 y\dot{y}}{\sqrt{b^2y^2 + y^4}} = \frac{1}{b}$ into $\frac{\frac{1}{2}b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} + \frac{\frac{1}{2}b^2\dot{y}}{\sqrt{b^2 + y^2}}$: Where, the Fluent of the firſt Term (of the Fluxion ſo transformed) being $= \frac{1}{2}\sqrt{b^2y^2 + y^4}$ (or $\frac{1}{2}y\sqrt{b^2 + y^2}$ by the common Rule; and that of the ſecond Term $= \frac{1}{2}b^2 \times$ hyp. Log. $\frac{y + \sqrt{b^2 + y^2}}{b}$, * it follows that the Length of the Curve will, in this Caſe, be $= \frac{\frac{1}{2}y\sqrt{b^2 + y^2}}{b} + \frac{1}{2}b \times$ hyp. Log. $\frac{y + \sqrt{b^2 + y^2}}{b}$.

* Art. 126.

E X-

EXAMPLE III.

139. *Let the Curve propoſed be the Involute of a Circle;* whoſe Nature is ſuch, that the Part PR of the Tangent intercepted by the Point of Contact and the Perpendicular CP, is every where equal to the Radius CO of the generating Circle: Therefore $\dot{z}$ ($= \frac{y\dot{y}}{t}$ *) being here $= \frac{y\dot{y}}{a}$, we firſt get $z = \frac{y^2}{2a}$; which corrected, by making $y = a$ ($=$ AC) becomes $\frac{y^2 - a^2}{2a}$ $\left(\frac{CP^2}{2CA}\right)$ the true Meaſure of the required Arch AR.

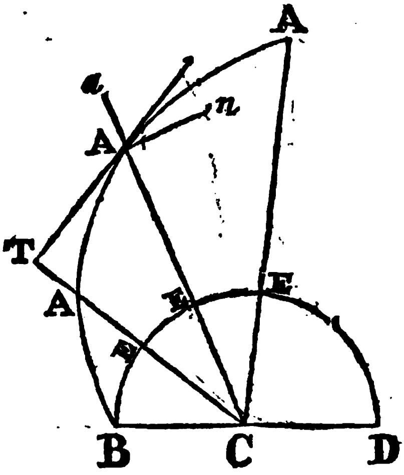


* Art. 136.

EXAMPLE IV.

140. *In which the Spiral of* Archimedes *is propoſed.*

Where, the Value of t (AT) being denoted by $\frac{by}{\sqrt{b^2+y^2}}$ (*Vid. Art.* 62.) we get $\dot{z}$ ($= \frac{y\dot{y}}{t}$) $= \frac{\dot{y}\sqrt{b^2+y^2}}{b}$: Which Fluxion being exactly the



ſame as that expreſſing the Arch of the common Parabola, found in *Article* 138. its Fluent will therefore be truly repreſented by the Meaſure of the ſaid Arch, or by $\frac{\frac{1}{2}y\sqrt{b^2+y^2}}{b} + \frac{1}{2}b \times$ *hyp. Log.* $\frac{y+\sqrt{b^2+y^2}}{b}$, the Value there exhibited.

EX-

EXAMPLE V.

141. *Let the Curve be a Spiral whoſe Equation is* $a^{m-1}x = y^m$ (*Vid. Art.* 136.)

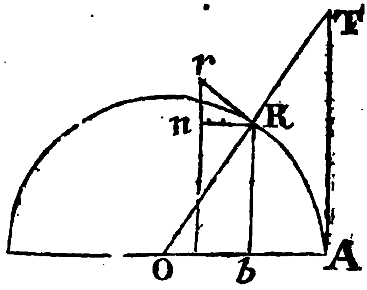
In which Caſe $\dot{x}$ being $= \frac{m\dot{y}y^{m-1}}{a^{m-1}}$, it is evident that $\dot{z}\ \left(= \sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{a^2}}\,^*\right) = \sqrt{\dot{y}^2 + \frac{m^2y^{2m}\dot{y}^2}{a^{2m}}} = \dot{y}\sqrt{1 + \frac{m^2y^{2m}}{a^{2m}}}$; and therefore $z = y + \frac{m^2y^{2m+1}}{\overline{2m+1} \times 2a^{2m}} - \frac{m^4y^{4m+1}}{\overline{4m+1} \times 8a^{4m}} + \frac{m^6y^{6m+1}}{\overline{6m+1} \times 16a^{6m}}$ &c. Which Value may be otherwiſe had, without an Infinite Series, when $\frac{1}{2m}$ is a whole poſitive Number, *Vid. Art.* 138.

* Art. 136.

EXAMPLE VI.

142. *Where, the Right-ſine, Verſed-ſine, Tangent, or Secant of an Arch of a Circle, being given, it is required to find the Length of the Arch itſelf in Terms thereof.*

Put the Verſed-ſine $Ab = x$, the Right-ſine $Rb = y$, the Tangent AT $= t$, the Secant OT $= s$, the Arch AR $= z$, and the Radius AO, or RO, $= a$; alſo let $Rn = \dot{x}$, $nr = \dot{y}$ and $Rr = \dot{z}$: Since the Angle rnR (= Right-angle) $= ObR$, and rRn (= Right-angle — nRO) $= ORb$, the Triangles rRn and ORb are



are therefore equi-angular; and it will be, Rb (y) : OR (a) :: Rn $(\dot{x})$: Rr $(\dot{z}) = \frac{a\dot{x}}{y} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$ (becauſe, by the Property of the Circle $\sqrt{2ax - xx} = y$.) Alſo, Ob $(\sqrt{a^2 - y^2})$: OR (a) :: nr $(\dot{y})$:Rr $(\dot{z}) = \frac{a\dot{y}}{\sqrt{a^2 - y^2}}$. Theſe two Values exhibit the Fluxion of the Arch in Terms of the Verſed-ſine and Right-ſine reſpectively: But, to get the ſame, in Terms of the Tangent and Secant, we have *(by ſim. Triangles)* OT $(= s = \sqrt{a^2 + t^2})$: OA (a) :: OR (a) : Ob $= \frac{a^2}{s} = \frac{a^2}{\sqrt{a^2 + t^2}}$: Hence A$b$ $= a - \frac{a^2}{s} = a - \frac{a^2}{\sqrt{a^2 + t^2}}$; whoſe Fluxion is therefore $= \frac{a^2\dot{s}}{s^2} = \frac{a^2t\dot{t}}{\overline{a^2 + t^2}|^{\frac{3}{2}}}$: Whence (again by ſimilar Triangles) AT $(= \sqrt{s^2 - a^2} = t)$: OT $(= s = \sqrt{a^2 + t^2})$ :: Rn : Rr $= \frac{a^2\dot{s}}{s\sqrt{s^2 - a^2}} = \frac{a^2\dot{t}}{a^2 + t^2} = \dot{z}$.

Now, from any one of the four Forms of Fluxions $\left(\frac{a\dot{x}}{\sqrt{2ax - xx}}, \frac{a\dot{y}}{\sqrt{a^2 - y^2}}, \frac{a^2\dot{t}}{a^2 + t^2}, \frac{a^2\dot{s}}{s\sqrt{s^2 - a^2}}\right)$ here found, the Value of the Arch itſelf (by taking the Fluent, in an Infinite Series) will likewiſe become known.

But the third Form, expreſſed in Terms of the Tangent, being intirely free from radical Quantities, will be the moſt ready in Practice, eſpecially where the required Arch is but ſmall; though the Series ariſing from the firſt Form, always, converges the faſteſt.

If, therefore, $\frac{a^2\dot{t}}{a^2+t^2}$ be now converted to an Infinite Series, we ſhall have $\dot{z} = \dot{t} - \frac{t^2\dot{t}}{a^2} + \frac{t^4\dot{t}}{a^4} - \frac{t^6\dot{t}}{a^6}$. &c. and conſequently $z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} + \frac{t^9}{9a^8}$ &c. = AR. Where, if (*for Example Sake*) AR be ſuppoſed an Arch of 30 Degrees, and AO (to render the Operation more eaſy) be put = Unity, we ſhall have $t = \sqrt{\tfrac{1}{3}} = .5773502$ (becauſe Ob $\sqrt{\tfrac{3}{4}}$: bR $(\tfrac{1}{2})$ :: OA (1) : AT $(t) = \sqrt{\tfrac{1}{3}}$)

Whence

$$t^3 \ \left(= t \times t^2 = t \times \tfrac{1}{3}\right) = .1924500$$

$$t^5 \ \left(= t^3 \times t^2 = \frac{t^3}{3}\right) = .0641500$$

$$t^7 \ \left(= t^5 \times t^2 = \frac{t^5}{3}\right) = .0213833$$

$$t^9 \ \left(= t^7 \times t^2 = \frac{t^7}{3}\right) = .0071277$$

$$t^{11} \ \left(= t^9 \times t^2 = \frac{t^9}{3}\right) = .0023759$$

$$t^{13} \ \left(= t^{11} \times t^2 = \frac{t^{11}}{3}\right) = .0007919$$

$$t^{15} \ \left(= t^{13} \times t^2 = \frac{t^{13}}{3}\right) = .0002639$$

&c.

And therefore AR $= .5773502 - \frac{.1924500}{3} + \frac{.0641500}{5} - \frac{.0213833}{7} + \frac{.0071277}{9} - \frac{.0023759}{11} +$

$$+ \frac{.0007919}{13} - \frac{.0002639}{15} + \frac{.0000879}{17} - \frac{.0000293}{19}$$

$$+ \frac{.0000097}{21} - \frac{.0000032}{23} = .5235987$$: Which multiplied by 6 gives 3.141592 + for the Length of the Semi-periphery of the Circle whoſe Radius is Unity.

At *Article* 126. certain Forms of Fluxions were pointed out, whoſe Fluents are explicable by means of hyperbolical Spaces, or a *Table of Logarithms:* Which Forms, it is obſervable, agree in every thing, but the Signs (and conſtant Quantities) with thoſe exhibited above, for the Arch of a Circle. And theſe laſt, like them, may ſerve as ſo many (other) Theorems for finding Fluents by means of a *Table of Sines, Tangents and Secants.* But, as ſuch a Table is uſually calculated to a Radius of 1,000000 *&c.* (or Unity) the following Equations, derived from thoſe above, being adapted to that Radius, will be rather more commodious.

$$\text{Thus, the Fluent of} \left\{ \begin{array}{c} \dfrac{\dot{w}}{\sqrt{2aw - w^2}} \\[2ex] \dfrac{\dot{w}}{\sqrt{a^2 - w^2}} \\[2ex] \dfrac{a\dot{w}}{a^2 + w^2} \\[2ex] \dfrac{a\dot{w}}{w\sqrt{w^2 - a^2}} \end{array} \right\} \text{is equal to the Arch whoſe} \left\{ \begin{array}{l} \text{Verſed-ſine} \\[2ex] \text{Right-ſine} \\[2ex] \text{Tangent} \\[2ex] \text{Secant} \end{array} \right\} \begin{array}{l} \text{is } \dfrac{w}{a}, \text{ and} \\ \text{Radius Unity.} \end{array}$$

The way of deducing theſe Expreſſions, from the foregoing ones, is extremely eaſy: For, if *A* be put to denote the Arch whoſe Radius is Unity, and whoſe Verſed-ſine, Right-ſine, Tangent, or Secant is $\frac{w}{a}$ (according to the different Caſes here ſpecified). Then, becauſe ſimilar Arcs, of unequal Circles, are as their

Radii, it will be $1 : a :: A : (aA)$ the Length of the Arch AR (*ſee the Figure.*) Therefore, the Fluent of $\frac{a\dot{x}}{\sqrt{2ax-xx}}$ (or $\frac{a\dot{w}}{\sqrt{2aw-w^2}}$, putting $w=x$) being $=aA$ (AR), that of $\frac{\dot{w}}{\sqrt{2aw-w^2}}$ muſt neceſſarily be $=A$: And in the very ſame Manner the other Forms are made out.

EXAMPLE VII.

143. *Let the propoſed Curve be the common Cycloid.*

Then, if the Radius AO of the generating Semi-circle* be denoted by a, we ſhall have BR $=\sqrt{2ax-x^2}$; and the Fluxion thereof $=\frac{a\dot{x}-x\dot{x}}{\sqrt{2ax-x^2}}$: Which being added to $\left(\frac{a\dot{x}}{\sqrt{2ax-xx}}\right)$ the Fluxion of AR or its Equal RS (given by the preceding Article) we thence get $\frac{2a\dot{x}-x\dot{x}}{\sqrt{2ax-x^2}}=\frac{\dot{x}\times\overline{2a-x}}{x^{\frac{1}{2}}\times\overline{2a-x}|^{\frac{1}{2}}}=\frac{\dot{x}}{x^{\frac{1}{2}}}\times\overline{2a-x}|^{\frac{1}{2}}$, for the true Fluxion of the Ordinate BS of the Cycloid.

Hence $\dot{z}$ $(\sqrt{\dot{x}^2+\dot{y}^2}$ †$)=\sqrt{\dot{x}^2+\frac{\dot{x}^2\times\overline{2a-x}}{x}}=\dot{x}\sqrt{\frac{2a}{x}}=\overline{2a}|^{\frac{1}{2}}\times x^{-\frac{1}{2}}\dot{x}$; and conſequently, by taking the Fluent, $z=\overline{2a}|^{\frac{1}{2}}\times\frac{x^{\frac{1}{2}}}{\frac{1}{2}}=2\sqrt{2ax}=$ the Arch AS of the Cycloid.

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* See Fig. Art. 132.

† Art. 135.

EXAMPLE VIII.

144. *Wherein it is required to determine the Length of the Arch of the common Hyperbola.*

In this Caſe (the Semi-tranſverſe Axis being repreſented by b, and the Semi-conjugate by c) we have $\frac{b^2y^2}{c^2} = 2bx + x^2$; and therefore $x = \frac{b\sqrt{c^2+y^2}}{c} - b$: Hence $\dot x = \frac{by\dot y}{c\sqrt{c^2+y^2}}$, and $\dot z$ $(= \sqrt{\dot y^2+\dot x^2})$ $\sqrt{\dot y^2 + \frac{b^2y^2\dot y^2}{c^2\times\overline{c^2+y^2}}} = \dot y\sqrt{1 + \frac{b^2y^2}{c^4+c^2y^2}}$; which, by converting $\frac{b^2y^2}{c^4+c^2y^2}$ into an *Infinite Series*, becomes $\dot y\sqrt{1 + \frac{b^2y^2}{c^4} - \frac{b^2y^4}{c^6} + \frac{b^2y^6}{c^8} - \frac{b^2y^8}{c^{10}}}$ &c. But ſtill we have the Square Root to extract; In order thereto let it be aſſumed $= 1 + Ay^2 + By^4 + Cy^6 + Dy^8$ &c. Then, by ſquaring, and tranſpoſing (*Vid. Art.* 98.) there ariſes

$$\left.\begin{array}{l} 1+2Ay^2+2By^4+2Cy^6+2Dy^8 \text{ &c.} \\ \qquad\quad +A^2y^4+2ABy^6+2ACy^8 \text{ &c.} \\ \qquad\qquad\qquad\quad +B^2y^8 \text{ &c.} \\ -1-\frac{b^2}{c^4}\times y^2+\frac{b^2}{c^6}\times y^4-\frac{b^2}{c^8}\times y^6+\frac{b^2}{c^{10}}\times y^8\text{&c.} \end{array}\right\} = 0$$

Hence $A = \frac{b^2}{2c^4}$; $B = -\frac{b^2}{2c^6} - \frac{1}{2}A^2 = -\frac{b^2}{2c^6} - \frac{b^4}{8c^6}$; $C = \frac{b^2}{2c^8} - AB = \frac{b^2}{2c^8} + \frac{b^4}{4c^{10}} + \frac{b^6}{16c^{12}}$, &c. &c. Therefore $\dot z$ $(= \dot y\sqrt{1 + \frac{b^2y^2}{c^4}\text{ &c.}} = \dot y \times \overline{1+Ay^2+By^4 \text{ &c.}}) = \dot y + \frac{b^2}{2c^4} \times y^2\dot y - \overline{\frac{b^2}{2c^6} + \frac{b^4}{8c^8}} \times y^4\dot y$

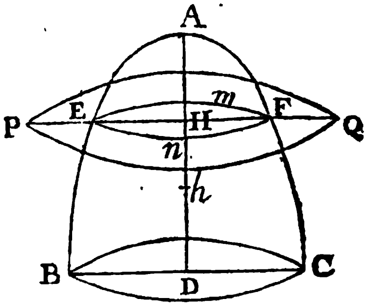
$y^4\dot{y} + \overline{\frac{b^2}{2c^8} + \frac{b^4}{4c^{10}} + \frac{b^6}{16c^{12}}} \times y^6\dot{y}$ &c. And conſequently $z = y + \frac{b^2y^3}{6c^4} - \overline{\frac{b^2}{c^2} + \frac{b^4}{4c^4}} \times \frac{y^5}{10c^4} + \overline{\frac{b^2}{c^2} + \frac{b^4}{2c^4} + \frac{b^6}{8c^6}} \times \frac{y^7}{14c^6}$ &c.

By the very ſame way of proceeding the Arch of an Ellipſis may be found, the Equations of the two Curves differing in nothing but their Signs.

SECTION IX.

The Application of FLUXIONS *in inveſtigating the Contents of Solids.*

145. LET ABC repreſent any Solid; conceived to be generated (or deſcribed) by a Plane PQ paſſing over it, with a parallel Motion: Let *Hh* (perpendicular to PQ) be taken to expreſs the Fluxion of AH (x) or the Velocity with which the generating Plane is carry'd; alſo let the Area of the Part, E*m*F*n*, of the Plane intercepted by, or contained in, the Solid, be denoted by A: Then it follows, from *Art.* 2 and 5. that the Fluxion of the Solid AEF, will be expreſſed by $A\dot{x}$. From whence, by expounding A in Terms of x, (according to the Nature of the Figure) and then taking the Fluent, the Content of



of the Solid (which we ſhall, always, hereafter repreſent by s) will be given.

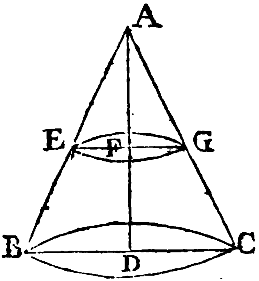
But, when the propoſed Solid is that ariſing from the Revolution of any given Curve AEB about AHD, as an Axis, the Fluxion ($\dot{s}$) of the Solidity may be exhibited in a Manner more convenient for Practice: For, putting the Area (3,141592 &c.*) of the Circle, whoſe Radius is Unity, $= p$, and the Ordinate EH $= y$, it will be $1^2 : y^2 :: p : (py^2)$ the Area of the Circle EmFn, which being wrote above inſtead of A, we have $\dot{s} = py^2\dot{x}$. The Uſe of which will be ſufficiently ſhewn in the following Examples.

* Art. 124.

EXAMPLE I.

146. *Let it be propoſed to find the Content of a Cone* ABC.

Put the given Altitude (AD) of the Cone $= a$, and the Semi-diameter (BD) of its Baſe $= b$: Then, the Diſtance (AF) of the Circle EG, from the Vertex A, being denoted by x, &c. we have, by ſimilar Triangles, as $a : b :: x :$ EF $(y) = \frac{bx}{a}$. Whence, in this Caſe, $\dot{s}$ $(= py^2\dot{x}) = \frac{pb^2x^2\dot{x}}{a^2}$; and conſequently $s = \frac{pb^2x^3}{3a^2}$; which, when $x = a$ (= AD) gives $\frac{pb^2a}{3}$ $(= p \times \text{BD}^2 \times \frac{1}{3}\text{AD})$ for the Content of the whole Cone ABC. Which appears, from hence, to be juſt $\frac{1}{3}$ of a Cylinder of the ſame Baſe and Altitude.



EXAMPLE II.

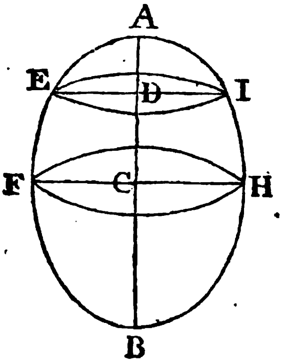
147. *Where, let the Solid propoſed be a parabolic Conoid, or that ariſing from the Revolution of any Kind of Parabola about its Axis.*

Then, from the Equation $a^{m-n}x^{n} = y^{m}$, of the generating Curve, we get $y = a^{\frac{m-n}{m}} \times x^{\frac{n}{m}}$, and $\dot{s}\ (= py^{2}\dot{x}) = pa^{\frac{2m-2n}{m}} \times \dot{x}x^{\frac{2n}{m}}$; and therefore $s = pa^{\frac{2m-2n}{m}} \times \dfrac{x^{\frac{2n}{m}+1}}{\frac{2n}{m}+1} = pa^{\frac{2m-2n}{m}} \times \dfrac{mx^{\frac{2n}{m}+1}}{2n+m} = pa^{\frac{2m-2n}{m}} \times x^{\frac{2n}{m}} \times \dfrac{mx}{2n+m} = py^{2} \times \dfrac{mx}{2n+m} =$ the Content of the Solid; which therefore is to $(py^{2}x)$ the Content of the circumſcribing Cylinder, as m to $2n+m$. Whence the Solid generated by the conical Parabola (where $m=2$, and $n=1$) appears to be juſt $\frac{1}{2}$ of its circumſcribing Cylinder.

EXAMPLE III.

148. *Let the propoſed Solid* AFBH *be a Spheroid.*

In which Caſe, putting the Axis AB, about which the Solid is generated, $=a$, and the other Axis FH, of the generating Ellipſis $= b$, it follows, from the Property of the Ellipſis, that $a^{2} : b^{2} :: x \times \overline{a-x}$ (AD × BD) : y^{2} (DE)^{2} $= \dfrac{b^{2}}{a^{2}} \times \overline{ax-xx}$: Whence we have $\dot{s}\ (= py^{2}\dot{x}\ ^{*}) = \dfrac{pb^{2}}{a^{2}} \times \overline{ax\dot{x} - x^{2}\dot{x}}$; and * Art. 145.
$s = \dfrac{pb^{2}}{a^{2}} \times \overline{\tfrac{1}{2}axx - \tfrac{1}{3}x^{3}} =$ the Segment AIE. Which, when



when AD (x) = AB (a), becomes $\left(\frac{pb^2}{a^2} \times \overline{\tfrac{1}{2}a^3 - \tfrac{1}{3}a^3}\right)$ $\frac{1}{6}pab^2$ = the Content of the whole Spheroid. Where, if b (FH) be taken $= a$ (AB) we ſhall alſo get $\frac{1}{6}pa^3$ for the true Content of the Sphere whoſe Diameter is a. Hence a Sphere, or a Spheroid, is $\frac{2}{3}$ of its circumſcribing Cylinder; for the Area of the Circle FH being expreſſed by $\frac{pb^2}{4}$, the Content of the Cylinder whoſe Diameter is FH, and Altitude AB, will therefore be $\frac{pb^2a}{4}$; of which $\frac{1}{6}pab^2$, is, evidently, two third Parts.

EXAMPLE IV.

149. *Let the Solid, whoſe Content you would find, be the hyperbolical Conoid.*

Then, from the Equation, $y^2 = \frac{b^2}{a^2} \times \overline{ax + xx}$, of the generating Hyperbola, we have $\dot{s}$ $(py^2\dot{x}) = \frac{pb^2}{a^2} \times \overline{ax\dot{x} + x^2\dot{x}}$, and conſequently $s = \frac{pb^2}{a^2} \times \overline{\tfrac{1}{2}ax^2 + \tfrac{1}{3}x^3}$ = the Content of the Conoid; which therefore is to $\left(\frac{pb^2}{a^2} \times \overline{ax + x^2} \times x\right)$ that of a Cylinder of the ſame Baſe and Altitude, as $\frac{1}{2}a + \frac{1}{3}x$ to $a + x$. This Ratio, if x be extremely ſmall, will become as 1 to 2 very nearly: Whence it may be inferr'd, that the Content of a very ſmall Part of any Solid, generated by a Curve, whoſe Ray of Curvature at the Vertex is a finite Quantity, is half that of a Cylinder of the ſame Baſe and Altitude, very nearly: Becauſe any ſuch Curve, for a ſmall Diſtance, will differ inſenſibly from an Hyperbola, whoſe Radius of Curvature, at the Vertex, is the ſame.

This might have been inferred, either, from the common parabolic Conoid, or the Spheroid, in the preceding Examples; but other Obſervations would not allow Room for it there.

EXAMPLE V.

150. *In which the propoſed Solid is that ariſing from the Rotation of the Ciſſoid of* Diocles, *about its Axis.*

Here, y^2 being $= \frac{x^3}{a-x}$, * we have $\dot{s}\ (py^2\dot{x}) = \frac{px^3\dot{x}}{a-x}$. But, in Caſes like this, (where the Denominator is rational and the variable Quantity in the Numerator of ſeveral Dimenſions) it will be neceſſary to divide the latter by the former, in order to obtain the Fluent, by leſſening the Number of Dimenſions: Thus, dividing $px^3\dot{x}$ by $-x+a$, according to the Manner of compound Quantities, the Work will ſtand thus:

* Art. 56.

$$\begin{array}{llll}
-x+a) & px^3\dot{x}-0 & & (-px^2\dot{x}-pax\dot{x}-pa^2\dot{x} \\
& px^3\dot{x}-pax^2\dot{x} & & \\
\hline
& +pax^2\dot{x}-0 & & \\
& +pax^2\dot{x}-pa^2x\dot{x} & & \\
\hline
& & +pa^2x\dot{x}-0 & \\
& & +pa^2x\dot{x}-pa^3\dot{x} & \\
\hline
& & & +pa^3\dot{x}
\end{array}$$

Where, the Quotient being $-px^2\dot{x}-pax\dot{x}-pa^2\dot{x}$, and the Remainder $pa^3\dot{x}$, the Value of the given Fraction $\frac{px^3\dot{x}}{a-x}$, will therefore be truly expreſſed by $-px^2\dot{x} - pax\dot{x} - pa^2\dot{x} + \frac{pa^3\dot{x}}{a-x}$: Whoſe Fluent, properly corrected, is $-\frac{1}{3}px^3 - \frac{1}{2}pax^2 - pa^2x + pa^3 \times$ *hyp. Log.* $\frac{a}{a-x}$. *Vid. Art.* 126.

EXAMPLE VI.

151. *Let the Solid be that ariſing from the Rotation of the Conchoid of* Nicomedes *about its Axis.*

The Sub-tangent $\frac{y\dot{x}}{\dot{y}}$ of this Curve being $= \frac{-a\,\overline{b^2-y^2}}{y\sqrt{b^2-y^2}}$ (*Vid. Art.* 48 *and* 57.) we have $\dot{x} = \frac{-ab^2\dot{y}-y^3\dot{y}}{y^2\sqrt{b^2-y^2}}$, and

* Art. 145. therefore $\dot{s}\ (py^2\dot{x}^*) = \frac{-pab^2\dot{y}-py^3\dot{y}}{\sqrt{b^2-y^2}} = -\frac{pab^2\dot{y}}{\sqrt{b^2-y^2}} - \frac{py^3\dot{y}}{\sqrt{b^2-y^2}}$. But, in order for the more eaſy finding the Fluent thereof, put $\sqrt{b^2-y^2} = u$; and then, y being $= \sqrt{b^2-u^2}$, and $\dot{y} = \frac{-u\dot{u}}{\sqrt{b^2-u^2}}$, we ſhall, by Subſtitution, get $\dot{s} = \frac{pab^2\dot{u}}{\sqrt{b^2-u^2}} + p \times \overline{b^2\dot{u}-u^2\dot{u}}$.

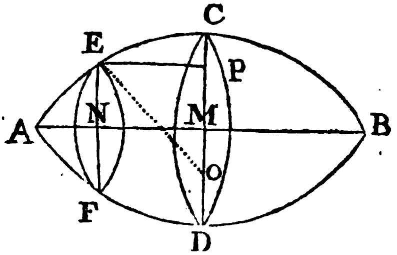
Whence, the Fluent of $\frac{\dot{u}}{\sqrt{b^2-u^2}}$ being expreſſed by the Arch (A) of the Circle whoſe Radius is Unity and

† Art. 142. Sine $\frac{u}{b}$ †, the Fluent of the whole Expreſſion will be $pab^2 \times A + p \times \overline{b^2u - \frac{1}{3}u^3}$. Which, when $y=0$, or $u=b$, gives $(pab^2 \times \frac{1}{2}p + p \times \frac{2}{3}b^3)$ $pb^2 \times \overline{\frac{1}{2}pa + \frac{2}{3}b}$ for the Content of the whole Solid, when its Axis becomes infinite.

EXAMPLE VII.

152. *Where it is required to find the Content of a parabolic Spindle; generated by the Rotation of a given Parabola* ACB *about its Ordinate* AB.

Put CM (the Abſciſſa of the given Parabola) $= a$, and the Semi-ordinate AM (or BM) $= b$; and, ſuppoſing ENF to be any Section of the Solid parallel to DC, let its Diſtance MN (or EP) from DC, be denoted by w: Then, by the Property of the Curve, we ſhall



have AM² (b^2) : EP² (w^2) :: CM (a) : CP $= \frac{aw^2}{b^2}$: Therefore EN $(= \text{CM} - \text{CP}) = a - \frac{aw^2}{b^2} = \frac{a \times \overline{b^2 - w^2}}{b^2}$, and conſequently $p \times \text{EN}^2 = \frac{pa^2}{b^4} \times \overline{b^4 - 2b^2w^2 + w^4} =$ the Area of the Section EF: Which multiply'd by $(\dot{w})$ the Fluxion of MN, gives $\frac{pa^2}{b^4} \times \overline{b^4\dot{w} - 2b^2w^2\dot{w} + w^4\dot{w}}$ for the Fluxion of the Solidity, * whoſe Fluent, $\frac{pa^2}{b^4} \times \overline{b^4w - \frac{2}{3}b^2w^3 + \frac{1}{5}w^5}$, when w becomes $= b$, is $\left(\frac{8pa^2b}{15}\right)$ half the Content of the Solid.

* Art. 145.

EXAMPLE VIII.

153. *Let the Solid* ACBD *(ſee the laſt Figure) be a Spindle, generated by the Rotation of the Segment of a Circle,* ACB, *about its Chord, or Ordinate,* AB.

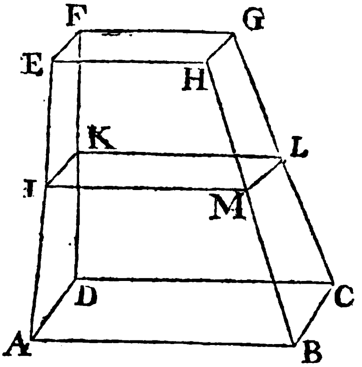
Then, if the Radius OE be put $= r$, OM $= d$, and EP $= w$ *&c.* (as before) we ſhall have OP ($= \sqrt{OE - EP^2}) = \sqrt{r^2 - w^2}$, and EN ($= OP - OM$) $= \sqrt{r^2 - w^2} - d$: Therefore $\dot{s}$, in this Caſe, is $= p\dot{w} \times \overline{\sqrt{r^2 - w^2} - d}\,^2 = p\dot{w} \times \overline{r^2 - w^2 + d^2 - 2d\sqrt{r^2 - w^2}} = p\dot{w} \times \overline{r^2 - d^2 - w^2} - p\dot{w} \times \overline{2d\sqrt{r^2 - w^2} - 2d^2}$: Whence, the Fluent of the *Part*, $p\dot{w} \times \overline{2d\sqrt{r^2 - w^2} - 2d^2}$ ($= 2dp \times \dot{w} \times \overline{\sqrt{r^2 - w^2} - d} = 2dp \times \dot{w} \times EN$) being expreſſed by $2dp \times$ *Area MNEC* * the Fluent of the *Whole*, or the true Value of s, will be expreſſed by $pw \times \overline{r^2 - d^2 - \frac{1}{3}w^2} - 2dp \times$ *Area MNEC*, or by its Equal $p \times MN \times \overline{AM^2 - \frac{1}{3}MN^2} - 2p \times OM \times$ *Area MNEC*: Which, when MN $=$ MA, gives $p \times \frac{2}{3} AM^3 - 2p \times OM \times$ *Area ACM*, for the Content of half the Solid: Where the *Area ACM* may be found by *Art.* 124. or more eaſily by the common Table of the Areas of the Segments of a Circle; to be met with in moſt Books of *Gauging*.

* Art. 112.

EXAMPLE IX.

154. *Let it be propoſed to find the Content of the Solid* AEGB; whoſe four Sides AH, AF, CH, CF are plane Surfaces, and its Ends ADCB, EFGH given Rectangles, parallel to each other.

Let the Sides AB and AD, of the Baſe, be denoted by a and b; and thoſe of the Top (EH and EF) by c and d reſpectively; moreover, let h expreſs the perpendicular Height of the Solid; and let x (conſider'd as variable) be the Diſtance of (IL) any Section thereof (parallel to the Baſe) from the Plane EG.



It is evident, from the Nature of the Figure, that the Section IL is a Rectangle; and that
$b : x :: \text{AB}-\text{EH} : \text{IM}-\text{EH} :: \text{BC}-\text{HG} : \text{ML}-\text{HG}$.

From theſe Proportions we have IM—EH $= \frac{\overline{a-c} \times x}{b}$ and ML—HG $= \frac{\overline{b-d} \times x}{b}$: Hence IM $= \frac{\overline{a-c} \times x}{b} + c$, and ML $= \frac{\overline{b-d} \times x}{b} + d$; and conſequently the Area of the Rectangle (IL) $= \frac{\overline{a-c} \times \overline{b-d}}{b^2} \times x^2 + \frac{ad-2cd+cb}{b} \times x + cd$: Which being multiply'd by $\dot{x}$, and the Fluent taken, there reſults $\frac{\overline{a-c} \times \overline{b-d} \times x^3}{3b^2} + \frac{\overline{ad-2cd+cb} \times x^2}{2b} + cdx$ for the Content of IFGL:

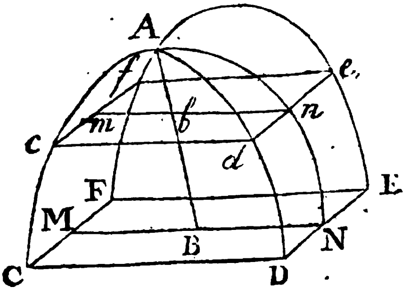
Which, when $x = b$, becomes $\Big(\frac{\overline{a-c}\times\overline{b-d}\times b}{3} + \frac{\overline{ad-2cd+cb}\times b}{2} + cdb = \overline{2ab+ad+bc+2cd}\times\frac{1}{6}b =\Big)$ $\overline{\mathrm{AB}\times\mathrm{AD}+\mathrm{EH}\times\mathrm{EF}+\overline{\mathrm{AB}+\mathrm{EH}}\times\overline{\mathrm{AD}+\mathrm{EF}}}\times\frac{1}{6}b =$ the Quantity propoſed to be found.

If EF (d) be ſuppoſed to vaniſh, and the Lines EH and FG to coincide, the Planes AEHB and DFGC will form an Angle or Ridge, at the Top of the Solid (reſembling the Rooſs of ſome Buildings, whoſe Ends as well as Sides run up ſloping) and, in this Caſe, the Content, found above, will become more ſimple, being then expreſſed by $\overline{2ab+bc}\times\frac{1}{6}b$, or its Equal $\overline{2\mathrm{AB}+\mathrm{EH}}\times\mathrm{AD}\times\frac{1}{6}b$.

But, if EF be ſuppoſed $=$ EH, and AD $=$ AB, the Solid will then be the Fruſtrum of a ſquare Pyramid; and its Content $= \overline{a^2+ac+c^2}\times\frac{1}{3}b, = \overline{\mathrm{AB}^2+\mathrm{AB}\times\mathrm{EH}+\mathrm{EH}^2}\times\frac{1}{3}b$: From whence, by taking EH $= 0$, the Content of the whole Pyramid whoſe Baſe is AB^2, and its Altitude b, will alſo be given, being $= \mathrm{AB}^2\times\frac{1}{3}b$.

EXAMPLE X.

155. *Let the propoſed Solid be that, commonly known by the Name of a Groin*; whoſe Sections parallel to the Baſe are, all, Squares, and whereof the two Sections perpendicular to the Baſe, through the Middle of the oppoſite Sides, are Semi-circles.



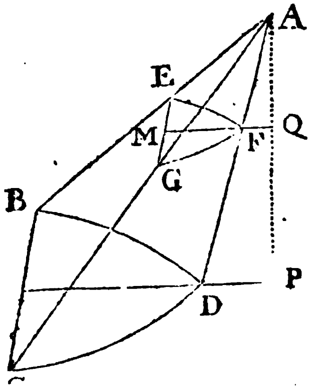
Let $bcdef$ be any Section parallel to the Baſe; and let its Diſtance Ab from the Vertex of the Solid, be denoted by x; alſo let a repreſent the Radius AB (or BN) of the circular Section ABNA, perpendicular to the Baſe. Then, bn being (by the Property of the Circle) $= \sqrt{2ax - xx}$, the Side of the Square df, will be $= 2\sqrt{2ax - xx}$, and therefore the Area $= 4 \times \overline{2ax - xx}$; whence $\dot{s} = 4\dot{x} \times \overline{2ax - xx}$, and conſequently $s = 4ax^2 - \frac{4x^3}{3}$: Which, when $x = a$, becomes $\frac{\overline{2a}|^3}{3} =$ the Content of the whole Solid.

If the Solid be a Groin of any other Kind, or ſuch, that its two Sections perpendicular to the Baſe, through the Middle of the oppoſite Sides, are any other Curves than Semi-circles, the Content may, ſtill, be found in the ſame Manner; and will be always in proportion to the Solid generated by the Revolution of the ſaid Curve about its Axis, as a Square, is to its inſcribed Circle. But, if the foreſaid perpendicular Sections be Curves of different Kinds, the Sections parallel to the Baſe will no longer be Squares, but Rectangles; whoſe Sides are the correſponding (double) Ordinates of the reſpective Curves. Thus, for Inſtance, let one Section be a Circle and the other a Parabola, whoſe Ordinates, to the common Abſciſſa, x, are expreſſed by $\sqrt{dx - xx}$ and $\sqrt{ax}$, reſpectively; then the Sides of the rectangular Section, parallel to the Baſe of the Groin, will be $2\sqrt{dx - xx}$ and $2\sqrt{ax}$: Whence the Area of that Section is $= 4x\sqrt{ad - ax}$, and therefore $\dot{s} = 4x\dot{x}\sqrt{ad - ax}$: Where, by taking the Fluent, * $s = \frac{16d^2\sqrt{ad} - a^{\frac{1}{2}} \times \overline{d - x}|^{\frac{3}{2}} \times \overline{16d + 24x}}{15} =$ the true Content of ſuch a Solid.

* Art. 83.

EXAMPLE XI.

156. *Where the Solid* BACD *propoſed is a kind of Cone, or Pyramid*; form'd by conceiving Right-lines to be drawn from every Point in the Perimeter of any given Plane BDC, to a given Point, or Vertex, A above that Plane.



Let EFG be any Section parallel to BDC, whoſe perpendicular Diſtance (AQ) from the Vertex let be denoted by x; moreover, let the whole given Altitude (AP) of the Solid be put $= a$, and the Area of the Baſe BDC (which is alſo ſuppoſed given) $= b$. In the firſt place, it is eaſy to conceive that the Planes BDC and EFG muſt be ſimilar: And therefore, ſince ſimilar Figures are to each other as the Squares of their like Sides, or Dimenſions, it follows that AP^2 (a^2) : AQ^2 (x^2) :: BDC (b) : EFG $= \frac{bx^2}{a^2}$. Whence $\dot{s} = \frac{bx^2\dot{x}}{a^2}$, and conſequently $s = \frac{bx^3}{3a^2} = \frac{ba}{3}$, when $x = a$. Therefore the Solidity of a Cone or Pyramid, let the Figure of its Baſe be what it will, is always had by multiplying the Area of the Baſe by $\frac{1}{3}$ of the Altitude.

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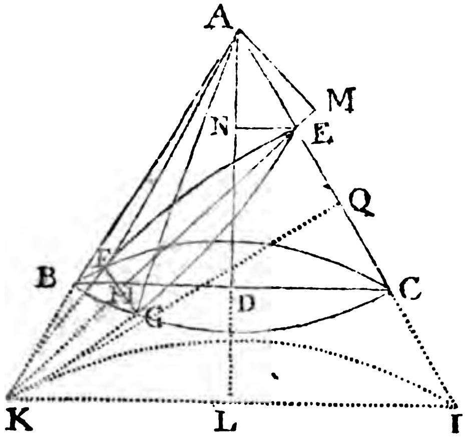
EXAMPLE XII.

157. *Where it is propoſed to find the Content of the Ungula* EFGC, cut off from a given Cone, ABC, by a Plane EFG paſſing through the Baſe thereof.



Let AD be the perpendicular Height of the Cone, alſo let AM be perpendicular to HE, the Axis of the Section FEG, and let FAG be another Section of the Cone, thro' FG and the Vertex A.

Since the Solids CAFG and EAFG, whoſe Baſes are FCG, and FEG, come under the Form ſpecified in the preceding Example, their Contents will therefore be expreſſed by FCG × $\frac{1}{3}$ AD and FEG × $\frac{1}{3}$ AM reſpectively: Whoſe Difference, $\frac{FCG \times AD - FEG \times AM}{3}$, is the Solidity of the *Ungula* CEFG: Where the Baſes FCG and FEG being conic Sections, their Areas will be given by *Art.* 115. 124 and 129. from whence the whole will be known. Thus, if HE be ſuppoſed parallel to AB, the Section FEG, then being a Parabola, its Area will be $= \frac{2}{3} \times FG \times EH$ *: Whence the Solidity of the Segment

* *Art.* 115.

Segment EFGA is $= \frac{2}{9} \times FG \times EH \times AM$: Which being deducted from that of CFGA (found by Help of the common Table of circular Segments) the Remainder will be the Content of the *Ungula*. But, if the Axis EH produced, cuts AB, the Section FEG will be a Segment of an Ellipſis EFKG; whoſe conjugate Axis (ſuppoſing EN and KL perpendicular to AD) is

* Art. 41. $= 2\sqrt{EN \times KL}$*. Now, in order to compute the Content, the eaſieſt way, in this Caſe, let the Ratio of EH to EK (which is given by Trigonometry) be expreſſed by that of *m* to Unity, and let the Ratio of CH to CB, be as *n* to Unity: And from the common Table of *Segments* (adapted to the Circle whoſe Diameter is Unity) let the Areas anſwering to the verſed Sines *m* and *n*, be taken and denoted by *M* and *N* reſpectively: Then, the Area of FEG being $= M \times EK \times$

† Art. 124 and 130. $2\sqrt{EN \times KL}$, and that of FCG $= N \times BC^2$†, the Content of the *Ungula*, by ſubſtituting theſe Values, will become $= \frac{1}{3} N \times BC^2 \times AD - \frac{1}{3} M \times EK \times AM \times 2\sqrt{EN \times KL}$: But, ſince AM : AE :: KQ (perpendicular to AC) : KE; and AN : AE :: KQ : KI, it follows, by Equality, that $AM \times KE = AN \times KI$; whence the Content of the *Ungula* is alſo expreſſed by $\frac{1}{3} N \times BC^2 \times AD - \frac{1}{3} M \times AN \times KI \times 2\sqrt{EN \times KL}$. Which, if H be ſuppoſed to coincide with B, and KI with BC, will become $\frac{(0.78539}{3}$ &c. $\times BC^2 \times AD - \frac{0.78539}{3}$ &c. $\times AN \times BC \times 2\sqrt{EN \times BD} = 0.26179$ &c. $\times BC \times \overline{BC \times AD - 2AN \times \sqrt{EN \times BD}}$.

When the Section EFG is an Hyperbola, its Area may be found by means of a Table of Logarithms (inſtead of a Table of Segments) whence the Content of the *Ungula* will likewiſe be had in that Caſe.

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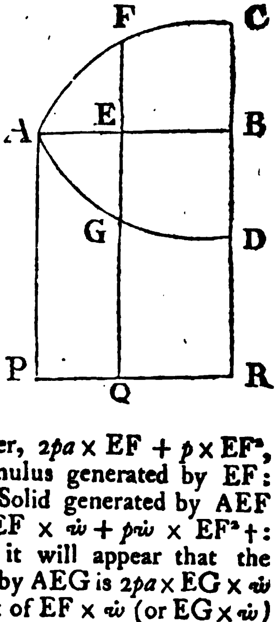
EXAMPLE XIII.

158. *Let* AFC, *or* AGD, *be a Curve of any Kind;* whoſe Area, and the Content of the Solid ariſing from its Rotation about its Axis, or Ordinate, AB, are both known; it is propoſed to find, from thence, the Content of the Solid generated by the Revolution of that Curve about any other Line PR parallel to the ſaid Axis or Ordinate AB,



Let AP, FQ, and CR be all perpendicular to AB and to the Axis of Motion PQR; alſo let AP (or EQ) $= a$, AE, conſidered as variable, $= w$, the Area AFE, or AEG $= M$, and the Solid, ariſing from its Revolution about AB, $= N$. It is plain that the Area of the Circle generated by QF will be $= p \times \mathrm{FQ}^2$ * $= p \times \overline{a + \mathrm{EF}}|^2 = pa^2 + 2pa \times \mathrm{EF} + p \times \mathrm{EF}^2$; from which deducting the Area, pa^2, generated by QE, the Remainder, $2pa \times \mathrm{EF} + p \times \mathrm{EF}^2$, will be the Area of the Annulus generated by EF: Whence the Fluxion of the Solid generated by AEF is truly repreſented by $2pa \times \mathrm{EF} \times \dot{w} + p\dot{w} \times \mathrm{EF}^2$ †: And, in the ſame manner, it will appear that the Fluxion of the Solid generated by AEG is $2pa \times \mathrm{EG} \times \dot{w} - p\dot{w} \times \mathrm{EG}^2$. But the Fluent of $\mathrm{EF} \times \dot{w}$ (or $\mathrm{EG} \times \dot{w}$) is $=$ the Area *(M)* of AEF (or AEG) ‡, and that of $p\dot{w} \times \mathrm{EF}^2$ (or $p\dot{w} \times \mathrm{EG}^2$) equal to *(N)* the given Solid ariſing from that Area §; therefore the Fluent of the *Whole*, or the Solidity required, is $2paM + N$, in the former Caſe, and $2paM - N$ in the latter; where $2pa$,

in

* Art. 145.

† Art. 145.

‡ Art. 112.

§ Art. 145.

in either Caſe, expreſſes the Periphery of the Cylinder deſcribed by AB, about the Axis of Rotation PR.

Hence, if ABC and ABD are equal and ſimilar to each other, then the Value of *M* &c. being the ſame in both Caſes, it follows that the Content of the Solid generated by AFG will be expreſſed by $2pa \times 2M$, or $2pa \times$ Area AFG.

Now, if (for Example ſake) ACD be ſuppoſed a Circle, whoſe Semi-diameter is d, the Area of that Circle being $= pd^2$, the Solid generated by its Revolution (repreſenting the Ring of an Anchor) will therefore be $= 2pa \times pd^2 = 2p^2ad^2$. But if you would know the Content of the Part generated by the upper Semicircle BAC, or the lower one BAD, let the Content $\left(\frac{4pd^3}{3}\right)$ * of a Sphere whoſe Semi-diameter is d, be wrote for N, in each of the two foregoing Expreſſions, and you will then get $p^2ad^2 + \frac{4pd^3}{3}$, and $p^2ad^2 - \frac{4pd^3}{3}$.

* Art. 148.

Again, if AFC, and AGD be taken as Right-lines, you will have $M = \frac{AB \times BC}{2}$ $\left(\text{or } \frac{AB \times BD}{2}\right)$ and $N = p \times BC^2 \times \frac{1}{3} AB$ (or $p \times BD^2 \times \frac{1}{3} AB$) †: Hence the Solid generated by the Triangle ABC is $\left(= 2pa \times \frac{AB \times BC}{2} + \frac{p}{3} \times BC^2 \times AB\right) = p \times AB \times BC \times \overline{RB + \frac{1}{3} BC}$; and that generated by ABD $\left(= 2pa \times \frac{AB \times BD}{2} - \frac{p}{3} \times BD^2 \times AB\right) = p \times AB \times BD \times \overline{RB - \frac{1}{3} BD}$.

† Art. 146.

Laſtly, let ABC (or ABD) be conſidered as a Parabola, whoſe Ordinate is AB, and Axis CB (or DB): Then M being here $= \frac{2}{3} AB \times BC$ (or $\frac{2}{3} AB \times BD$) ‡ and $N = \frac{8p}{15} \times AB \times BC^2$ § (or $\frac{8p}{15} \times AB \times BD^2$) it

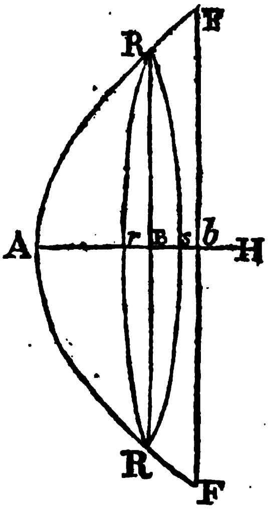
‡ Art. 115.

§ Art. 152.

it follows that the Solid generated by ABC will be $(= 2pa \times \frac{2}{3}AB \times BC + \frac{8p}{15} \times AB \times BC^2) = 4p \times AB \times BC \times \frac{5BR + 2BC}{15}$, and that generated by ABD $= 4p \times AB \times BD \times \frac{5BR - 2BD}{15}$.

SECTION X.

The Uſe of Fluxions in finding the Superficies of ſolid Bodies.



159. LET FAF repreſent a Solid generated by the Revolution of any given Curve AF about its Axis AH; alſo let a Circle, whoſe Diameter is the variable Line (or Ordinate) RBR, be conceived to move uniformly from A towards FF, and to dilate itſelf ſo, on all Sides, at the ſame time, as to generate, by its Periphery, the propoſed Superficies RAR: Then the Length of that Periphery, or the generating Line, being expreſſed by 3,141592 * *&c.* × RR $(= 2py)$ and the Celerity with which it moves by $\dot{z}$ † the Fluxion of the Superficies RAR, or the Space that

* Art. 142.

† Art. 135.

would be uniformly generated in the time of deſcribing $\dot{z}$, will therefore be truly repreſented by $2py\dot{z}$.

Hence, if w be taken to repreſent the whole Surface RAR, generated from the beginning (according to the Method obſerved in the three laſt Sections) we ſhall have $\dot{w} = 2py\dot{z} = 2py\sqrt{\dot{x}^2 + \dot{y}^2}$ * ; whence w itſelf may be found.

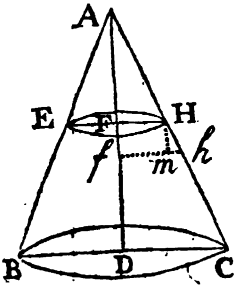
* Art. 135.

EXAMPLE I.

160. *Let it be propoſed to determine the convex Superficies of a Cone* ABC.

Then, the Semi-diameter of the Baſe (BD, or CD) being put $= b$, the ſlanting Line, or Hypothenuſe, AC $= c$, and FH (parallel to DC) $= y$ *&c.* we ſhall, from the Similarity of the Triangles ADC and H*mh*, have $b : c :: \dot{y}$ (mh) : $\dot{z}$ $(Hh) = \frac{c\dot{y}}{b}$: Whence $\dot{w}$ $(2py\dot{z}$ †$)$ $= \frac{2pcy\dot{y}}{b}$; and conſequently $w = \frac{pcy^2}{b}$. This, when $y = b$, becomes $= pcb = p$ x DC x AC $=$ the convex Superficies of the whole Cone ABC: Which therefore is equal to a Rectangle under half the Circumference of the Baſe and the ſlanting Line.

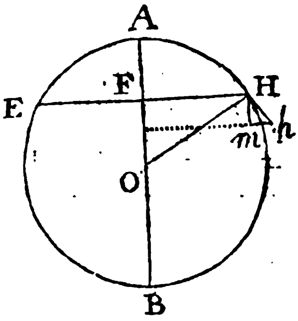
† Art. 159.



EXAMPLE II.

161. *Let the Solid, whoſe Surface you would find, be a Sphere* AEBH.

In which Caſe, putting the Radius OH $=a$, AF $=x$, Hm $=\dot{x}$, *&c.* we ſhall (by reaſon of the ſimilar Triangles OHF and Hmb *) have y (FH) : a (OH) :: $\dot{x}$ (Hm) : $\dot{z}$ (Hb) $=\frac{a\dot{x}}{y}$: Therefore $\dot{w}$ $(2py\dot{z})=2pa\dot{x}$; and conſequently the Superficies (w) itſelf $=2pax=$ AF × *Periph.* AEBH. Which, if the whole Sphere be taken, will become AB × *Periph.* AEBH = four times the Area BEAHO.



Hence the Superficies of a Sphere is equal to four times the Area of its greateſt Circle: And the convex Superficies of any Segment thereof, is to that of the *Whole*, as the Axis (or Thickneſs) of the Segment to the Diameter of the Sphere.

* Art. 6[illegible].

EXAMPLE III.

162. *Wherein let the parabolic Conoid be propoſed.*

The Equation of the generating Parabola being $ax=y^2$, or $x=\frac{y^2}{a}$, we have $\dot{x}=\frac{2y\dot{y}}{a}$, and therefore $\dot{z}\ (=\sqrt{\dot{y}^2+\dot{x}^2}\,\dagger)=\sqrt{\dot{y}^2+\frac{4y^2\dot{y}^2}{a}}=\frac{\dot{y}\sqrt{a^2+4y^2}}{a}$: Hence $\dot{w}$ $(2py\dot{z})=\frac{2py\dot{y}}{a}\times\overline{a^2+4y^2}\big|^{\frac{1}{2}}$; whereof the

† Art. 135.

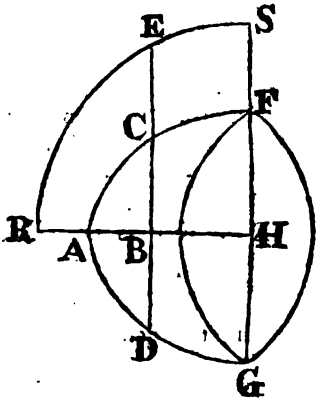
Fluent is $\frac{p\times\overline{a^2+4y^2}|^{\frac{3}{2}}}{6a}$; which corrected (by ſup-

* Art. 79. poſing $y=0$ *) gives $\frac{p\times\overline{a^2+4y^2}|^{\frac{3}{2}}}{6a}-\frac{pa^2}{6}$, for the Superficies ſought.

EXAMPLE IV.

163. *Let it be required to determine the Superficies of a Spheroid.*

Let ACFHG repreſent one half of the propoſed Spheroid, generated by the Rotation of the Semi-ellipſis FAG, about its Axis AH; put $AH=a$, FH (or HG) $=c$, $BH=x$, $BC=y$, $FC=z$, and the Superficies generated by FC (or GD) $=w$: Then, from the Na-



ture of the Ellipſis, we have $y=\frac{c}{a}\sqrt{a^2-x^2}$; whence

† Art. 135. $\dot{y}=-\frac{cx\dot{x}}{a\sqrt{a^2-x^2}}$, and conſequently $\dot{z}$ $(=\sqrt{\dot{x}^2+\dot{y}^2}$ †)

=

$= \sqrt{\dot{x}^2 + \frac{c^2x^2\dot{x}^2}{a^2\times\overline{a^2-x^2}}} = \frac{\dot{x}\sqrt{a^4-\overline{aa-cc}\times xx}}{a\sqrt{aa-xx}} = \frac{\dot{x}\sqrt{a^4-b^2x^2}}{a\sqrt{a^2-x^2}}$ = (by putting (the Excentricity) $\sqrt{a^2-c^2}=b$) $= \frac{b\dot{x}\sqrt{\frac{a^4}{bb}-x^2}}{a\sqrt{a^2-x^2}}$: Therefore, in this Caſe, $\dot{w}\ (2py\dot{z}) = \frac{2pbc\dot{x}}{aa}\sqrt{\frac{a^4}{bb}-x^2}$; whoſe Fluent, in an Infinite Series, is $2pcx \times \overline{1 - \frac{b^2x^2}{2.3a^4} - \frac{b^4x^4}{2.4.5a^8} - \frac{3b^6x^6}{2.4.6.7a^{12}}}$. But the ſame Fluent may be, *otherwiſe*, very eaſily exhibited by means of the Area of a Circle: For, if from the Center H, with a Radius equal to $\frac{aa}{b}$, a Circle SER be deſcribed, and the Ordinate BC be produced to interſect it in E, it is evident that BE $= \sqrt{\frac{a^4}{bb}-xx}$, and that the Fluxion of the Area ESHB will be expreſſed by $\dot{x}\sqrt{\frac{a^4}{bb}-x^2}$; which being to $\frac{2pbc\dot{x}}{aa}\times\sqrt{\frac{a^4}{bb}-x^2}$, the Fluxion before found, in the conſtant Ratio of 1 to $\frac{2pbc}{a^2}$, their Fluents muſt therefore be in the ſame Ratio; and ſo the latter, expreſſing the Superficies CFGD, will conſequently be $= \frac{2pbc}{aa}\times$ BESFH $= 2p \times \frac{\text{FH}}{\text{HS}} \times$ BESFH.

This Solution, it may be obſerved, obtains only in Caſe of an *oblong* Spheroid, generated by the Rotation of the Ellipſis about its greater Axis; for, in an *oblate*

Spheroid, generated about the leſſer Axis, the Value of b $(\sqrt{a^2 - c^2})$ will be impoſſible; ſince, in this Caſe HF is greater than HA. But, if we, *here*, put $b = \sqrt{c^2 - a^2}$, and $d = \frac{a^2}{b}$, the Value of $\dot{w}$ (found above) will become $= \frac{2pbc\dot{x}}{a^2} \sqrt{\frac{a^4}{bb} + x^2} = \frac{2pc\dot{x}}{d} \sqrt{d^2 + x^2}$ $= \frac{2pc}{d} \times \dot{x} \sqrt{a^2 + x^2}$: Whoſe Fluent may be brought out by help of a Table of Logarithms: For, let the variable Part $\dot{x} \sqrt{d^2 + x^2}$ be tranſformed to $\left(\frac{\dot{x} \times \overline{d^2 + x^2}}{\sqrt{d^2 + x^2}} = \frac{d^2\dot{x} + x^2\dot{x}}{\sqrt{d^2 + x^2}} = \frac{d^2x\dot{x} + x^3\dot{x}}{\sqrt{d^2x^2 + x^4}} = \right) \frac{\frac{1}{2}d^2x\dot{x} + x^3\dot{x}}{\sqrt{d^2x^2 + x^4}} + \frac{\frac{1}{2}d^2x\dot{x}}{\sqrt{d^2x^2 + x^4}}$, ſo that the Numerator of the firſt Term $\frac{\frac{1}{2}d^2x\dot{x} + x^3\dot{x}}{\sqrt{d^2x^2 + x^4}}$ (now in a given Ratio to the Fluxion of the Quantity under the radical
* Art. 77. Sign) may be had by the common Rule *; by which means we get $\frac{1}{2} \sqrt{d^2x^2 + x^4}$, for the true Fluent of the ſaid Term; to which adding the Fluent of the other Term $\frac{\frac{1}{2}d^2x\dot{x}}{\sqrt{d^2x^2 + x^4}}$, or $\frac{\frac{1}{2}d^2\dot{x}}{\sqrt{d^2 + x^2}}$ (given by *Art.* 126.) there ariſes $\frac{1}{2} x \sqrt{d^2 + x^2} + \frac{1}{2} d^2 \times$ hyp. Log. $x + \sqrt{d^2 + x^2}$, for the Fluent of $\dot{x} \sqrt{d^2 + x^2}$: And
† Art. 78. this, corrected † and multiplied by $\frac{2pc}{d}$, gives $\frac{pcx}{d} \sqrt{d^2 + x^2} + pcd \times$ hyp. Log. $\frac{x + \sqrt{dd + xx}}{d}$, for the Superficies in this Caſe, where the propoſed Spheroid is an oblate One.

E X-

EXAMPLE V.

164. *Let the Solid, whoſe Superficies is ſought, be the hyperbolical Conoid.*

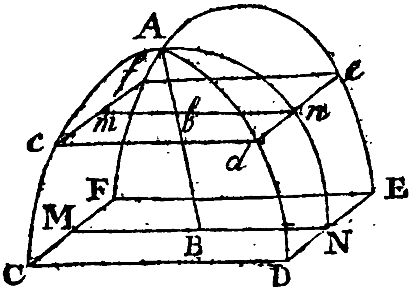
Let the ſemi-tranſverſe Axis, of the generating Hyperbola, $=a$, the ſemi-conjugate $=c$, and the Diſtance of any Ordinate from the Center thereof $=x$; then from the Nature of the Curve you will have $y = \frac{c}{a}\sqrt{x^2 - a^2}$; whence $\dot{y} = \frac{c x \dot{x}}{a\sqrt{xx - aa}}$, $\dot{z} = \frac{\dot{x}\sqrt{\overline{a^2+c^2}\times x^2 - a^4}}{a\sqrt{xx-aa}}$, and $\dot{s}$ $(2py\dot{z}) = \frac{2pc\dot{x}}{aa} \times \sqrt{\overline{aa+cc}\times xx - a^4}$; which laſt Value, if d^2 be put $= \frac{a^4}{a^2+c^2}$, will be more commodiouſly expreſſed by $\frac{2pc\dot{x}}{d}\sqrt{x^2-d^2}$: whereof the Fluent, by proceeding as in the latter Part of the foregoing Example, will come out $= \frac{pcx\sqrt{xx-dd}}{d} - pcd \times$ *hyp. Log.* $x + \sqrt{x^2 - d^2}$: Which corrected (by taking $x=q$) becomes $\frac{pcx}{d}\sqrt{xx-dd} - pc^2, - pcd \times$ *hyp. Log.* $\frac{x+\sqrt{x^2-d^2}}{a+\frac{cd}{a}}$, the true Meaſure of the required Superficies.

EXAMPLE VI.

165. *Let it be propoſed to find the Superficies of the Solid called a Groin.* (Vid. Art. 155.)

Let *bcdef* be any Section of the Solid parallel to the Baſe thereof, and let x denote its Diſtance from the

Vertex A, alſo put *z* equal to the correſponding Arch A*n* of the ſemi-circular Section N*n*A *&c.* whoſe Radius AB or BN let be denoted by *a*.



It appears from *Art.* 161. that $\dot{z} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$:

* Art. 159. Which Value, multiplied by $(2\sqrt{2ax - xx})$ that of *de* $(= 2bn)$ gives $2a\dot{x}$ * for the Fluxion of one of the four equal convex Superficies by which the Solid is bounded. Hence the whole Superficies (excluding the Baſe) comes out $= 8a^2$: Which therefore is exactly equal to twice the Baſe.

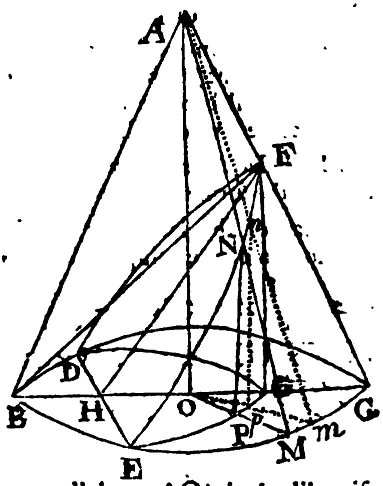
If the Solid be ſuppoſed a Groin of any other Kind, ſuch that its two equal Sections, through the Middle of the oppoſite Sides, are other Curves than Circles, the Superficies may *ſtill* be had in the ſame manner; and will be always in proportion to the Superficies ariſing from the Revolution of either of the ſaid equal Curves about its Axis, as a Square is to its inſcribed Circle. Thus, the Superficies of a parabolic Conoid being $=$ $\frac{p \times \overline{aa + 4yy}|^{\frac{3}{2}}}{6a} - \frac{pa^2}{6}$ (by *Art.* 162.) the convex Superficies of the Groin, ſuppoſing the generating Curve A*n*N to be a Parabola, will therefore be $=$ $\frac{4 \times \overline{a^2 + 4yy}|^{\frac{3}{2}}}{6a} - \frac{4a^2}{6}$.

EX-

EXAMPLE VII.

166. *Wherein let it be required to find the convex Superficies of a conical* Ungula ECFD; formed by a Plane DFE paſſing through the Baſe of the Cone.

Let a right-angled Triangle AOM (whoſe Baſe OM is the Radius of the Circle BDCE) be ſuppoſed to revolve about the Axis AO; whilſt a Right-line NP, drawn perpendicular to OM from the interſection of AM and the Arch EFD, traces out, upon the Baſe of the Cone, the Curve-line EPGD.



If MPOAN and *mp*OA*n* be conſidered as two Poſitions of the generating Triangle indefinitely near to each other, it is evident that the Space MA*m*, generated by AM, will be to the Space MO*m*, generated by OM, as AM to OM, or OB. Whence, MN and MP being proportional Parts of AM and OM (becauſe NP is parallel to AO) it is likewiſe plain that the Spaces MN*nm* and MP*pm*, generated by thoſe Parts, will be to each other in the ſame Ratio of AM to OB. And ſince this every where holds, it follows that the whole Space (ENM) *&c.* generated by MN, will be to that (EPM) generated by PM, as AM to OB: And ſo the whole required Superficies (generated by AM) is truly repreſented by $\frac{AM}{OB} \times$ Area EPGDCE.

But now, to find this Area, EPGDCE, it is obſervable that the Area of the Plane DFE (being the Segment of a Conic-ſection) is given, *by Art.* 115. 129 or 130. And it is very eaſy to apprehend and demonſtrate that the Area ſo given will be to that of EGDH, as the Radius to the Co-ſine of the Angle of the Inclination of the ſaid Plane to the Baſe, or as HF to HG. Therefore, ſeeing EGDH is $= \frac{HG}{HF} \times EFD$, we have EPGDCE $(= ECDHE - EGDH) = ECDHE - \frac{HG}{HF} \times EFD$; and conſequently $\frac{AM}{OB} \times EPGCDE = \frac{AM}{OB} \times ECDHE - \frac{AM \times HG}{OB \times HF} \times EFD =$ the convex Superficies that was to be found.

If the Point H be ſuppoſed to coincide with B, ECDHE will become the *whole* Circle CB; and EDF will become a whole Ellipſis, whoſe greater Axis is BF, and its leſſer Axis $= 2\sqrt{OB \times OG}$.* Therefore, the Area of the former Figure will be expreſſed by $p \times BO^2$†, and that of the latter by $p \times \frac{1}{2} BF \times \sqrt{OB \times OG}$; and ſo the convex Superficies of the Part BFC will be $(= \frac{AM}{OB} \times p \times BO^2 - \frac{AM \times BG}{OB \times BF} \times p \times \frac{1}{2} BF \times \sqrt{OB \times OG}) = p \times AM \times OB - p \times AM \times \frac{1}{2} BG \times \sqrt{\frac{OG}{OB}}$: Which being deducted from $(p \times AM \times OB)$ the Superficies of the whole Cone BAC, there reſts $p \times AM \times \frac{1}{2} BG \times \sqrt{\frac{OG}{OB}}$, for the Superficies of the oblique Cone BAF; which from hence is alſo given.

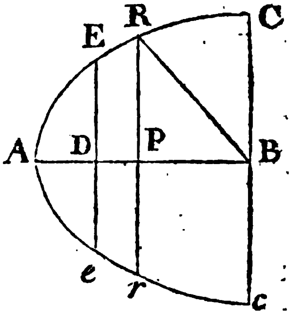
* Art. 41.

† Art. 124.

SCHO-

SCHOLIUM.

167. In moſt of the Examples, delivered in the four laſt Sections, the Part of the propoſed *Figure* next the Vertex, whether, a Curve, Solid, or Superficies, is firſt found; from whence, by taking the Altitude (x) of that Part equal to (a) the Altitude given, the Content of the *Whole* is deduced: But, if the Content of the lower Segment (BCED) of any Figure (ABC) ariſing by taking away a Part (ADE) next the Vertex, be required; then the Difference between the *Whole* and the Part taken away (found as before explained) will be the Quantity ſought.



Thus, for Example, let ABC be the common *Parabola*, and let it be propoſed to find the Content of the Part, BCED, included between any two Ordinates BC (b) and DE (c) at a given Diſtance BD (d) from each other: Then, the Equation of the Curve being $ax = y^2$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, and therefore $y\dot{x}$ * $= \frac{2y^2\dot{y}}{a}$, whoſe Fluent $\frac{2y^3}{3a}$ is a general Expreſſion for the Area comprehended between the Vertex and the Ordinate y: Whence, expounding y, by b and c ſucceſſively, we get $\frac{2b^3}{3a}$ and $\frac{2c^3}{3a}$ for the correſponding Values of ABC and ADE; whoſe Difference $\frac{2b^3 - 2c^3}{3a}$ is the required Area BCED: But, to expreſs the ſame independent of a, it will be, by the Property of the Curve, $b^2 : c^2 :: \text{AB} : \text{AD}$; whence,

* Art. 112.

whence, by Diviſion, $b^2 : b^2 - c^2 :: \mathrm{AB} : \mathrm{BD}$ (d) and conſequently $\frac{b^2 - c^2}{d} = \frac{b^2}{\mathrm{AB}} = a$; which firſt Value being wrote inſtead of a, there reſults $\mathrm{BCED} = \frac{\overline{2b^3 - 2c^3} \times d}{3b^2 - 3c^2}$ $= \frac{2d}{3} \times \frac{b^2 + bc + c^2}{b + c}$.

After the ſame Manner, the Segments of other Figures may be found; but in many Caſes they will be more readily had from a direct Inveſtigation, without either finding the Whole or the Part taken away.

Thus, in the Caſe above, if the Exceſs of any Ordinate RP above DE (c) be denoted by w, we ſhall have, by the Property of the Curve, $b^2 - c^2$ $(\mathrm{BC}^2 - \mathrm{DE}^2) : \overline{c + w}|^2 - c^2$ $(\mathrm{RP}^2 - \mathrm{DE}^2) :: \mathrm{DB}$ $(d) : \mathrm{DP} = \frac{d \times \overline{2cw + w^2}}{b^2 - c^2}$; whoſe Fluxion $\left(d \times \frac{2c\dot{w} + 2w\dot{w}}{b^2 - c^2}\right)$ multiplied by $c + w$ $(= \mathrm{PR})$ gives $d \times \frac{2c^2\dot{w} + 4cw\dot{w} + 2w^2\dot{w}}{b^2 - c^2}$, for the Fluxion of the Area DPRE: Whereof the Fluent $\left(\text{which is } 2dw \times \frac{c^2 + cw + \frac{1}{3}w^2}{b^2 - c^2}\right)$ will, when $w = b - c$ (or $\mathrm{RP} = \mathrm{BC}$) be truly expounded by $\frac{2d \times \overline{b - c} \times \overline{\frac{1}{3}b^2 + \frac{1}{3}bc + \frac{1}{3}c^2}}{b^2 - c^2}$ or its Equal, $\frac{2d}{3} \times \frac{b^2 + bc + c^2}{b + c}$; *the ſame as before.*

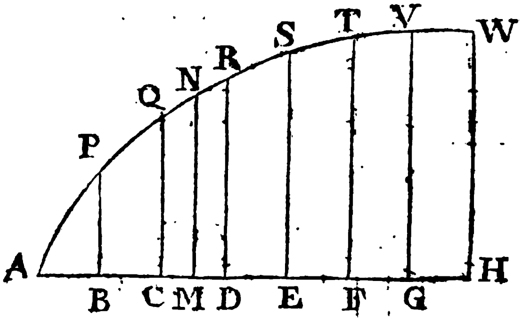
Again, for another Example, let CED*ec* be conſidered as the lower Fruſtrum of an Hemiſphere, whoſe Center is the Point B: Then, BP being *here*, denoted by w, we ſhall have y^2 $(= \mathrm{BR}^2 - \mathrm{BP}^2) = b^2 - w^2$, and conſequently $py^2\dot{w}$ * $= p \times \overline{b^2\dot{w} - w^2\dot{w}}$; whoſe Fluent

* Art. 145.

Fluent ($p \times \overline{b^2w - \frac{1}{3}w^3} = \frac{1}{3}pw \times \overline{3b^2 - w^2} = \frac{1}{3}pw \times \overline{2b^2 + b^2 - w^2} = \frac{1}{3}pw \times \overline{2b^2 + y^2} = \frac{1}{3}p \times BP \times \overline{2BC^2 + PK^2}$) is the true Content of the Part CED&c; which will alſo hold when the Figure is a Spheroid.

This laſt Method, of finding the Content of a Portion of a Figure, remote from the Vertex, will be of Service, when the general Value, for the *Whole*, cannot be expreſſed without an infinite Series; becauſe ſuch a Series, in that Caſe, not converging, becomes uſeleſs *.

By dividing the whole propoſed Figure, AHW, into a Number of ſuch Portions, HV, GT, FS, &c. the Content thereof may be obtained, when to find it at once, by a Series, commencing from the Vertex, would be altogether impracticable.



But, to render ſuch an Operation as ſhort and eaſy as may be, it will be proper to find each Part (DQ, &c.) of the Figure, by means of a Series proceeding both Ways, from the middle Ordinate (MN) between the two correſponding Extremes (CR and DR).

Thus, let the Value of MN (found by the Property of the Curve) be denoted by a; and let the Value of DR, in a Series, be repreſented by $a + bx + cx^2 + dx^3 + ex^4 + fx^5 +$ &c. where $x = MD$; then the Area MDRN will be repreſented by the Fluent of $\dot{x} \times \overline{a + bx + cx^2 + dx^3 +}$

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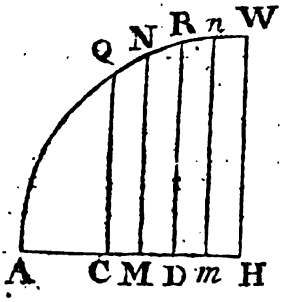
* Art. 93.

&c. or by $x \times \overline{a + \frac{bx}{2} + \frac{cx^2}{3} + \frac{dx^3}{4}}$ + &c. And by writing $-x$ inſtead of x, the Ordinate CQ will be expreſſed by $a - bx + cx^2 - dx^3$ &c. and the Area MCQN, by $x \times \overline{a - \frac{bx}{2} + \frac{cx^2}{3} - \frac{dx^3}{4} + \frac{ex^4}{5}}$ &c. whence the Area CDRQ is $= 2x \times \overline{a + \frac{cx^2}{3} + \frac{ex^4}{5} + \frac{gx^6}{7}}$ + &c. Therefore, if DE, EF, FG, and GH be ſuppoſed, each, = BC ($2x$) and the Areas DS, ET, &c. (found as above) be denoted by $2x \times \overline{a' + \frac{c'x^2}{3} + \frac{e'x^4}{5}}$ &c. and $2x \times \overline{a'' + \frac{c''x^2}{3} + \frac{e''x^4}{5}}$ &c. reſpectively, it follows that the Area CR + DS + ET will be repreſented by $2x \times \overline{a + a' + a''}$ &c. $+ \frac{2}{3}x^3 \times \overline{c + c' + c''}$ &c. $+ \frac{2}{5}x^5 \times \overline{e + e' + e''}$ &c.



An Example will ſhew the Uſe of this laſt Expreſſion: Let CHWQ be a Portion of a Quadrant HAW of a Circle, whoſe Baſe HC (conceived to be divided into four equal Parts) is equal half the Radius AH, repreſented by *Unity*. Then, putting CM (= DM = Dm = mH = $\frac{1}{8}$) = x, HM (= $\frac{3}{8}$) = p, and Hm (= $\frac{1}{8}$) = q, we have, by the Property of the Circle, a (MN) = $\sqrt{\text{HN}^2 - \text{HM}^2} = \sqrt{1 - pp}$, and

DR $(=\sqrt{HR^2-HD^2}) = \sqrt{1-\overline{p-x}^2} = \sqrt{1-p^2+2px-x^2} = \sqrt{a^2+2px-x^2}$; which, in a Series, is $\left(=a+\frac{2px-x^2}{2a}-\frac{\overline{2px-x^2}^2}{8a^3}+\text{\&c.}\right) = a+\frac{px}{a}-\overline{\frac{1}{2a}+\frac{p^2}{2a^3}}\times x^2$ &c. Therefore, in this Caſe, $b=\frac{p}{a}$, $c=-\overline{\frac{1}{2a}+\frac{p^2}{2a^3}}$, &c. Which Value of c, by writing $1-a^2$ for its Equal p^2, will be reduced to $-\frac{1}{2a^3}$. From whence it is alſo evident that $\overset{\prime}{c}=-\frac{1}{2\overset{\prime}{a}{}^3}$ (ſuppoſing $\overset{\prime}{a}$ (mn) $=\sqrt{1-q^2}$)

Conſequently $2x\times\overline{a+\overset{\prime}{a}+\overset{\prime\prime}{a}\text{ \&c.}}+\frac{2}{3}x^3\times\overline{c+\overset{\prime}{c}+\overset{\prime\prime}{c}\text{ \&c.}}+\frac{2}{5}x^5\times\overline{e+\overset{\prime}{e}+\overset{\prime\prime}{e}\text{ \&c.}}\left(=\overline{a+\overset{\prime}{a}}\times 2x+\overline{c+\overset{\prime}{c}}\times\frac{2x^3}{3}\right)=\overline{\sqrt{\frac{55}{64}}+\sqrt{\frac{63}{64}}}\times\frac{1}{4}-$

$$\overline{\frac{1}{\frac{2\times55}{64}\sqrt{\frac{55}{64}}}+\frac{1}{\frac{2\times63}{64}\sqrt{\frac{63}{64}}}}\times\frac{2}{64\times8\times3}=$$

$$\frac{\sqrt{55}+\sqrt{63}}{32}-\frac{1}{3\times55\sqrt{55}}-\frac{1}{3\times63\sqrt{63}}=$$

$$\frac{\sqrt{55}+\sqrt{63}}{32}-\frac{\sqrt{55}}{3\times55\times55}-\frac{\sqrt{63}}{3\times63\times63}=$$

0,48730 = the Area, CHWQ, that was to be found.

This Example, choſen as an Illuſtration of the foregoing Method, may indeed be wrought the common Way; whence the very ſame Concluſion is brought out (*Vide*

(*Vide Art.* 124.) But that Method is alſo applicable to any other Caſe, whether the Part propoſed be near to the Vertex, or remote from it; and whether the Figure itſelf be a Curve, Solid or Superficies; ſince the Meaſure thereof may, always, be expreſſed by the Area of a Curve.

There is another Way, well known to Mathematicians, whereby the Area of a Curve may be determined, by means of a Number of equidiſtant Ordinates; which Method, derived from *that of Differences*, may, alſo, be uſed to good Purpoſe, in Caſes like thoſe above ſpecified: But, it having been treated of by ſeveral *others*, and alſo in my *Diſſertations*, the Reader will excuſe me, if no further Notice is taken of it here.

SECTION XI.

Of the Uſe of FLUXIONS *in finding the Centers of Gravity, Percuſſion, and Oſcillation of Bodies.*

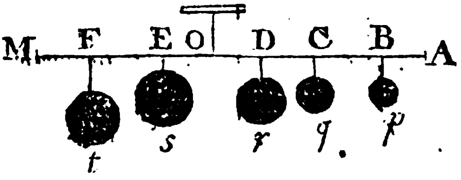
168. THE Center of Gravity is that Point of a Body, by which, if it were ſuſpended, it would reſt *in Equilibrio*, in any Poſition.

LEMMA.

169. *Let p, q, r, s, &c. be any Number of given Weights, hanging at an inflexible Line (or Rod)* AM *ſuſpended* in Equilibrio, *in an horizontal Poſition, at the Point* O; *to determine the Poſition of that Point.*

Since (*by Mechanics*) the Force of any Weight (p) to raiſe the oppoſite End (M) of the Balance, is as that Weight drawn into its Diſtance (BO) from the Fulcrum,

crum, we ſhall, from the Equality of theſe Forces, have $p \times OB + q \times OO + r \times OD = s \times OE + t \times OF$.



that is $p \times \overline{AO - AB} + q \times \overline{AO - AC} + r \times \overline{AO - AD} = s \times \overline{AE - AO} + t \times \overline{AF - AO}$, and conſequently $AO = \dfrac{p \times AB + q \times AC + r \times AD + s \times AE + t \times AF}{p+q+r+s+t}$.

From which it appears, that, *if each Weight be multiply'd by its Diſtance from the End (or any given Point) of the Axis, the Sum of all the Products divided by the Sum of all the Weights, will give the Diſtance of the Center of Gravity from that End (or Point.)*

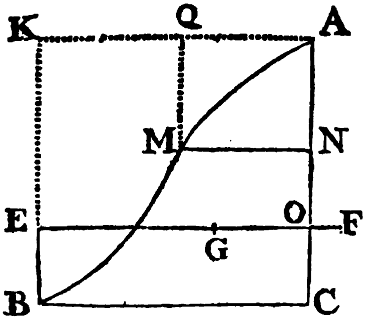
Note. The Products here mentioned are, uſually, call'd *the Forces*, of their reſpective Weights; not in reſpect to their Action at the Center O (which is expreſſed by a different Quantity) but with regard to the Effects they have in the Concluſion, or the Value of AO; which appear to be in that Ratio.

PROPOSITION I.

170. *To determine the Center of Gravity of a Line, Plane, Superficies, or Solid* (admitting the three former capable of being affected by Gravity.)

Let AMBC be the propoſed Figure, and G the Center of Gravity thereof; thro' which, parallel to the Horizon, let the Line EF be drawn, interſecting AC, at Right-angles, in O; alſo let AK and NM be perpendicular to AC, and parallel to EF.

171. Caſe



171. Caſe 1. *If the Figure* AMBC *be a Plane*; let it be ſuppoſed to reſt *in Equilibrio* upon the Line EF; and then, if the Line MN be conſider'd as a Weight, its Force (*defined above*) will be expreſſed by MN drawn into its Diſtance (AN) from the End of the Axis AC; that is by yx (ſuppoſing, as uſual, AN$=x$ and MN$=y$.) This, therefore, multiply'd by the Fluxion of AN, gives $yx\dot{x}$ for the Fluxion of the Force of the Plane AMN; whoſe Fluent, when $x=$AC, expreſſes the Force of the whole Plane, or the Sum of all the Products of the Ordinates (or Weights) by their reſpective Diſtances from AK: Which Fluent being, therefore, divided by the Area ABC, or the Fluent of $y\dot{x}$ (according to the foregoing Lemma) the Quotient $\left(\frac{Flu.\ yx\dot{x}}{Flu.\ y\dot{x}}\right)$ will give (AO) the Diſtance of the Center of Gravity from the Line AK.

172. Caſe 2. *If the Figure be a Solid*; let MN be a Section thereof by a Plane perpendicular to the Horizon; then, the Area of that Section being denoted by A, the Force thereof (conſidered as above) will be expreſſed by Ax, and the Fluxion of the Force of the Solid AMN by $Ax\dot{x}$; whoſe Fluent, divided by the Content of the Body, or the Fluent of $A\dot{x}$, gives AO, in this Caſe. But, if the Solid be the half (or the whole) of that ariſing from the Rotation of a Curve AMB about its Axis AC; then (putting p for the Area of the Circle whoſe Radius is Unity) A will become $= \frac{1}{2}py^2$ *; and conſequently AO $= \frac{Flu.\ \frac{1}{2}py^2x\dot{x}}{Flu.\ \frac{1}{2}py^2\dot{x}} = \frac{Flu.\ y^2x\dot{x}}{Flu.\ y^2\dot{x}}$.

173. Caſe

* Art. 145.

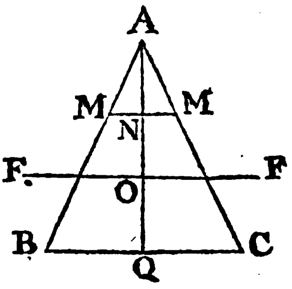
173. Caſe 3. *If the Figure propoſed be the Curve-line* AMB; then, the Force of a Particle at M being expreſſed by AN or MQ (x) we ſhall (putting AM $= z$) have $\frac{Flu.\, x\dot{z}}{z} = AO$.

174. Caſe 4. *But if the Figure given be the Superficies generated by the Rotation of* AMB *about* AC.

Then, the Periphery of the Circle generated by the Point M being $= 2py$, it follows that $\frac{Flu.\, 2pyx\dot{z}}{Flu.\, 2py\dot{z}} = \frac{Flu.\, yx\dot{z}}{Flu.\, y\dot{z}} = AO$.

EXAMPLE I.

175. *Let the Figure propoſed be the iſoſceles Triangle* ABC.



It is evident the Center of Gravity (O) will be ſomewhere in the Perpendicular AQ: And, if AQ $=a$, BC $=b$, AN $=x$, and MM $=y$; then y being $= \frac{bx}{a}$, we ſhall have, *by Caſe* 1, AO $\left(= \frac{Flu.\, yx\dot{x}}{Flu.\, y\dot{x}}\right) = \frac{Flu.\, x^2\dot{x}}{Flu.\, x\dot{x}} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2} = \frac{2x}{3} = \frac{2}{3}$ AQ, when $x =$ AQ; and conſequently OQ $= \frac{AQ}{3}$.

In the very ſame manner, the Center of Gravity of any other (plane) Triangle will appear to be at $\frac{1}{3}$ of the Altitude of the Triangle.

EX-

EXAMPLE II.

176. *Let the Figure propoſed be a Parabola of any Kind;*

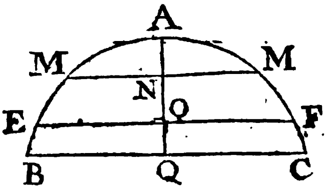
whereof the Equation is $y = \frac{x^n}{a^{n-1}}$.

Here, $\frac{\text{Flu. } yx\dot{x}}{\text{Flu. } y\dot{x}} = \frac{\text{Flu. } x^{n+1}\dot{x}}{\text{Flu. } x^n\dot{x}} = \frac{n+1}{n+2} \times x =$ the Diſtance of the Center of Gravity from the Vertex of the Curve.

EXAMPLE III.

177. *Let* BAC *be a Segment of a Circle.*

Then, if the Radius thereof be put $= r$, we ſhall have y (NM) $= \sqrt{2rx - xx}$: Whence the Fluent of $yx\dot{x}$ $(x\dot{x}\sqrt{2rx-xx})$ will, by *Art.* 163. be found $= -\frac{\overline{2rx-xx}|^{\frac{3}{2}}}{3} + r \times$ Area ANM; which divided by ANM, gives $r - \frac{\text{NM}^3}{3 \times \text{Area ANM}} = \text{AO}$*. This, therefore, when BAC is a Semi-circle, becomes $= \frac{576}{1000} \times r$, nearly.



But, with reſpect to the Center of Gravity of the Arch BAC; we have, *Flu.* $x\dot{z}$, *(by Caſe 3.)* = Fluent of $\frac{rx\dot{x}}{\sqrt{2rx-xx}} = r \times \overline{\text{AM} - \text{MN}}$; and conſequently AO *here* $= r - \frac{r \times \text{MN}}{\text{AM}}$.

* Art. 171.

EXAMPLE IV.

178. *Let* ABC (ſee the preceding Figure) *repreſent a Segment of a Sphere, or Spheroid.*

In which Caſe, denoting the Axis of the Sphere, or Spheroid, by a, and the other Axis of the generating Curve, when an Ellipſis, by b, we have $y^2 = \frac{bb}{aa} \times \overline{ax - xx}$;

and therefore $\frac{Flu.\ y^2 x\dot{x}}{Flu.\ y^2 \dot{x}}$ * $= \frac{Flu.\ \overline{ax - xx} \times x\dot{x}}{Flu.\ \overline{ax - xx} \times \dot{x}}$

$$= \frac{\frac{1}{3} ax^3 - \frac{1}{4} x^4}{\frac{1}{2} ax^2 - \frac{1}{3} x^3} = \frac{\frac{1}{3} ax - \frac{1}{4} x^2}{\frac{1}{2} a - \frac{1}{3} x} = \frac{x \times \overline{4a - 3x}}{6a - 4x} = \text{AO}.$$

If the Solid be an hyperbolical Conoid, the Diſtance (AO) of its Center of Gravity from the Vertex, will alſo be exhibited by the Expreſſion here brought out, when the negative Signs are changed to poſitive ones.

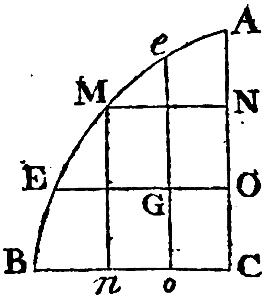
179. In thoſe Caſes where the Figure cannot be divided into two Parts, equal and like to each other (as a Curve is by its Axis, &c.) the Poſition of two Lines EO, *eo* (*ſee the enſuing Figure*) muſt be determined, *as above*; in whoſe Interſection (G) the Center of Gravity will be found.

* Art. 177.

EXAMPLE V.

Let ABC *be a Semi-parabola of any Kind*; whereof the Equation is $y = \frac{x^{n}}{a^{n-1}}$.

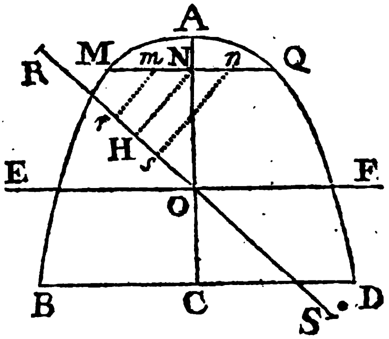
It appears, from *Ex.* 2. that (AO) the Diſtance of EGO from the Vertex, is expreſſed by $\frac{n+1}{n+2} \times$ AC: But to find the Poſition of *o*G*e* (perpendicular to EO) let M*n* be parallel to *eo*, or AC; then, AN being $= x$, and NM $(y) = \frac{x^n}{a^{n-1}}$; if AC be denoted by b, we



ſhall have $\mathrm{M}n = b - x$, and $\mathrm{M}n \times \mathrm{NM} \times \dot{y} = \overline{b - x} \times \frac{x^n}{a^{n-1}} \times \frac{nx^{n-1}\dot{x}}{a^{n-1}} = \frac{nbx^{2n-1}\dot{x} - nx^{2n}\dot{x}}{a^{2n-2}}$, for the Fluxion of the Sum of the Forces in this Caſe (*Vid. Art.* 171.) whoſe Fluent $\left(\frac{nbx^{2n}}{2na^{2n-2}} - \frac{nx^{2n+1}}{\overline{2n+1} \times a^{2n-2}}\right.$ $= \frac{x^{2n}}{a^{2n-2}} \times \overline{\frac{b}{2} - \frac{nx}{2n+1}} = y^2 \times \overline{\frac{b}{2} - \frac{nx}{2n+1}} = y^2 \times \frac{b}{4n+2}$, or $\frac{\mathrm{BC}^2 \times \mathrm{AC}}{4n+2}$, when $x = b$) divided by the Area ABC $\left(= \frac{\mathrm{BC} \times \mathrm{AC}}{n+1}\right)$ gives $\frac{n+1}{4n+2} \times$ BC for the true Value of Co, or OG. Which, in caſe of the common Parabola, where $n = \frac{1}{2}$, and where AO $\left(\frac{n+1}{n+2} \times \mathrm{AC}\right) = \frac{3}{5}$AC, will become $= \frac{3}{8}$CB.

Before I leave this Subject it may not be improper to take notice, *that*, whatever Line you found your Calculations upon, by ſuppoſing the Figure to reſt, *in Equilibrio*,

Equilibrio, upon that Line, the very ſame Point, for the Place of the Center of Gravity, will be determined.



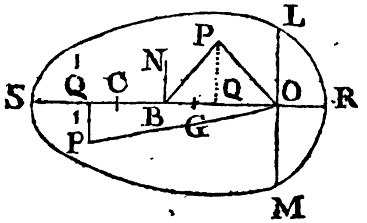
180. Thus, let O be the Point in the Axis AC, of a given Curve B A D, determined, as above, by ſuppoſing the Figure to reſt upon EF perpendicular to AC; and let RS be any other Line paſſing through the Point O; then I ſay the Sum of the *Momenta* of the Particles on each Side of RS will, *alſo*, be equal. For, if from two Points, in any Ordinate MQ, equally diſtant from the middle Point N, two Perpendiculars *mr* and *ns* be let fall upon RS, the Efficacy of thoſe two Points, in reſpect to RS, will be repreſented by $mr + ns$, or its Equal 2NH (ſuppoſing NH alſo perpendicular to RS.) Whence the Efficacy of all the Particles in MQ, will be expreſſed by their Number multiplied by NH, or by $MQ \times NH$: Which is to their Efficacy ($MQ \times ON$) when referred to the Line EF, in the conſtant Ratio of NH to ON, or of the Sine of the Angle RON to Radius. Whence it is evident that the Force of all the Ordinates (or the whole Curve) in the former Caſe, muſt be to that in the latter, in the ſame Ratio: But the ſaid Force, in the one Caſe, is equal to nothing by Hypotheſis, therefore it muſt be likewiſe ſo in the other: And conſequently the Sum of the *Momenta* of the Particles, on each Side of RS, equal to each other.

Much after the ſame manner the thing may be proved, in a Solid: Whence it will appear that there is actually ſuch a (fixed) Point in a Body as the Center of Gravity is defined to be: Which, however evident from mechanical Conſiderations, is not ſo eaſy to demonſtrate, *geometrically*, from the Reſolution of Forces.

PROPOSITION II.

181. *To determine the Center of Percuſſion of a Body.*

The Center of Percuſſion is that Point, in the Axis of Suſpenſion of a vibrating (or revolving) Body, at which it may be ſtopt, by an immoveable Obſtacle, ſo as to reſt thereon *in Equilibrio* as it were, without acting upon the Center of Suſpenſion.



Let O be the Point of Suſpenſion, G the Center of Gravity, and SLM a Section of the Body, by the Plane wherein the Axis of Suſpenſion OGS performs its Motion; to which Section let all the Particles of the Body be conceived to be transferred in ſuch Parts thereof where they would be projected into *(orthographically)* by Lines parallel to the Axis of Motion; which Suppoſition will neither affect the Place of the Center of Gravity nor the angular Motion of the Body.

Since the angular Velocity of any Particle P is as the Diſtance, or Radius, OP, its Force in the Direction, PB, perpendicular to OP, will be expreſſed by P × OP. Therefore the Efficacy of that Force upon the Axis, at B, in the perpendicular Direction BN (ſuppoſing the Axis ſtopt at C the Center of Percuſſion) will be $P \times OP \times \frac{OP}{OB}$, whoſe Power to turn the Body about the Point C is therefore as $P \times OP \times \frac{OP}{OB} \times BC = P \times \frac{OP^2 \times BC}{OB} = P \times \frac{OP^2 \times \overline{OC - OB}}{OB} = P \times \frac{OP^2 \times OC}{OB} - P \times OP^2$; which, if PQ be made Perpendicular to

OS, will at laſt (becauſe $\frac{OP^2}{OB} = OQ$) be reduced to $P \times OQ \times OC - P \times OP^2$. By the very ſame Argument, the Force of any other Particle $\acute{P}$ will be denoted by $\acute{P} \times O\acute{Q} \times OC - \acute{P} \times O\acute{P}^2$ *&c. &c.* But, as all theſe Forces muſt deſtroy one another (by the Nature of the Problem) the Sum of all the Quantities $P \times OQ \times OC$, $\acute{P} \times O\acute{Q} \times OC$, *&c.* muſt therefore be = the Sum of all the Quantities $P \times OP^2$, $\acute{P} \times O\acute{P}^2$ *&c.* and conſequently $OC = \frac{P \times OP^2 + \acute{P} \times O\acute{P}^2 + \text{\&c. \&c.}}{P \times OQ + \acute{P} \times O\acute{Q} + \text{\&c. \&c.}}$. But (by the preceding Propoſition) the Sum of all the Quantities $P \times OQ + \acute{P} \times O\acute{Q} +$ *&c.* is equal to $OG \times$ by the Content of the Body. Therefore OC is likewiſe = $\frac{P \times OP^2 + \acute{P} \times O\acute{P}^2 + \text{\&c. \&c.}}{OG \times \textit{Body}}$.

The ſame otherwiſe.

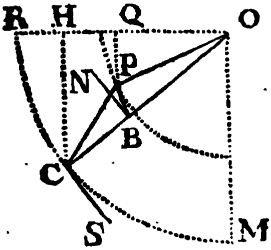
Since the Force of the Particle P, in the perpendicular Direction NB, is defined by $P \times \frac{OP^2}{OB}$, or its Equal, $P \times OQ$, the Sum of all the Quantities $P \times OQ$, $\acute{P} \times O\acute{Q}$, *&c. &c.* will expreſs the Force which, acting at C perpendicular to OS, is ſufficient to ſtop the Body, without the Center of Suſpenſion O being any way affected: This Sum, therefore, drawn into OC ($= OC \times \overline{P \times OQ + \acute{P} \times O\acute{Q} + \text{\&c. \&c.}}$) is as the Efficacy of the ſaid Force to turn the Body about the Point O. But the Force of the Particle P, in the Direction BN being $P \times \frac{OP^2}{OB}$, its Efficacy to turn the Body about the ſame

Point (the contrary way) is as $P\times OP^2$; and conſequently the Efficacy of all the Particles as the Sum of all the Quantities $P\times OP^2$, $\acute{P}\times O\acute{P}^2$ &c. &c. Therefore (Action and Re-action being equal) we have $OC\times\overline{P\times OQ+\acute{P}\times O\acute{Q}+\&c.}=P\times OP^2+\acute{P}\times O\acute{P}^2+\&c.$ *the ſame as before.*

For the Center of Oſcillation, it will be requiſite to premiſe the following

LEMMA.

182. *Suppoſe two exceeding ſmall Weights* C *and* P, *acting on each other by means of an inflexible Line* (*or Wire* PC) *to vibrate in a vertical Plane* ROPCM, *about the Center* O; *it is required to determine how much the Motion of the one is affected by the other.*



Let CH and PQ be perpendicular to the horizontal Line OR; alſo let PB and CS be perpendicular to OP and OC reſpectively.

If the Force of Gravity be denoted by Unity, the Forces acting in the Directions CS and PB, whereby the Weights, in their Deſcent, are accelerated, will, according to the Reſolution of Forces, be repreſented by $\frac{OH}{OC}$ and $\frac{OQ}{OP}$. Moreover, ſince the Velocities are always in the Ratio of the Radii OC and OP, if the foreſaid Forces were to be in that Ratio, or that of P was to become $\frac{OH}{OC}\times\frac{OP}{OC}$, inſtead of $\frac{OQ}{OP}$. I ſay, in that Caſe, it is plain the Weights would continue their Motion without

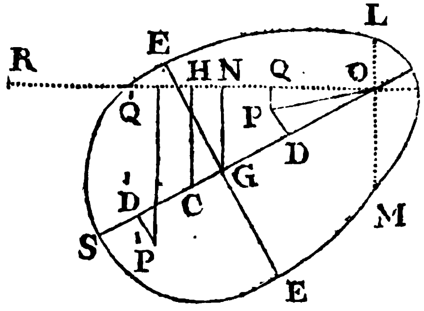
out affecting each other, or acting at all on the Line of Communication PC (or PB). Whence, the Exceſs of $\frac{OQ}{OP}$ above $\frac{OH}{OC} \times \frac{OP}{OC}$ muſt be the accelerative Force whereby the Weight P acts upon the Line (or Wire) OC, in the Direction PB; which multiply'd by the Weight P gives $P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}}$ for the abſolute Force in that Direction: Which therefore, in the perpendicular Direction NB, is $P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}} \times \frac{OP}{OB}$; whereof the Part acting upon C, being to the Whole as OB to OC, is truly defined by $P \times \overline{\frac{OQ}{OC} - \frac{OH \times OP^2}{OC^3}}$. *Q. E. I.*

If P be ſuppoſed to act upon C by means of PC (inſtead of PB) the Concluſion will be no way different: For, let F (to ſhorten the Operation) be put to denote the Force (P $\times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}}$) in the Direction PB, found above, then the Action thereof upon PC (according to the Principles of Mechanics) will be expreſſed by $F \times \frac{\textit{Radius}}{\textit{Co-ſ.}\ CPB}$; which therefore in the Direction SC, perpendicular to OC, is $F \times \frac{\textit{Radius}}{\textit{Co-ſ.}\ CPB} \times \frac{S.\ PCO}{\textit{Radius}} = \frac{S.\ PCO}{\textit{Co-ſ.}\ CPB} = \frac{S.\ PCO}{S.\ OPC} = F \times \frac{OP}{OC}$, *the very ſame as before.*

PROPOSITION III.

183. *To determine the Center of Oſcillation of a Body.*

The Center of Oſcillation is that Point, in the Axis (or Line) of Suſpenſion of a vibrating Body, into which if the whole Body was contracted, the angular Velocity and the Time of Vibration would remain unaltered.



Let LMS be a Section of the Body by a Plane, perpendicular to the Horizon and the Axis of Motion, paſſing thro' the Center of Gravity G and the Point of Suſpenſion O; and ſuppoſe all the Particles of the Body to be transferred to this Section, in ſuch Places of it, as they would be projected into *(orthographically)* by Perpendiculars falling thereon. (Which Suppoſition will no way affect the Concluſion, the angular Motion continuing the ſame.) Moreover let C be the Center of Oſcillation, or that Point in the Axis OS where a Particle of Matter (or a ſmall Weight) may be placed ſo as to be neither accelerated nor retarded by the Action of the other Particles of Matter ſituate in the Plane. Then, if, from C and any other Point P in the Plane LMS, two Perpendiculars CH and PQ be let fall upon the horizontal Line OR, the Force of a Particle (or Weight) at P to accelerate the Weight at C, will *(according to the foregoing Lemma)* be repreſented by P ×

$\overline{\frac{OQ}{OC} - \frac{OH \times OP^2}{OC^3}}$: Which, ſuppoſing GN perpendicular to OR, will alſo be expreſſed by $P \times \overline{\frac{OQ}{OC} - \frac{ON}{OG} \times \frac{OP^2}{OC^2}}$, or, its Equal $P \times \frac{OQ \times OG \times OC - ON \times OP^2}{OC^2 \times OG}$. In the very ſame manner the Force of any other Particle P' will be repreſented by $P' \times \frac{OQ' \times OG \times OC - ON \times OP'^2}{OC^2 \times OG}$ *&c. &c.*

Therefore the Forces of all the Particles deſtroying each other *(by Hypotheſis)* the Sum of all the Quantities $\overline{P \times OG \times OQ \times OC - ON \times OP^2} + \overline{P' \times OG \times OQ' \times OC - ON \times OP'^2}$ *&c. &c.* muſt be equal to nothing: Whence $P \times OG \times OQ \times OC + P' \times OG \times OQ' \times OC$ *&c. &c.* $= P \times ON \times OP^2 + P' \times ON \times OP'^2$ *&c. &c.* and conſequently $OC = \frac{ON}{OG} \times \frac{P \times OP^2 + P' \times OP'^2 + \&c.}{P \times OQ + P' \times OQ' + \&c.}$. But *(by Art.* 171. and 172.) the Sum of all the Quantities $P \times OQ + P' \times OQ'$ *&c.* is equal to the Content of the Body multiplied by the Diſtance (ON) of the Center of Gravity G from the Line LM (perpendicular to OC); whence OC is alſo $= \frac{ON}{OG} \times \frac{P \times OP^2 + P' \times OP'^2 \&c. \&c.}{ON \times \mathit{Body}} = \frac{P \times OP^2 + P' \times OP'^2 \&c. \&c.}{OG \times \mathit{Body}}$ Which Expreſſion continuing the ſame in all Inclinations of the Axis OS, the Point C, thus determined is a fixed Point, agreeable to the Definition; and appears to be the ſame with the Center of Percuſſion; *ſee Art.* 181.

COROLLARY.

184. If PD, $\acute{P}\acute{D}$ &c. be perpendicular to OS, the Numerator of the Fraction found above, will become $P\times\overline{OG^2+GP^2-2OG\times GD}+\acute{P}\times\overline{OG^2+G\acute{P}^2+2OG\times G\acute{D}}+$ &c. &c. (ſince $OP^2=OG^2+GP^2-2OG\times GD$ &c.) Which, becauſe all the Quantities $P\times-2OG\times GD+\acute{P}\times 2OG\times G\acute{D}$ &c. or $P\times-GD+\acute{P}\times G\acute{D}$ &c. (by the Nature of the Center of Gravity) deſtroy one another, will be *barely* $=P\times\overline{OG^2+GP^2}+\acute{P}\times\overline{OG^2+G\acute{P}^2}+$ &c. &c. $=\overline{P+\acute{P}+\text{\&c.}}\times OG^2+P\times GP^2+\acute{P}\times G\acute{P}^2+$ &c. $=$ *Maſs* $\times\, OG^2+P\times GP^2+\acute{P}\times G\acute{P}^2+$ &c. Whence it is evident that OC is, alſo,

$$\left(=\frac{\textit{Maſs}\times OG^2,+P\times GP^2+\acute{P}\times G\acute{P}^2+\text{\&c. \&c.}}{\textit{Maſs}\times OG}\right)$$

$$=OG+\frac{P\times GP^2+\acute{P}\times G\acute{P}^2+\text{\&c.}}{\textit{Maſs}\times OG};$$ and conſequently

$$CG=\frac{P\times GP^2+\acute{P}\times G\acute{P}^2+\text{\&c. \&c.}}{\textit{Maſs}\times OG},$$

Whence it appears that, *if a Body be turned about its Center of Gravity, in a Direction, perpendicular to the Axis of the Motion, the Place of the Center of Oſcillation will remain unaltered*; becauſe the Quantities $P\times GP^2$, $\acute{P}\times G\acute{P}^2$ are no way affected by ſuch a Motion of the Body.

It alſo appears *that the Diſtance of the Center of Gravity from that of Oſcillation* (if the Plane of the Body's Motion remains unalter'd) *will be reciprocally as the Diſtance of the former from the Point of Suſpenſion. Therefore, if that Diſtance be found when the Point of Suſpenſion is in the Vertex, or ſo poſited, that the Operation may become the moſt ſimple, the Value thereof in any other propoſed Poſition of that Point will likewiſe be given, by one ſingle Proportion.*

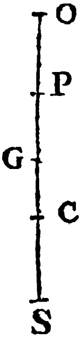
185. But now, to ſhew how theſe Concluſions may be reduced to Practice, we muſt firſt of all obſerve, that *the Product of any Particle of the Body by the Square of its Diſtance from the Axis of Motion is* (here) *called the Force thereof* (its Efficacy to turn the Body about the ſaid Axis being in that Ratio.) According to which, and the firſt general Value of OC, it appears that, *if the Sum of all the Forces be divided by the Product of the Body into the Diſtance of the Center of Gravity from the Point of Suſpenſion, the Quotient thence ariſing will give the Diſtance of the Center of Percuſſion, or Oſcillation from the ſaid Point of Suſpenſion.*

The Manner of computing the Diviſor has been already explained; it remains therefore to ſhew how the Sum of all the Forces in the Numerator may be collected: Which will admit of ſeveral Caſes. Wherein, to avoid a Multiplicity of Words, I ſhall always expreſs the Diſtance of the Center of Gravity from the Point of Suſpenſion by g, and the Diſtance of the Center of Percuſſion, or Oſcillation, from the ſame Point, by C.

186. Caſe 1. *Let OS be a Line ſuſpended at one of its Extremes.*

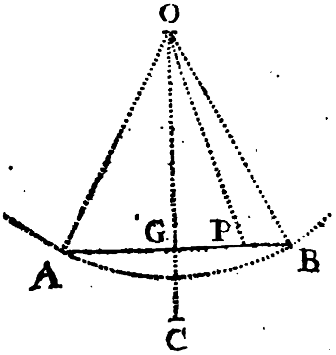
Then, if the Part OP (conſidered as variable) be denoted by x, the Force of $\dot{x}$ Particles, at P, will *(as above)* be defined by $\dot{x} \times x^2$: Whoſe Fluent ($\frac{1}{3}x^3$) therefore expreſſes the Force of all the Particles in OP (or the Sum of all the Products, under each Particle, and the Square of its Diſtance from O the Point of Suſpenſion. This Quantity therefore (when x be-

comes



comes = OS) being divided by OS × $\frac{1}{2}$OS (according to the foregoing Rule or Obſervation) we get $\left(\frac{\frac{1}{3}OS^3}{\frac{1}{2}OS^2} =\right) \frac{2}{3}$ OS for the Value of *C*, the true Diſtance of the Center of Oſcillation (or Percuſſion) from the Point of Suſpenſion.

187. Caſe 2. *Let* AB *be a Line, vibrating in a vertical Plane, having its two Extremes* A *and* B *equally diſtant from the Point of Suſpenſion* O.

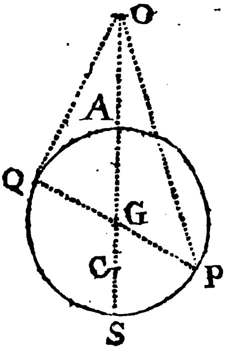


If OG (perpendicular to AB) be put $=a$, and GP $= x$, the Force of $\dot{x}$ Particles at P, will be denoted by $\dot{x} \times \overline{a^2+x^2} = \dot{x} \times OP^2$ *: Whoſe Fluent, divided by ax (or PG × OG) gives $\left(\frac{a^2x + \frac{1}{3}x^3}{ax}\right) a + \frac{x^2}{3a} = OG + \frac{BG^2}{3OG} = C$, when x becomes = GB.

* Art. 185.

188. Caſe 3. *Let* APSQ *be a Circle, vibrating in a vertical Plane.* Let PQ be any Diameter thereof; then $OP^2 + OQ^2$ being $= 2OG^2 + 2PG^2$, the Sum of the Forces of two Particles at P and Q, (putting OG $= a$, and AG $= r$) will be $= \overline{a^2+r^2} \times 2$; whence it is evident that the Sum of the Forces of all the Particles, in the whole Periphery, will be expreſſed by *their Number* $\times \overline{a^2+r^2}$, or by $\overline{a^2+r^2} \times$ *Periph.* APSQ: Which,

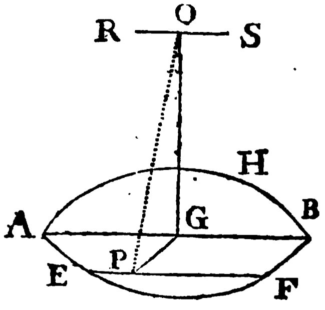
if p be put $=$ 3.141 &c. will be $= \overline{a^2+r^2} \times 2pr = 2pa^2r + 2pr^3$. Hence the Force of the Circle itſelf is alſo given, being $=$ Fluent of $\overline{2pa^2r+2pr^3} \times \dot{r} = pa^2r^2+\frac{1}{2}pr^4 = \overline{a^2+\frac{1}{2}r^2}$ $\times$ *Circle* APSQ. Now, if the two Expreſſions thus found be divided by $a \times$ *Periph.* APSQ, and $a \times$ *Circle* APSQ reſpectively *, we ſhall have



$a + \frac{r^2}{a}$ and $a + \frac{r^2}{2a}$, for the two correſponding Values of C.

189. Caſe 4. *Let* AHBE *be a Circle having its Plane (always) perpendicular to the Axis of Suſpenſion* OG.

Let AGB be that Diameter of the Circle which is parallel to the Axis of Motion RS; and let EF be any Chord parallel to AB and RS; whoſe Diſtance, GP, from the Center of the Circle, let be denoted by x; putting OG $= a$, and AG $= r$:



Then, by the Nature of the Circle, EF $= 2\sqrt{r^2-x^2}$; whoſe Diſtance OP, from the Axis of Motion RS, is alſo given $= \sqrt{a^2+x^2}$. Hence it appears that the Force of all the Particles in the Line EF (*defined in Art.* 185.) will be repreſented by $\overline{a^2+x^2} \times 2\sqrt{r^2-x^2}$. Therefore $\dot{x} \times \overline{a^2+x^2} \times 2\sqrt{r^2-x^2}$ is the Fluxion of the Force of the Plane ABFE; whoſe Fluent (when $x=r$)

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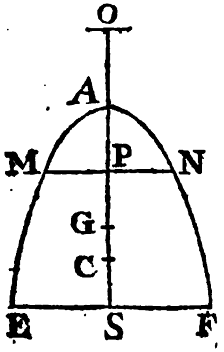
* Art. 185.

$x = r$) is $= \overline{a^2 + \frac{1}{4}r^2} \times$ *Area AEFBG*; which, if p be put for the Area of the Circle whoſe Radius is Unity, will be $= \overline{a^2+\frac{1}{4}r^2} \times \frac{1}{2}pr^2$; whereof the Double ($pa^2r^2 + \frac{1}{4}pr^4$) is the Force of the whole Circle AEFH: whoſe Fluxion $2par\dot{r} + pr^3\dot{r}$ (ſuppoſing r variable) being divided by $\dot{r}$, we likewiſe get $2pa^2r + pr^3$ ($= \overline{a^2+\frac{1}{2}r^2} \times$ *Periph. AEFH*) for the Force of the Periphery AEFH. But the Center of Gravity, whether we regard the Circle itſelf or its Periphery, is in the Center of the Circle; therefore the Diſtance of the Center of Oſcillation from the Point of Suſpenſion, will in theſe two Caſes be exhibited by $a + \frac{r^2}{4a}$ and $a + \frac{r^2}{2a}$ reſpectively.

If the Circle, inſtead of being perpendicular to GO, either coincides, or makes a given Angle with it, the Value of C will come out exactly the ſame; provided the Diameter AB ſtill continues parallel to the Axis of Motion RS: This appears from *Art.* 184. and may be, otherwiſe, very eaſily demonſtrated.

190. Caſe 5. *Let the Figure propoſed be a Curve* AEF, *moving (flat-ways, as it were) ſo that the Plane deſcribed by the Axis* OAS *may be perpendicular to that of the Curve.*



Here, putting AP $= x$, PN $= y$, AN $= z$, OA $= d$, OG $= g$, and AG $= a$, the Force of the Particles in MN will be defined by $2y \times \overline{a+x}|^2$. Therefore the Fluent of $2y\dot{x} \times \overline{d+x}|^2$ will be as the whole Force of the Plane NAM (or AEF, when $x =$ AS) and conſequently $C = \frac{Flu.\ \overline{d+x}|^2 \times y\dot{x}}{Flu.\ \overline{d+x} \times y\dot{x}}$: Which, therefore,

fore, when the Point of Suſpenſion is in the Vertex A, will become $C = \frac{Flu.\, yx^2\dot{x}}{Flu.\, yx\dot{x}}$. Let this Value be denoted by v; then, the Diſtance of the Centers of Gravity and Oſcillation being $v-a$, we have (by *Art.* 184.) $g : a :: v-a : \left(\frac{a \times \overline{v-a}}{g}\right)$ the Diſtance of the ſame Centers, when the Point of Suſpenſion is at O, and conſequently C, in that Caſe, $= g + \frac{a \times \overline{v-a}}{g}$: Which Form will be found more commodious than the foregoing in moſt Caſes.

After the ſame Manner the Value of C, with reſpect of the Arch AEF, will appear to be $= \frac{Flu.\, \overline{d+x}|^2 \times \dot{z}}{Flu.\, \overline{d+x} \times \dot{z}}$ $= g + \frac{a \times \overline{v-a}}{g}$, ſuppoſing $v = \frac{Flu.\, x^2\dot{z}}{Flu.\, x\dot{z}}$.

It may not be improper to give an Example or two of the Uſe of the foregoing Theorems.

191. Let therefore EAF be, firſt, conſider'd as an iſoſceles Triangle: In which Caſe AP (x) and PN (y) being in a conſtant Ratio, we have $y = \frac{bx}{c}$ (ſuppoſing SF$=b$ and AS$=c$.) Hence $C \left(= \frac{Flu.\, \overline{d+x}|^2 \times y\dot{x}}{Flu.\, \overline{d+x} \times y\dot{x}}\right)$ $= \frac{Flu.\, d^2x\dot{x} + 2dx^2\dot{x} + x^3\dot{x}}{Flu.\, dx\dot{x} + x^2\dot{x}} = \frac{\frac{1}{2}d^2 + \frac{2}{3}dx + \frac{1}{4}x^2}{\frac{1}{2}d + \frac{1}{3}x} =$ $\frac{6d^2 + 8dx + 3x^2}{6d + 4x}$: Or (according to the ſecond Form) becauſe $v \left(\frac{Flu.\, yx^2\dot{x}}{Flu.\, yx\dot{x}}\right) = \frac{3x}{4}$, and a is known to

be

* Art. 175. be $= \frac{2x}{3}$*, we have $C\left(= g + \frac{a \times \overline{v-a}}{g}\right) = g + \frac{x^2}{18g}$, where $g\ (= d+a) = d + \frac{2}{3}x$.

Again, becauſe $\dot{z}$ and $\dot{x}$ are in a conſtant Ratio, we alſo have $\frac{Flu.\ \overline{d+x}|^2 \times \dot{z}}{Flu.\ \overline{d+x} \times \dot{z}} = \frac{Flu.\ \overline{d+x}|^2 \times \dot{x}}{Flu.\ \overline{d+x} \times \dot{x}} = \frac{d^2 + dx + \frac{1}{3}x^2}{d + \frac{1}{2}x}$; whence the Center of Oſcillation of the Lines EH and AF is given.

192. For a ſecond Example, let EAF be ſuppoſed a Parabola of any Kind, whoſe Equation is $y = \frac{x^n}{c^{n-1}}$: Then (*according to Form* 2.) we ſhall firſt have $v\left(= \frac{Flu.\ yx^2\dot{x}}{Flu.\ yx\dot{x}}\right) = \frac{Flu.\ x^{n+2}\dot{x}}{Flu.\ x^{n+1}\dot{x}} = \frac{\overline{n+2} \times x}{n+3}$: Whence,

† Art. 176. a being $= \frac{\overline{n+1} \times x}{n+2}$†, we alſo get $C\left(= g + \frac{a \times \overline{v-a}}{g}\right) = g + \frac{\overline{n+1} \times x^2}{\overline{n+2}|^2 \times \overline{n+3} \times g}$; where $g = d + \frac{\overline{n+1} \times x}{n+2}$.

But, with reſpect to the Arch of the Curve, $v\left(= \frac{Flu.\ x^2\dot{z}}{Flu.\ x\dot{z}}\right)$ is $= \frac{Flu.\ x^2\dot{x}\sqrt{c^{2n-2} + nnx^{2n-2}}}{Flu.\ x\dot{x}\sqrt{c^{2n-2} + nnx^{2n-2}}}$: From which Value (found by infinite Series, and even with-

‡ Art. 138. out in ſome Caſes ‡) that of C will alſo be given.

193. Caſe. 6. *Let the propoſed Figure be a Curve vibrating (edge-ways) ſo that the Motion of the Axis may be in the Plane of the Curve.*

Then (*by Caſe* 2.) the Force of all the Particles in the Line PN (*ſee the preceding Figure*) being defined by $OP^2 \times PN + \frac{1}{3}PN^3$, or $\overline{d+x}|^2 \times y + \frac{1}{3}y^3$ (retaining the No-

Notation above) we have $C = \frac{Flu.\ \overline{d+x}|^2 \times y\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ \overline{d+x} \times y\dot{x}}$: Which, when the Point of Suſpenſion is in the Vertex A, will become $\frac{Flu.\ yx^2\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ yx\dot{x}}$: Let this (when found) be denoted by v; then, it appears from the preceding Caſe, that the general Value of C will, *alſo*, be repreſented by $g + \frac{a \times \overline{v-a}}{g}$.

In the ſame manner the Value of C, with reſpect to the Arch EAF, will be expounded by $\frac{Flu.\ \overline{\overline{d+x}|^2 + y^2} \times \dot{z}}{Flu.\ \overline{d+x} \times \dot{z}}$, or by $g + \frac{a \times \overline{v-a}}{g}$, ſuppoſing $v = \frac{Flu.\ \overline{x^2+y^2} \times \dot{z}}{Flu.\ x\dot{z}}$.

194. Example. *Let the Equation of the given Curve be* $y = \frac{x^n}{c^{n-1}}$: Then $v \left(= \frac{Flu.\ yx^2\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ yx\dot{x}}\right) =$

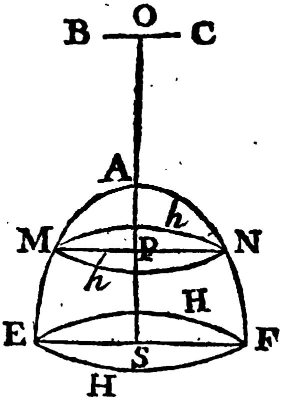
$$\frac{Flu.\ c^{1-n}x^{n+2}\dot{x} + \frac{1}{3}c^{3-3n}x^{3n}\dot{x}}{Flu.\ c^{1-n}x^{n+1}\dot{x}} = \frac{\overline{n+2} \times x}{n+3} +$$

$$\frac{\frac{1}{3}c^{2-2n}x^{3n+1} \times \overline{n+2}}{\overline{3n+1} \times x^{n+2}} = \frac{\overline{n+2} \times x}{x+3} + \frac{\overline{n+2} \times c^{2-2n} \times x^{2n-1}}{3 \times \overline{3n+1}}$$

$= \frac{n+2}{n+3} \times x + \frac{\overline{n+2}}{3 \times \overline{3n+1}} \times \frac{y^2}{x}$: From which the Value of C is alſo given; and from whence it appears, that if n be expounded by 0, v will become $= \frac{2x}{3} + \frac{2y^2}{3x} = \frac{2}{3} \times \frac{x^2+y^2}{y}$; in which Caſe the Figure will degenerate to a Rectangle: But, if n be interpreted by 1, the Figure EAF will then be an iſoſceles

Triangle, and $v = \frac{3x}{4} + \frac{y^2}{4x}$: Laſtly, if n be taken $= \frac{1}{2}$, the Curve will be the common Parabola, and $v = \frac{5x}{7} + \frac{c}{3}$.

195. *Caſe* 7. *Let the Figure* AEFH *be a Solid generated by the Rotation of a Curve* EAF *about its Axis* AS ; *having its Baſe* HH *parallel to the Axis of Motion* BOC.



It appears, *from Caſe* 4. that the Force of all the Particles in the circular Section *hh* (parallel to HH) will be expreſſed by $\overline{OP^2 + \frac{1}{4}PN^2} \times$ *Circle hh*, or $\overline{OP^2 \times PN^2 + \frac{1}{4}PN^4} \times p$ (*p* being = 3.1415 *&c.*) which, in algebraic Terms, is $\overline{\overline{d+x}|^2 \times y^2 + \frac{1}{4}y^4} \times p$. Hence we have $C =$

$$\frac{Flu.\ \overline{\overline{d+x}|^2 \times y^2 + \frac{1}{4}y^4} \times p\dot{x}}{Flu.\ \overline{a+x} \times py^2\dot{x}} {}^{*} = \frac{Flu\ \overline{d+x}|^2 \times y^2\dot{x} + \frac{1}{4}y^4\dot{x}}{Flu.\ \overline{d+x} \times y^2\dot{x}}.$$

* Art. 185.

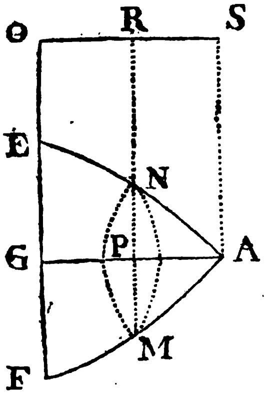
Which, therefore, when the Point of Suſpenſion is in the Vertex A, becomes $\frac{Flu.\ y^2x^2\dot{x} + \frac{1}{4}y^4\dot{x}}{Flu.\ y^2x\dot{x}} = v$; and conſequently $C = g + \frac{a \times \overline{v-a}}{g}$, as in the preceding Caſes.

But, with regard to the Superficies of the Solid ; it is found, *in Caſe* 4. that the Force of the Particles in the Periphery M*b*N*b* is expreſſed by $\overline{OP^2 + \frac{1}{2}PN^2} \times$ *Periph. Mb Nb* $= \overline{d+x}|^2 \times 2py + py^3$.

Hence the Fluent of $\overline{\overline{d+x}|^2 \times 2py+py^3} \times \dot{z}$, divided by that of $\overline{d+x} \times 2py\dot{z}$ $\left(= \dfrac{Flu.\ \overline{d+x}|^2 \times 2y\dot{z} + y^3\dot{z}}{Flu.\ d+x \times 2y\dot{z}}\right)$ will give the true Value of C with reſpect to the curve Surface E*b*A*b*F. Which, putting $v = \dfrac{Flu.\ 2yx^2\dot{z}+y^3\dot{z}}{Flu.\ 2yx\dot{z}}$, is likewiſe expreſſed by $g + \dfrac{a \times \overline{v-a}}{g}$.

196. Ex. 1. *Let* EAF *be conſidered as a Cone*; then, putting AS $=f$, SF $=b$ and AF $=c$, we have $y = \dfrac{bx}{f}$, $z = \dfrac{cx}{f}$; and therefore C $\left(= \dfrac{Flu.\ \overline{d+x}|^2 \times y^2\dot{x} + \frac{1}{4}y^4\dot{x}}{Flu.\ \overline{d+x} \times y^2\dot{x}}\right)$ $= \dfrac{20d^2+30fd+12f^2+3b^2}{20d+15f}$, when $x = f$. But, with reſpect to the convex Superficies, C will be found $= \dfrac{12d^2+16df+6f^2+3b^2}{12d+8f}$.

197. Ex. 2. *Let* EAF *&c. be conſidered as a Sphere whoſe Center is* S, *and Radius* AS$=r$; in which Caſe, y^2 being $= 2rx-x^2$, we have v $\left(= \dfrac{Flu.\ y^2x^2\dot{x}+\frac{1}{4}y^4\dot{x}}{Flu.\ y^2x\dot{x}}\right)$ $= \dfrac{Flu.\ r^2x^2\dot{x}+rx^3\dot{x}-\frac{3}{4}x^4\dot{x}}{Flu.\ 2rx^2\dot{x}-x^3\dot{x}} = \dfrac{\frac{1}{3}r^2+\frac{1}{4}rx-\frac{3}{20}x^2}{\frac{2}{3}r-\frac{1}{4}x}$; whence C is alſo given. But, when $x = 2r$ (or the whole Sphere is taken) $v = \dfrac{7r}{5}$: Therefore a being $=r$, and $g =$ OS, in this Caſe, we have C $\left(= g + \dfrac{a \times \overline{v-a}}{g}\right) = g + \dfrac{r \times 2r}{5g} = g + \dfrac{2r^2}{5g}$.



198. Caſe 8. *Let the Figure propoſed be a Solid, as in the preceding Caſe, but let its Axis* AG *be, here, parallel to the Axis of Motion* ORS.

Then, if RP (OG) be put $= g$, 3,1459 &c. $= p$, AP $= x$ &c. the Force of the Particles in the Circle NM (parallel to EF) will be exhibited by $\overline{g^2 + \frac{1}{2}y^2} \times py^2$, or $pg^2y^2 + \frac{1}{2}py^4$ (*Vid. Caſe* 3.) Hence we have $C =$

* Art. 185.
† Art. 145.

$$\frac{Flu.\ pg^2y^2\dot{x} + \frac{1}{2}py^4\dot{x}}{g \times Solid} * = \frac{Flu.\ pg^2y^2\dot{x} + \frac{1}{2}py^4\dot{x}}{g \times Flu.\ py^2\dot{x}} † = g + \frac{Flu.\ \frac{1}{2}y^4\dot{x}}{g \times Flu.\ y^2\dot{x}}.$$

Moreover, with reſpect to the Superficies; the Force of the Particles in the Periphery of the ſaid Circle MN being $2pg^2y + 2py^3$ ‡, we have, in this Caſe, $C =$

‡ Art. 185.

$$\frac{Flu.\ \overline{2pg^2y + 2py^3} \times \dot{z}}{g \times Superficies.} = \frac{Flu.\ 2pg^2y\dot{z} + 2py^3\dot{z}}{g \times Flu.\ 2py\dot{z}} = g + \frac{Flu.\ y^3\dot{z}}{g \times Flu.\ y\dot{z}}.$$

199. Ex. 1. *Let* EAF *be a Segment of a Sphere, whoſe Radius is r*; then y^2 being $= 2rx - x^2$, we ſhall have

$$C\left(g + \frac{Flu.\ \frac{1}{2}y^4\dot{x}}{g \times Flu.\ y^2\dot{x}}\right) = g + \frac{Flu.\ 2r^2x^2\dot{x} - 2rx^3\dot{x} + \frac{1}{2}x^4\dot{x}}{g \times Flu.\ 2rx\dot{x} - x^2\dot{x}}$$

$$= g + \frac{\frac{2}{3}r^2x - \frac{1}{2}rx^2 + \frac{1}{10}x^3}{g \times \overline{r - \frac{1}{3}x}} = g + \frac{\overline{20r^2 - 15rx + 3x^2} \times x}{\overline{30r - 10x} \times g}.$$

Which, when x is expounded, either, by r or $2r$, becomes $= g + \frac{2r^2}{5g}$, for the true Value of C, when either the Hemiſphere, or whole Sphere, is taken. But, with reſpect to the Center of Oſcillation of the Superficies thereof, we have $\dot{z}$ in this Caſe $= \frac{r\dot{x}}{\sqrt{2rx - xx}}$ * $= \frac{r\dot{x}}{y}$: And therefore $g + \frac{Flu.\ y^3\dot{z}}{g \times Flu.\ y\dot{z}} = g + \frac{Flu.\ \overline{2rx - xx} \times r\dot{x}}{g \times Flu.\ r\dot{x}} = g + \frac{rx - \frac{1}{3}x^2}{g}$: Which, when $x = r$, or $x = 2r$, becomes $g + \frac{2r^2}{3g}$.

* Art. 142.

200. Ex. 2. *Let the Solid* EAF *be a Paraboloid, whoſe generating Curve is defined by the Equation* $y = \frac{x^n}{c^{n-1}}$:

Then $C = g + \frac{Flu.\ \frac{1}{2}y^4\dot{x}}{g \times Flu.\ y^2\dot{x}} = g + \frac{Flu.\ \frac{1}{2}x^{4n}\dot{x} \times c^{4-4n}}{g \times Flu.\ x^{2n}\dot{x} \times c^{2-2n}} = g + \frac{\overline{2n+1} \times x^{2n}}{\overline{4n+1} \times 2g \times c^{2n-2}} = g + \frac{\overline{2n+1} \times y^2}{\overline{4n+1} \times 2g}$. Where, if n be taken $= 0$, the Figure will become a Cylinder, and $C = g + \frac{y^2}{2g}$: But if n be expounded by 1, the Figure will be a Cone, and $C = g + \frac{3y^2}{10g}$. Laſtly, if n be taken $= \frac{1}{2}$, the Figure will be the Solid generated from the common Parabola and $C = g + \frac{y^2}{3g}$.

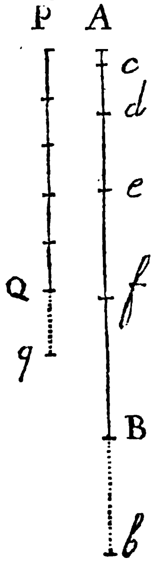
SECTION XII.

Of the Uſe of FLUXIONS *in determining the Motion of Bodies affected by centripetal Forces.*

PROPOSITION I.

201. THE *Motion, or Velocity, acquired by a Body freely deſcending from Reſt, by the Force of an uniform Gravity, is proportional to the Time of its Deſcent; and the Space gone over, as the Square of that Time.*

The firſt Part of the Propoſition is almoſt ſelf-evident: For, ſince any Motion is proportional to the Force by which it is generated, that generated by the Force of an uniform Gravity muſt be as the Time of Deſcent; becauſe the whole Effect of ſuch a Force, acting equally every Inſtant, is as that Time.



Let, now, the Velocity acquired during a Deſcent of one Second of Time, be ſuch as would carry the Body uniformly over any Diſtance b in one Second; and let AB (x) denote the Diſtance deſcended in any propoſed Time t; which Time let be denoted by PQ; making $Bb = \dot{x}$ and $Qq = \dot{t}$: Then it will be, as $1' : t :: b : (bt)$ the Diſtance that would be uniformly deſcribed in $1''$, with the Velocity at B: Alſo $1'' : \dot{t} ::$ the ſaid Diſtance (bt) to $bt\dot{t} = \dot{x}$ *. By taking the Fluent whereof we get $\frac{1}{2}bt^2$

* Art. 24.

$\frac{1}{2}bt^2=x$. Therefore the Diſtance deſcended ($\frac{1}{2}bt^2$) is as the Square of the Time. *Q. E. D.*

Otherwiſe, without Fluxions.

Conceive the Time (PQ) of falling thro' AB to be divided into an indefinite Number of very ſmall equal Particles, repreſented, each, by m; and let the Diſtance deſcended in the firſt of them be A*c*, in the ſecond *cd*, in the third *de*, &c. &c. Then, the Velocity being always as the Time from the Beginning of the Deſcent, it will in the Middle of the firſt of the ſaid Particles be defined by $\frac{1}{2}m$; in the Middle of the ſecond by $1\frac{1}{2}m$; in the Middle of the third by $2\frac{1}{2}m$, &c. &c. But, ſince the Velocity at the Middle of any Particle of Time, is a Mean between thoſe at the two Extremes, or betwixt any other two equally remote from it, the correſponding Particle of the Diſtance AB may, therefore, be conſidered as deſcribed by that mean Velocity. And ſo, the Spaces *Ac*, *cd*, *de*, *ef*, &c. deſcribed in equal Times, being reſpectively as the ſaid mean Celerities $\frac{1}{2}m$, $1\frac{1}{2}m$, $2\frac{1}{2}m$, $3\frac{1}{2}m$, &c. it follows, by Addition, that the Diſtances, A*c*, A*d*, A*e*, A*f*, &c. gone over from the Beginning, are to one another as $\frac{m}{2}, \frac{4m}{2}, \frac{9m}{2}, \frac{16m}{2}$, &c. or 1, 4, 9, 16, 25, &c. that is, as the Squares of the Times. *Q. E. D.*

COROLLARY I.

202. Since the Diſtance that might be uniformly run over in one Second, with the Velocity at B, is expreſſed by bt, the Diſtance that might be deſcribed with the ſame Velocity in the Time t will therefore be expreſſed by $bt\times t$, or bt^2: Whence it appears, that the Space AB ($\frac{1}{2}bt^2$) thro' which the Body falls in any given Time t, is juſt the half of that which would be uniformly deſcribed with the Celerity at B, in the ſame Time.

Therefore, ſince it is found from Experiment, that a Body near the Earth's Surface (where the Gravity may be taken as uniform) deſcends about $16\frac{1}{12}$ Feet in the firſt Second, it follows that the Value of *b* (is in this Caſe) $=2\times16\frac{1}{12}=32\frac{1}{6}$: And conſequently the Number of Feet deſcended in *t* Seconds, equal to $16\frac{1}{12}\times t^2$.

COROLLARY 2.

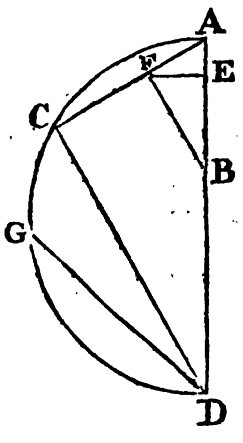
203. It is evident, whatever Force the Body deſcends by, the Value of *b* will always be as that Force; ſince a double Force, in the ſame time, generates a double Velocity; a treble Force, a treble Velocity, &c. Therefore, ſeeing our Equation $\frac{1}{2}bt^2=x$, alſo gives $t=\sqrt{\frac{x}{\frac{1}{2}b}}$, and $b=\frac{x}{\frac{1}{2}t^2}$, it follows,

1. That the Diſtance deſcended is, univerſally, as the Force and the Square of the Time conjunctly.

2. That the Time is always as the Square-root of the Diſtance applied to the Force.

3. And that the Force is as the Diſtance apply'd to the Square of the Time——What is above demonſtrated with reſpect to the Times, holds alſo in the Velocities, when the accelerating Forces are equal.

PROPOSITION II.



204. *To determine the Velocity, and Time of Deſcent, of a Body along an inclined Plane* AC.

From any Point F, in AC, draw FE perpendicular to the vertical Line AD, and make FB and CD perpendicular to AC, meeting AD in B and D. Becauſe (by the Principles of Mechanics) the Force of Gravity in the Direction CF, whereby the Body is made to deſcend along the Plane, is to the abſolute Force thereof, as AF to

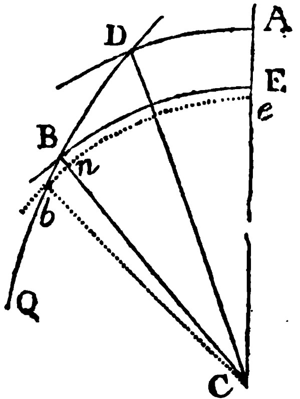
AB, or as AC to AD; and ſince (*by Caſe* 1. *Art.* 203.) the Diſtances deſcended in equal Times are as the Forces, it follows that the Time of Deſcent thro' AF will be equal to the Time of the perpendicular Deſcent thro' AB: And conſequently the Time of Deſcent thro' AC equal to that thro' AD; which is given by *Prop.* 1. Moreover, becauſe the Velocities at F and B, acquired in equal Times, are as the Forces, or as AF to AB; and it appears from *Prop.* 1. that the Velocity at E is to that at B, as $\sqrt{AE} : \sqrt{AB}$, or as $\sqrt{AE \times AB}$ $(=AF) : \sqrt{AB \times AB}$ $(= AB)$ it follows, by Equality, that the Celerity at F is equal to that at E; which is therefore given, by the preceding Propoſition. *Q. E. I.*

COROLLARY.

205. Hence the Time of Deſcent along the Chord AC of a Semi-circle ACD is equal to the Time of Deſcent along the vertical Diameter AD: And, if the Chord DG be of the ſame Length with AC (its Inclination to the Horizon being alſo the ſame) the Time of Deſcent along it will alſo be equal to that along the vertical Diameter.

PROPOSITION III.

206. *If, from two Points* A *and* D, *equally remote from the Center of Attraction* C, *two Bodies move, with equal Celerities, the one along the Right-line* AC, *the other in a Curve-line* DBQ, *their Celerities at all other equal Diſtances from the Center, will be equal.*



For, let CB and CE be any two ſuch Diſtances; let the Arch BE be deſcribed,

ſcribed, from the Center C, and alſo eb, indefinitely near to it, cutting CB in n: Let the centripetal Force at the Diſtance of CB, or CE, be repreſented by f, and the Velocity at B, by v.

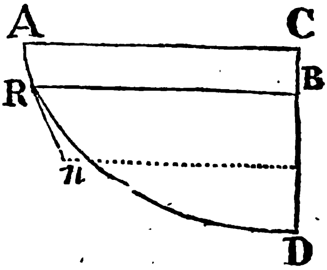
By the Reſolution of Forces, the Efficacy of the Force (f) in the Direction Bb, whereby the Velocity of the Body is accelerated, will be $\frac{Bn}{Bb} \times f$: Alſo the Time of moving over Bb (being as the Diſtance apply'd to the Velocity) is repreſented by $\frac{Bb}{v}$: Therefore the Increaſe of Velocity, in moving thro' Bb, being as the Force and Time conjunctly, will be defined by $\frac{Bn}{Bb} \times f \times \frac{Bb}{v}$, or its Equal $\frac{Bn}{v} \times f$. In the ſame Manner, the Velocity at E being denoted by w, the Time of falling thro' Ee will be repreſented by $\frac{Ee}{w}$, and the Velocity generated in that Time by $\frac{Ee}{w} \times f$: Which is to that $\left(\frac{Bn}{v} \times f\right)$ acquired in falling thro' the Arch Bb, as $\frac{1}{w}$ to $\frac{1}{v}$. Therefore, ſeeing the correſponding Increments of Velocity are always, reciprocally, as the Velocities themſelves, it is manifeſt, if thoſe Velocities are equal, in any two correſponding Poſitions of the Bodies, they will be ſo in all others, being always increaſed alike. But they are equal at A and D by Suppoſition: Therefore, *&c.* *Q. E. D.*

PROPOSITION IV.

207. *To find the Ratio of the Velocities, and Times of Deſcent, of Bodies, in Curves; the Force of Gravity being conſidered as uniform.*

Let ARD repreſent a Curve of any Kind, along which a Body deſcends, by the Force of its own Gravity yity from A; let AC, RB, *&c.* be parallel, and CD perpendicular, to the Horizon; moreover, let Rn touch the Curve at R; and let CB $= u$, AR $= w$, and R$n = \dot{w}$ *.

* Art. 135.



Since the Points B and R (as well as C and A) may be looked upon as equally remote from the Earth's Center (to which the Gravitation tends), the Velocity acquired in deſcending thro' the Arch AR will (*by the laſt Propoſition*) be equal to that acquired by falling freely through the Right-line CB; which laſt Velocity (*by Prop.* 1.) is always as $\sqrt{CB}$ (or $u^{\frac{1}{2}}$). Therefore the Celerity, whether the Body moves in a Right-line, or a Curve, is always in the ſubduplicate Ratio of the perpendicular Deſcent; and ſo, the Time in which Rn ($\dot{w}$) would be uniformly deſcribed, with that Celerity, will be univerſally as $\frac{\dot{w}}{u^{\frac{1}{2}}}$; whoſe Fluent is as the Time of falling through AR. *Q. E. I.*

EXAMPLE.

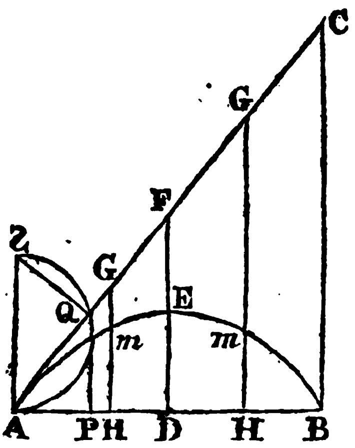
208. Let the Curve ARD be any Portion of the common Cycloid; whereof the Vertex is D and Axis DC; and whoſe Nature (putting DC $= c$, and the Ray of Curvature at D $= a$) is defined by the Equation $2a \times DB = DR^2$. Here, we have DR ($= \sqrt{2a} \times \sqrt{DB}$) $= \sqrt{2a} \times \overline{c-u}|^{\frac{1}{2}}$; whoſe Fluxion $- \sqrt{2a} \times \frac{\frac{1}{2}\dot{u}}{\overline{c-u}|^{\frac{1}{2}}}$, with a contrary Sign, is the Value of Rn or $\dot{w}$;

and

therefore $\frac{\dot{w}}{u^{\frac{1}{2}}} = \sqrt{2a} \times \frac{\frac{1}{2}\dot{u}}{\sqrt{cu-uu}}$: Whoſe Fluent, at the loweſt Point D, where u becomes $= c$, will (*by Art.* 142.) be equal to $\sqrt{2a}$ multiplied by $\left(\frac{3.14159 \&c.}{2}\right)$ half the Meaſure of the Periphery of the Circle whoſe Diameter is Unity. Which Fluent (and conſequently the Time of Deſcent) will therefore continue the ſame, let the Arch DA be what it will.

PROPOSITION V.

209. *To determine the Paths of Projectiles near the Earth's Surface; (neglecting the Reſiſtance of the Atmoſphere.)*



Let a Body be projected from the Point A, in the Direction of the Line AC, with a Velocity ſufficient to carry it uniformly over the Diſtance d in the Time t; and let the Space through which it would freely deſcend, by its own Gravity, in that time, be denoted by b; alſo let the Sine of the Angle of Elevation BAC (to the Radius r) be put $= s$, its Co-ſine $= c$, and the Diſtance of the Point A from the Ordinate Hm (conſidered as moving parallel to itſelf along with the Body) $= x$; then, *by Trig.* HG (perpendicular to AB) will be $= \frac{sx}{c}$, and AG $= \frac{rx}{c}$.

Becauſe the Projectile is turned aſide, continually, from a rectilinear Path, by the Earth's Attraction, it muſt deſcribe a Curve-line A*m*E*m*B, to which AC is a Tangent at the Point A: But that Attraction, acting always in a Direction (H*m*) perpendicular to the Horizon, can have no Effect upon that Part of the Velocity with which the Body approaches the Line BC, parallel to H*m*; therefore the Right-line HG (in which the Body is always found) will continue to move uniformly towards BC, the ſame as if Gravity was not to act; and the Diſtance G*m* deſcended from the Tangent AC, by means of the Attraction, will be the very ſame as if the Body was to deſcend from Reſt along the Line GH. This being premiſed, it is evident, that as d : AG $\left(\frac{rx}{c}\right) :: t : \left(\frac{rx}{cd} \times t\right)$ the Time of deſcribing A*m*; and, as $t^2 : \frac{r^2x^2}{c^2d^2} \times t^2 :: b : \left(\frac{br^2x^2}{c^2d^2}\right)$ the Space (G*m*) through which a Body would freely deſcend in that Time (*by Prop.* 1.)

Hence $\frac{sx}{c} - \frac{br^2x^2}{c^2d^2}$, or $\frac{csd^2x - br^2x^2}{c^2d^2}$ is a general Value for the Ordinate H*m*: By putting which $= 0$, we get $x = \frac{csd^2}{br^2} =$ AB $=$ the Amplitude of the Projection. But, by putting its Fluxion equal to nothing, we have $x = \frac{csd^2}{2br^2}$; which ſubſtituted for x in the Value of H*m*, gives $\frac{s^2d^2}{4br^2}$ for the Altitude DE of the Projection. *Q.E.I.*

COROLLARY.

210. If another Body be projected, with the ſame Celerity, in the vertical Direction AS; then, s becoming $= r$, the Altitude of that Projection $\left(\frac{s^2d^2}{4br^2}\right)$ will become

come $\frac{d^2}{4h} = AS$; which call *h*, and let this Value be ſubſtituted in thoſe of AB and DE, and they will become $\frac{4hcs}{r^2}$ and $\frac{hs^2}{r^2}$ reſpectively.

Hence, if from the Point Q where the Line of Direction AC cuts a Semi-circle deſcribed upon AS, the Lines SQ and QP be drawn, the latter perpendicular to AB, the Triangles ASQ and AQP being ſimilar, we ſhall have

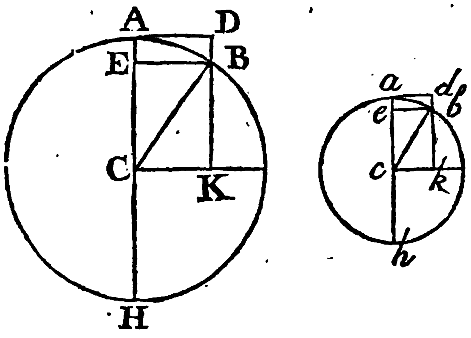
$$r : s :: h\ (AS) : \frac{sh}{r} = AQ$$

$$r : s :: \frac{sh}{r}\ (AQ) : \frac{s^2h}{r^2} = PQ = DE$$

$$r : c :: \frac{sh}{r^2}\ (AQ) : \frac{sch}{r} = AP = \tfrac{1}{4}\ AB$$

PROPOSITION VI.

211. *To determine the Ratio of the Forces, whereby Bodies, tending to the Centers of given Circles, are made to revolve in the Peripheries thereof.*



Let ABH and *abh* be any two propoſed Circles, whereof let AB and *ab* be ſimilar Arcs; in which, let the Velocities of the revolving Bodies be reſpectively as V to v; make DBK and *dbk* parallel to the Radii AC and *ac*, putting AC $= R$, *ac* $= r$, and the Ratio of the centripetal Force in ABH to that in *abh*, as F to f.

It is plain, becauſe the Angles ABD and *abd* are equal, that the Velocities at B and *b*, in the Directions BK and *bk*, with which the Bodies recede from the Tangents AD and *ad*, are to each other as the abſolute Celerities V and v*. But thoſe Velocities, being the Effects of the centripetal Forces acting in correſponding, ſimilar, Directions during the Times of deſcribing AB and *ab*, will therefore be as the Forces themſelves when the Times are equal; but when unequal, as the Forces and Times conjunctly. Therefore, the Times being univerſally as $\frac{AB}{V}$ to $\frac{ab}{v}$, or as $\frac{R}{V}$ to $\frac{r}{v}$ (becauſe the Arcs AB and *ab* are ſimilar) we have, as $F \times \frac{R}{V} : f \times \frac{r}{v} :: V : v$; whence (multiplying the Antecedents by $\frac{V}{R}$ and the Conſequents by $\frac{v}{r}$) it will be, as $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$: Therefore the Forces are as the Squares of the Velocities directly, and as the Radii inverſely. * Art. 35.

Otherwiſe.

Let the indefinitely little Arch AB be the Diſtance that the Body moves over in a given, or conſtant Particle of Time; and let the centripetal Force at B be meaſured by twice the Subtenſe or Space AE through which the Body is drawn from the Tangent AD in that Time †.

Then,

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† *The Velocity which any Force, uniformly continued, is capable of generating, in a given Body, in a given Time, is the proper Meaſure of the Intenſity of that Force*. But this Velocity is itſelf meaſured by the Space the Body would move uni-* * Art. 203.

formly

Then, by the Nature of the Circle, $AB^2 = AH \times AE = AC \times 2AE$, and conſequently $2AE = \frac{AB^2}{AC}$: Therefore, the Force is as the Square of the Velocity applied to the Radius of the Circle *(as before)*.

COROLLARY I.

212. Becauſe, $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$, it follows that

$$V : v :: \sqrt{RF} : \sqrt{rf}, \text{ and}$$

$$R : r :: \frac{V^2}{F} : \frac{v^2}{f}.$$

COROLLARY II.

213. If the Ratio of the periodic Times be denoted by that of P to p; then the Ratio of the Velocities V, v being as $\frac{R}{P}$ to $\frac{r}{p}$, we ſhall have, by Equality $\sqrt{RF} : \sqrt{rf} :: \frac{R}{P} : \frac{r}{p}$; whence alſo

$$F : f :: \frac{R}{P^2} : \frac{r}{p^2}, \text{ and}$$

$$R : r :: FP^2 : fp^2.$$

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formly over in a given Time; which Space is always the double of that through which the Body would freely deſcend, from Reſt, in the ſame time[*]. *Therefore* 2AE *is the proper Meaſure of the centripetal Force, according as we have aſſumed it.—— It is true, when the Forces to be compared are all computed in the ſame Manner, from the Naſcent, or indefinitely ſmall Subtenſes of contemporaneous Arcs, it matters not whether we conſider thoſe Subtenſes, or their Doubles, as the Meaſures of the Forces, the Ratio being the ſame in both Caſes. But when the Forces ſo found are to be compared with others derived from a fluxional Calculus, it is abſolutely neceſſary to take the double Subtenſe for the Meaſure of the Force.—— This Note is inſerted, that the Learner may avoid the Errors, which ſome very conſiderable Mathematicians have fallen into by not properly attending to this Particular.*

[*] Art. 202.

COROLLARY III.

214. If the Meaſure of the Force, or the Velocity that might be uniformly generated in a given Time (1) be expounded by any Power a^n of the Radius AC (a); then the Diſtance through which a Body would freely deſcend in the ſame Time, by that Force, uniformly continued, will be expreſſed by $\frac{1}{2}a^n$ *. Therefore, * Art. 202.
the Diſtances deſcended, by means of the ſame Force, uniformly continued, being as the Squares of the Times †, it is evident, if the Time of moving through † Art. 201.
AB be denoted by t, that the Diſtance AE deſcended in that Time, will be denoted by $\frac{t^2}{1^2}\times\frac{1}{2}a^n$: And ſo we ſhall have AB $(\sqrt{2AE\times AC}) = \frac{t}{1}\times a^{\frac{n+1}{2}}$; which being the Diſtance deſcribed by the revolving Body in the Time t, it follows that the Space gone over in the given Time (1) will be $a^{\frac{n+1}{2}}$: Which, therefore, is the true Meaſure of the Celerity in this Caſe. The ſame concluſion might have been derived in much fewer Words from *Corol.* 1. but, as a thorough underſtanding hereof is abſolutely neceſſary in what follows hereafter, I have endeavoured to make it as plain as poſſible.

COROLLARY IV.

215. Hence the Time of Revolution is alſo derived; for it will be as $a^{\frac{n+1}{2}} : 3.14159$ &c. $\times\ 2a$ (the whole Periphery) :: $1 : \frac{3.14\ \&c.\ \times\ 2a}{a^{\frac{n+1}{2}}}$ or 3.14159 &c. $\times\ 1a^{\frac{1-n}{2}}$, the true Meaſure of the periodic Time.

COROLLARY V.

216. Therefore, if n be expounded by 1, 0, — 1, — 2 and — 3 ſucceſſively, then the Velocity correſponding will be as a, $a^{\frac{1}{2}}$, 1, $a^{-\frac{1}{2}}$, and a^{-1}; and the Time of Revolution, as 1, $a^{\frac{1}{2}}$, a, $a^{\frac{3}{2}}$ and a^2 reſpectively.

SCHOLIUM.

217. From the preceding Propoſition, and its ſubſequent Corollaries, the Velocity and periodic Time of a Body revolving in a Circle at any given Diſtance from the Earth's Center, by means of its own Gravity, may be deduced: For let d be put for the Space thro' which a heavy Body, at the Surface of the Earth, deſcends in the firſt Second of Time, then $2d$ will be the Meaſure of the Force of Gravity at the Surface: And therefore, the Radius of the Earth being denoted by r, the Velocity, *per* Second, in a Circle at its Surface, will be $\sqrt{2rd}$; and the Time of Revolution $= \frac{3.14159 \text{ \&c.} \times 2r}{\sqrt{2rd}}$ $= 3.14159 \text{ \&c.} \times \sqrt{\frac{2r}{d}}$ (*Seconds*); which two Expreſſions, becauſe r is $= 21000000$ Feet and $d = 16\frac{1}{12}$ will therefore be nearly equal to 26000 Feet and 5075 *Seconds*, reſpectively. Let R be now put for the Radius of any other Circle deſcribed by a Projectile about the Earth's Center: Then, becauſe the Force of Gravitation above the Surface is known to vary according to the Square of the Diſtance inverſely, we have (*by Caſe* 4. *Corol.* 5.) $r^{-\frac{1}{2}} : R^{-\frac{1}{2}} :: (26000^{F})$ the Velocity (*per* Second) at the Surface, to $26000 \times \sqrt{\frac{r}{R}}$, the Ve-

locity in the Circle whoſe Radius is *R*: And $r^{\frac{3}{2}} : R^{\frac{3}{2}}$ :: ($5075^{\text{S.}}$) the periodic Time at the Surface : to $5075 \times \sqrt{\frac{R^3}{r^3}}$, the Time of Revolution in the Circle *R*. Which, if *R* be aſſumed equal to (60*r*) the Diſtance of the Moon from the Earth, will give $2360000^{\text{S.}}$, or 27.3^{D} nearly, for the periodic Time at that Diſtance.

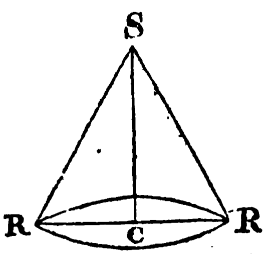
In like ſort the Ratio of the Forces of Gravitation of the Moon, towards the Sun and Earth, may be computed. For, the centrifugal Forces in Circles, being univerſally as the Radii apply'd to the Squares of the Times of Revolution, it will be as $\left(\frac{81000000}{1}\right)$ the Semi-diameter of the *Magnus Orbis* divided by the Square of one Year (the periodic Time of the Earth and Moon about the Sun) is to (240000×178) the Diſtance of the Moon from the Earth divided by $\frac{1}{178}$, the Square of the periodic Time of the Moon about the Earth, ſo is 1,9 to 1 nearly; and ſo is the Gravitation of the Moon towards the Sun to her Gravitation towards the Earth.

Alſo, after the ſame Manner, the centrifugal Force of a Body at the Equator, ariſing from the Earth's Rotation, is derived. For ſince it is found above, that 5075 Seconds is the Time of Revolution, when the centrifugal Force would become equal to the Gravity, and it appears (*by Caſe* 2. *Corol.* 2.) that the Forces, in Circles having the ſame Radii, are inverſely as the Squares of the periodic Times, we therefore have, as $\overline{86160}|^{2}$ (the Square of the Number of Seconds in ($23^{\text{H}}\ 56^{\text{M}}$) one whole Rotation of the Earth) to $\overline{5075}|^{2}$ (the Square of the Number of Seconds above given) ſo is the Force of

Gravity (which we will denote by Unity) to $\frac{1}{289}$, the centrifugal Force of a Body at the Equator ariſing from the Earth's Rotation.

But, to determine, in a more general Manner, the Ratio of the Force of a Body revolving in any given Circle, to its Gravity, we have already given 3.14 &c. × $\sqrt{\frac{2r}{d}}$ for the Time of Revolution at the Surface of the Earth, when the Gravity and centrifugal Force are equal: Therefore, if the Time of Revolution in any Circle whoſe Radius is a, be denoted by t, it follows, from *Corol.* 2. *laſt Prop.* that, $\frac{r}{\overline{3.14}|^{2}\ \&c. \times \frac{2r}{d}} : \frac{a}{t^2}$ :: the Gravity of the Body : to its centrifugal Force in that Circle; which, therefore, is as Unity to $\frac{\overline{3.14}|^{2}\ \&c. \times 2a}{dt^2}$; or as 1 to $1.228 \times \frac{a}{t^2}$ very nearly: where a denotes the Number of Feet in the Radius of the propoſed Circle, and t the Number of Seconds in one intire Revolution. So that, if the Length of a Sling, by which a Stone is whirled about, be two Feet, and the Time of Revolution $\frac{1}{2}$ of a Second, the Force by which the Stone endeavours to fly off, will be to its Weight as 9.824 to Unity.

From this general Proportion, the centrifugal Force and periodic Time of a Pendulum deſcribing a conical Surface may likewiſe be deduced.



For let SR, the Length of the Pendulum, be denoted by g; the Altitude CS of the Cone, by c; the Semi-diameter CR of the Baſe by a; and the Time of Revolution by t: Then, the Force of Gravity being re-

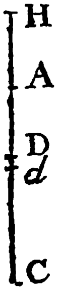
repreſented by Unity, the Force with which the revolving Body at R, the End of the Pendulum, tends to recede from the Center C, will be defined by $\frac{\overline{3.14\,\&c.}|^2 \times 2a}{dt^2}$, as has been already ſhewn. Therefore, becauſe the Body is retained in the Circle RR by the Action of three different Powers, *i. e.* the centrifugal Force $\left(\frac{\overline{3.14\,\&c.}|^2 \times 2a}{dt^2}\right)$ in the Direction CR, the Force of Gravity (1) in a Direction parallel to SC, and the Force of the Thread or Wire RS, compounded of the former two; it follows, from the Principles of Mechanics, that as SC (c) to CR (g), ſo is the Weight of the Body at R, to the Force with which it acts upon the Thread or Wire RS; and as $1 : \frac{\overline{3.14\,\&c.}|^2 \times 2a}{dt^2} :: CS\ (c) : CR\ (a)$: Whence $dt^2 = \overline{3.14\,\&c.}|^2 \times 2c$, and $t = 3.14\ \&c. \times \sqrt{\frac{2c}{d}} = 1,108\sqrt{c}$ nearly. Becauſe dt^2, or its Equal $\overline{3.14\,\&c}|^2 \times 2c$, expreſſes the Space a heavy Body will deſcend, by its own Gravity, in the Time t *, and ſince $1^2 : \overline{3.14\,\&c.}|^2 :: 2c : \overline{3.14\,\&c.}|^2 \times 2c\ (=dt^2)$ it therefore appears that, as the Square of the Diameter of any Circle, is to the Square of its Periphery, ſo is twice the perpendicular Altitude of the Cone, to the Diſtance a heavy Body will freely deſcend in the Time of one whole Gyration of the Pendulum, let the Baſe of the Cone and the Length of the Pendulum be what they will.

* Art. 202.

PROPOSITION VII.

218. *To determine the Ratio of the Velocities of Bodies deſcending, or aſcending, in Right-lines, when accelerated, or retarded, by Forces, varying according to a given Law.*

Suppoſe a Body to move in the Right-line CH, and let the Force whereby it is urged towards C, or H, be as any variable Quantity F: Moreover, let the Velocity of the Body be repreſented by v; putting its Diſtance CD, from the Point C $=x$, and D$d = \dot{x}$.



Then, ſince the Time wherein the Space Dd ($\dot{x}$) would be uniformly deſcribed, with the Velocity at D, is known to be as $\frac{\dot{x}}{v}$, the Velocity that would be uniformly generated, or deſtroyed, in that Time by the Force F (being as the Time and Force conjunctly) will conſequently be as $\frac{F\dot{x}}{v}$: Which therefore muſt be equal to, $\pm\dot{v}$, the uniform Increaſe or Decreaſe of Celerity in that Time; and conſequently $\pm v\dot{v} = F\dot{x}$. From whence, when the Value of F is given in Terms of x, or v, the Value of v will likewiſe be known. *Q. E. I.*

COROLLARY I.

219. Hence, the Law of the Velocity being given, that of the Force is deduced: For, ſince $F\dot{x} = \pm v\dot{v}$, it is evident that $F = \pm \frac{v\dot{v}}{\dot{x}}$.

COROLLARY II.

220. Hence, alſo, the Ratio of the Velocity at D to that whereby a Body might revolve in the Periphery of a Circle about the Center C, at the Diſtance of CD, will be known: For, if this laſt Velocity be denoted by

*Art. 212. w, the Value of F will be rightly expreſſed by $\frac{w^2}{x}$ *: Whence, by Subſtitution, we have $\pm v\dot{v} = \frac{w^2\dot{x}}{x}$, or

$\pm v^2 \times \frac{\dot{v}}{v} = w^2 \times \frac{\dot{x}}{x}$: Whence $w^2 : u^2 : \pm\frac{\dot{v}}{v} : \frac{\dot{x}}{x}$, and conſequently $w : v :: \sqrt{\pm\frac{\dot{v}}{v}} : \sqrt{\frac{\dot{x}}{x}}$. Where, as well as above, the Sign of $\dot{v}$ muſt be taken + or — according as the Body is urged from, or towards the Center C.

PROPOSITION VIII.

221. *Suppoſing a Body, let go from a given Point* A *with a given Celerity* (*c*) *along a Right-line* CH, *to be urged, either way, in that Line, by a Force varying as any Power* (*n*) *of the Diſtance from a given Point* C; *to find, not only, the Relation of the Velocities, and Spaces gone over, but alſo the Times of Aſcent and Deſcent.*

The Conſtruction of the preceding Problem being retained, F will here be expounded by x^n, and we ſhall therefore have $\pm v\dot{v}$ $(=F\dot{x}) = x^n\dot{x}$; and conſequently, by taking the Fluent thereof, $\pm\frac{vv}{2} = \frac{x^{n+1}}{n+1}$; but to correct the Fluent thus found, let x be taken $=$ CA (which we will call a) then v being $= c$, the Fluent in that Circumſtance will become $\pm\frac{c^2}{2} = \frac{a^{n+1}}{n+1}$: Therefore the Fluent duly corrected is $\pm\frac{v^2}{2} \mp \frac{c^2}{2} = \frac{x^{n+1} - a^{n+1}}{n+1}$*, or $v^2 \backsim c^2 = \frac{2x^{n+1} \backsim 2a^{n+1}}{n+1}$: Whence v will come out $= \sqrt{c^2 + \frac{\mp 2a^{n+1} \pm 2x^{n+1}}{n+1}}$: Where the Signs of v and x^{n+1} muſt be alike, when both Quantities increaſe, or decreaſe, at the ſame time; that is,

* Art. 78.

* Art. 220. when the Force, from C, is a repulſive one *; but, unlike, when one increaſes while the other decreaſes, or the Force, tending to C, is an attractive one. In the former Caſe we therefore have $v = \sqrt{c^2 + \frac{2x^{n+1} - 2a^{n+1}}{n+1}}$; and, in the latter, $v = \sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}$.

The Value of v being thus obtained, let the required Time of moving over the Space AD be now denoted by T; then, ſince $\dot{T}$ is univerſally $= \frac{\dot{x}}{v}$, we have $\dot{T} = \frac{\dot{x}}{\sqrt{c^2 + \frac{2x^{n+1} - 2a^{n+1}}{n+1}}}$, or $\dot{T} = \frac{\dot{x}}{\sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}}$ according to the two foreſaid Caſes reſpectively: From whence, by finding the Fluent, the Time itſelf will be known. *Q. E. I.*

COROLLARY.

222. If the Body proceeds from Reſt at A, c will be $= 0$, and we ſhall have $\dot{T} = \frac{\overline{1+n}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{2x^{n+1} - 2a^{n+1}}}$, or $\dot{T} = \frac{\overline{1+n}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{2a^{n+1} - 2x^{n+1}}}$.

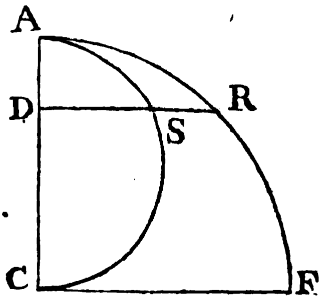
SCHOLIUM.

223. Although, the Fluents of the Expreſſions given above cannot be exhibited, in a general Manner, neither, in finite Terms, nor by means of circular Arcs and Logarithms; yet, in ſome of the moſt uſeful

Caſes that occur in Nature, they may be obtained with great Facility.

Thus, if in $\frac{\overline{1+n}|^{\frac{1}{2}}\dot{x}}{\sqrt{2a^{n+1}-2x^{n+1}}}$ (expreſſing the Fluxion of the Time of Deſcent along AD) n be expounded by 1, 0, − 2, and − 3 ſucceſſively, the Fluxion itſelf will become equal to $\frac{\dot{x}}{\sqrt{a^2-x^2}}$, $\frac{\dot{x}}{\sqrt{2a-2x}}$, $\frac{\sqrt{\frac{1}{2}a}\times x\dot{x}}{\sqrt{ax-xx}}$, and $\frac{ax\dot{x}}{\sqrt{a^2-x^2}}$ reſpectively: Whence, if ARF be a Quadrant of a Circle whoſe Center is C, and ASC a Semi-circle whoſe Diameter is AC, and DSR be perpendicular to AC; then it will appear,

1°. That, when $n=1$, and $\dot{T}=\frac{\dot{x}}{\sqrt{a^2-x^2}}$, the Velocity $(\sqrt{a^2-x^2})$ at D will be repreſented by DR, and the Fluent ſought by $\frac{AR}{AC}$*.



* Art. 142.

2°. That, when $n=0$, and $\dot{T}=\frac{\dot{x}}{\sqrt{2a-2x}}$, the Velocity at D, and the Time of Deſcent thro' AD, will each be defined by $\sqrt{2AD}$.

3°. That, when $n=-2$, and $\dot{T}=\frac{\sqrt{\frac{1}{2}a}\times x\dot{x}}{\sqrt{ax-xx}}$, the Velocity $\left(\frac{\sqrt{ax-xx}}{x\sqrt{\frac{1}{2}a}}\right)$ will be as $\frac{DS}{CD\sqrt{\frac{1}{2}AC}}$, and the Time of Deſcent thro' AD, as $\sqrt{\frac{1}{2}AC}\times\overline{AS+DS}$.

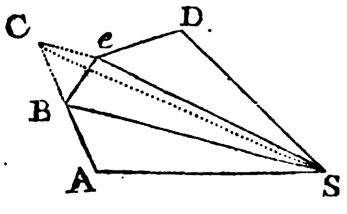
4°. And that, when $n = -3$, and $\dot{T} = \frac{ax\dot{x}}{\sqrt{a^2 - x^2}}$, the Velocity will be as $\frac{DR}{AC \times CD}$, and the Time as $AC \times DR$.

Hence the Time of the whole Deſcent thro' the Radius AC, appears to be as $\frac{AF}{AC}$, $\sqrt{2AC}$, $\sqrt{\frac{1}{2}AC} \times AF$, or AC^2. But the Time of one whole Revolution in the Periphery ARF *&c.* was found to be as $\frac{4AF}{AC^{\frac{n+1}{2}}}$*; which in the four Caſes above ſpecified is $\frac{4AF}{AC}$, $\frac{4AF}{\sqrt{AC}}$, $4AF \times \sqrt{AC}$, and $4AF \times AC$: Therefore, if the Time of moving over the Quadrant AF be denoted by $\mathcal{Q}$, it follows that the Time of Deſcent thro' the Radius AC, will be truly defined by $\mathcal{Q}$, $\mathcal{Q} \times \frac{AC\sqrt{2}}{AF}$, $\mathcal{Q} \times \sqrt{\frac{1}{2}}$, or $\mathcal{Q} \times \frac{AC}{AF}$ according to the foreſaid Caſes reſpectively.

* Art. 215.

LEMMA.

224. *The Areas which a revolving Body deſcribes, by Rays drawn to the Center of Force, are proportional to the Times of their Deſcription.*



For, let a Body, in any given Time, deſcribe the Right-line AB, with an uninterrupted uniform Motion; but upon its Arrival at B let it be impelled towards the Center S, ſo that, inſtead of proceeding along ABC, it may, after the Impulſe, deſcribe the Right-line B*e*.

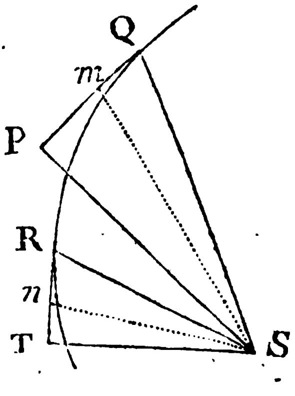
Becauſe the Force, acting in the Line SB, can neither add to, nor take from, the Celerity which the Body has in a Direction perpendicular to that Line, the Diſtance of the Body from the ſaid Line, at the end of a given Time, will therefore be the very ſame as if no Force had acted; and conſequently the Area B*e*S equal to the Area BCS, which would have been deſcribed in the ſame time, had the Body proceeded uniformly along BC; becauſe Triangles, having the ſame Baſe and Altitude, are equal.

Therefore ſeeing no Impulſe, however great, can affect the Quantity of the Area deſcribed about the Center S, in a given Time, and becauſe the Areas ASB, BSC, deſcribed about that Point, when no Force acts, are as the Baſes AB, BC, or the Times of their Deſcription, the Propoſition is manifeſt.

COROLLARY.

225. Hence the Velocity of a revolving Body, at any Point Q or R, is inverſely as the Perpendicular SP or ST, falling from the Center of Force upon the Tangent at that Point.

For, let two other Bodies *m* and *n* be ſuppoſed to move uniformly from Q and R, along the Tangents QP and RT, with Velocities reſpectively equal to thoſe of the revolving Body at Q and R; then the Diſtances Q*m* and R*n*, gone over in the ſame Time, will be to each other as thoſe Velocities; and the Areas QS*m* and RS*n* will be equal, being equal to

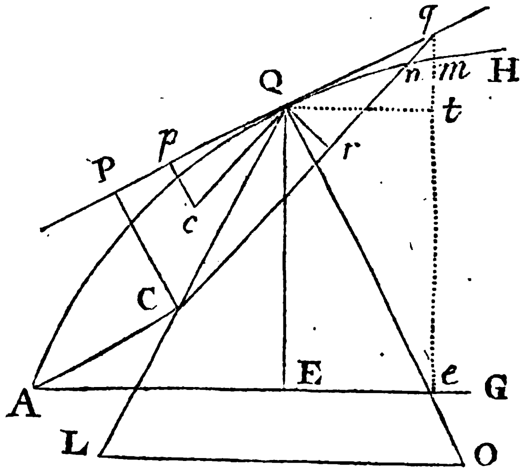


to thoſe deſcribed by the revolving Body in the ſame time *: Whence $Qm \times SP$ being $= Rn \times ST$, it follows that $Qm : Rn :: ST : SP :: \frac{1}{SP} : \frac{1}{ST}$.

* Art. 113.

PROPOSITION IX.

226. *To determine the Law of the centripetal Force, tending to a given Point* C, *whereby a Body may deſcribe a given Curve* AQH.



Let QP be a Tangent to the Curve at any Point Q; upon which, from the Center C, let fall the Perpendicular CP; put $CQ = s$, $CP = u$; and let the Velocity of the Projectile at Q be denoted by v.

Therefore, ſince v^2 is always as $\frac{1}{u^2}$ (by the *Corol.* to *Lemma*) it is evident, by taking the Fluxions of both Quantities, that $v\dot{v}$ will alſo be as $\frac{-\dot{u}}{u^3}$: But the centripetal Force, whether the Body moves in a Right-line or a Curve, is always as $-\frac{v\dot{v}}{\dot{s}}$ (by *Art.* 219. and 206.) Therefore the centripetal Force is likewiſe as $\frac{\dot{u}}{u^3\dot{s}}$. *Q.E.I.*

The ſame otherwiſe.

227. Let the Ray of Curvature QO be denoted by *R*: Then, becauſe the centripetal Forces in Circles are known to be as the Squares of the Velocities directly and the Radii inverſely *, it follows that the Force, tending to the Point O, whereby the Body might be retained in its Orbit at Q, or in the Circle whoſe Radius is QO, will be defined by $\frac{1}{u^2}\times\frac{1}{R}$: Whence (by the Reſolution of Forces) it will be CP (u) : CQ (s) :: $\frac{1}{u^2 R}$ (the Force in the Direction QO) : $\frac{s}{u^3 R}$, the Force in the Direction QC: Which, becauſe $R=\frac{s\dot{s}}{\dot{u}}$ † will alſo be expreſſed by $\frac{\dot{u}}{u^3\dot{s}}$. *Q. E. I.*

Another Way.

228. Let nq be the indefinitely ſmall Part of the Right-line Cq, intercepted by the Curve and the Tangent Qq, expreſſing the Effect of the centripetal Force in the Time of deſcribing the Area QCn. Now theſe Effects, or the Diſtances deſcended by means of Forces uniformly continued, are known to be in the duplicate Ratio of the Times ‡, or of the Areas denoting thoſe Times §: Therefore, the centripetal Force at Q, or the Diſtance deſcended by means thereof in a given Time, will be as nq applied to the ſecond Power of the Area QCq, or as $\frac{nq}{CP^2\times Qq^2}$. This Expreſſion is the ſame

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* Art. 212.

† Art. 73.

‡ Art. 201.

§ Art. 224.

with that given by Sir *Iſaac Newton*, in his *Principia*, *Book* 1. *Prop*. 6. But, to adapt it to a fluxional *Calculus*; let QE be an Ordinate to the principal Axis AG; and let (as uſual) AE $= x$, EQ $= y$, AQ $= z$, E*e* (or Q*t*) $= \dot{x}$, Q*q* $= \dot{z}$; ſuppoſing *eq* (parallel to EQ) to interſect the Curve and the Tangent in *m* and *q*.

Since Q*q* is conceived indefinitely ſmall (or in its naſcent State) the Triangle *nmq* may be taken as rectilineal *; alſo the Angle $n =$ CQP and the Angle $m =$ Q*qt*: Whence, it will be (by Trigonometry) as *S.* CQP (n) : *S.* Q*qt* (m) :: *mq* : *nq*; that is, as $\frac{CP}{CQ} : \frac{Qt}{Qq}$ :: *mq* : $nq = \frac{CQ \times Qt \times mq}{CP \times Qq}$: Which ſubſtituted above gives $\frac{CQ \times Qt \times mq}{CP^3 \times \overline{Qq}^3}$ for the Meaſure of the centripetal Force at Q: But *mq* (ſuppoſing *x* to flow uniformly) is known to be as $-\ddot{y}$: Therefore the Force at Q, is as $\frac{CQ \times Qt \times -\ddot{y}}{CP^3 \times Qq^3}$, or its Equal $\frac{-s\dot{x}\ddot{y}}{u^3\dot{z}^3}$; where the Diviſor $(u^3\dot{z}^3)$ is as the Cube of (QCq) the Fluxion of the Area AQC.

* Art. 136.

The very ſame Theorem may likewiſe be deduced from that given by our ſecond Method: For, ſince (R) the Ray of Curvature at Q is univerſally * $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, the Value of $\frac{s}{u^3 R}$ (there found) will here, by Subſtitution, become $= \frac{-s\dot{x}\ddot{y}}{u^3\dot{z}^3}$.

† Art. 68.

This Expreſſion, tho' in appearance leſs ſimple than $\frac{\dot{u}}{u^3 \dot{s}}$, firſt found, is, for the general part, more commodious in Practice.

COROLLARY I.

229. If the Point C be ſo remote that all Right-lines drawn from thence to the Curve may be conſider'd as parallel to each other, the Force will then (making Q*r* perpendicular to C*q*) be as $\frac{-s\dot{x}\ddot{y}}{\overline{Cq \times Qr}\rceil^{3}}$, or barely as $\frac{-\dot{x}\ddot{y}}{\overline{Qr}\rceil^{3}}$; ſince *s* (C*q*) in this Caſe may be rejected.

From this Expreſſion, which is general, in all Caſes where the Force acts in the Direction of parallel Lines, it appears that the Force, which always acting in the Direction of the Ordinate QE, would retain the Body in its Orbit, is every where as $\frac{-\ddot{y}}{\dot{x}^{2}}$; becauſe QC here coincides with QE, and Q*r* becomes $\doteq \dot{x}$.

COROLLARY II.

230. Becauſe the Force, tending to the Point C, is univerſally as $\frac{CQ}{CP^{3} \times QO}$ (or $\frac{s}{u^{3}R}$) the Force to any other Point *c*, will, by the ſame Argument, be as $\frac{cQ}{\overline{cp}\rceil^{3} \times QO}$. Hence the Forces, to different Centers C and *c* (about which equal Areas are deſcribed in the ſame time) are to each other in the Ratio of $\frac{CP^{3}}{CQ}$ to $\frac{\overline{cp}\rceil^{3}}{CQ}$ inverſely.

COROLLARY III.

231. Moreover, the Ratio of the Velocity at Q to the Velocity whereby the Body might revolve in a Circle about the Center at C, at the Diſtance CQ, is eaſily deduced from hence: For, ſince the Celerity at Q is that

whereby the Body might revolve in a Circle about the Center O, and the Forces tending to the Centers O and C are to each other as u (CP) and s (CQ); it therefore follows, if the Ratio ſought be aſſumed as v to w, that $\frac{v^2}{QO} : \frac{w^2}{QC} :: u : s$ (*by Art.* 212.) Whence alſo $v^2 : w^2 :: u \times QO\,(uR) : s \times QC\,(s^2)$ and conſequently $v : w :: \sqrt{\frac{uR}{ss}} : 1 :: \sqrt{\frac{u\dot{s}}{s\dot{u}}} : 1 :: \sqrt{\frac{\dot{s}}{s}} : \sqrt{\frac{\dot{u}}{u}}$ (becauſe $R = \frac{s\dot{s}}{\dot{u}}$).

The ſame Proportion may alſo be derived from *Corol.* 2. *Prop.* 7. For it is there proved that $v : w :: \sqrt{\frac{\dot{s}}{s}} : \sqrt{-\frac{\dot{v}}{v}}$; and it appears from above, that $-\frac{\dot{v}}{v} = \frac{\dot{u}}{u}$: Whence the whole is manifeſt.

If OL be made perpendicular to QC, QL will be $\left(= \frac{CP \times QO}{CQ}\right) = \frac{uR}{s}$, and $\frac{QL}{CQ} = \frac{uR}{s^2}$; and therefore $v : w :: QL^{\frac{1}{2}} : CQ^{\frac{1}{2}}$: Which is another Proportion of the propoſed Celerities.

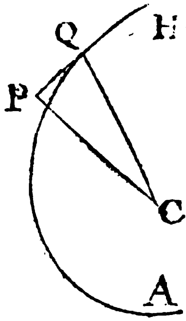
COROLLARY IV.

232. Laſtly, the Law of centripetal Force being given, the Nature of the Trajectory AQ may from hence be found; for ſince the Force (F) is univerſally defined by $\frac{\dot{u}}{u^3\dot{s}}$, it is evident that $\frac{-1}{2u^2}$ will be = the Fluent of $F\dot{s}$; which, when F is given in Terms of s, will become known; and then, the Relation between u and s being given, the Curve itſelf is known.

EXAMPLE I.

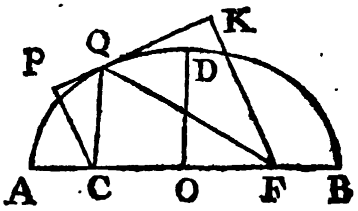
233. *Let the given Curve* AQH *be the logarithmic Spiral, and* C *the Center thereof:* Then *u* (CP) being in this Caſe $= \frac{bs}{a}$ *, we have $\frac{\dot{u}}{u^3\dot{s}}$ † $\left(= \frac{b\dot{s}}{a\dot{s}} \times \frac{a^3}{b^3s^3}\right)$ $= \frac{a^2}{b^2s^3}$, and $\sqrt{\frac{u\dot{s}}{s\dot{u}}}$ ‡ $\left(= \sqrt{\frac{bs\dot{s}}{a} \times \frac{a}{bs\dot{s}}}\right)$ = Unity. Hence it appears that the Force is inverſely as the Cube of the Diſtance; and the Velocity, every where, equal to that whereby the Body might revolve in a Circle at the ſame Diſtance.

* Art. 61.
† Art. 227.
‡ Art. 231.



EXAMPLE II.

234. *Let it be required to find the Law of the centripetal Force, whereby a Body, tending to the Focus* C, *is made to revolve in the Periphery of an Ellipſis* AQDB.



From the other Focus F draw FK parallel to CP meeting the Tangent PQ (at Right-angles) in K, join F, Q; putting the tranſverſe Axis AB = *a*, the Semi-conjugate OD = $\frac{1}{2}b$, and the Parameter $\left(\frac{b^2}{a}\right)$ $= p$: Then, CQ and CP being denoted as above*, we have FQ (=AB—CQ) $= a - s$; whence, by reaſon of the ſimilar Triangles CQP and FQK, it will be

* Art. 231.

$s : u :: a - \dot{s} : \mathrm{FK} = \frac{\overline{a-s}\cdot\times u}{s}$. But FK × CP is $= \mathrm{OD}^2$ (by the Nature of the Curve.) Hence we get $\frac{\overline{a-s}\times u^2}{s} = \frac{1}{4}b^2$; and conſequently $\frac{1}{u^2} = \frac{4a}{b^2 s} - \frac{4}{b^2}$; whereof the Fluxion being $-\frac{2\dot{u}}{u^3} = -\frac{4a\dot{s}}{b^2 s^2}$, we obtain

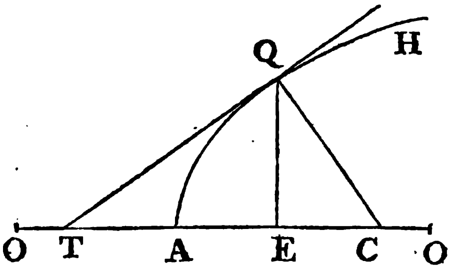
* Art. 227. † Art. 231. $\frac{\dot{u}}{u^3\dot{s}}$ * $= \frac{2a}{b^2} \times \frac{1}{s^2} = \frac{2}{ps^2}$, and $\sqrt{\frac{u\dot{s}}{s\dot{u}}}$ † $= \sqrt{\frac{2\times\overline{a-s}}{a}}$ $= \sqrt{\frac{\mathrm{FQ}}{\mathrm{AO}}}$. Hence, it appears that the centripetal Force is, in this Caſe, as the Square of the Diſtance inverſely; and the Velocity at Q is to that whereby the Body might deſcribe a Circle at the Diſtance CQ, every where, in the Ratio of $\mathrm{FQ}^{\frac{1}{2}}$ to $\mathrm{AO}^{\frac{1}{2}}$.

If the Curve had been an Hyperbola; then $\frac{a+s}{s} \times a^2$ (inſtead of $\frac{a-s}{s} \times u^2$) would have been $= \frac{1}{4}b^2$; and ſo $\frac{\dot{u}}{u^3\dot{s}} = \frac{2a}{b^2} \times \frac{1}{s^2} = \frac{2}{ps^2}$, the very ſame as before,

But, had it been a Parabola, the Equation would have been $\frac{a+0}{s} \times u^2 = \frac{1}{4}b^2$, or $\frac{u^2}{s}\left(=\frac{b^2}{4a}\right) = \frac{1}{4}p$; and the Force *ſtill*, as $\frac{2}{ps^2}$. But, the Meaſure of the Velocity $\left(\sqrt{\frac{u\dot{s}}{s\dot{u}}} = \sqrt{\frac{2a-2s}{a}}\right)$ in this Caſe becoming barely $= \sqrt{2}$, it follows that the Velocity in a Parabola is to that whereby the Body might deſcribe a Circle at the ſame Diſtance from the Center, in the conſtant Ratio of $\sqrt{2}$ to Unity.

EXAMPLE III.

235. *Let it be required to find the Law of the centripetal Force, by which a Body, tending to any given Point* C, *in the Axis, is made to deſcribe a conic Section* AQH.



Put the ſemi-tranſverſe Axis (OA) $= a$, the ſemi-conjugate $= b$, and the given Diſtance of the Point C from the Vertex A $= c$: Put alſo the Abſciſſa AE, $= x$, the Ordinate EQ $= y$, and CQ $= s$ *(as before)*.

The Area of the Triangle ECQ being ($= \frac{1}{2}$EC×EQ) $= \frac{cy - xy}{2}$, its Fluxion is therefore $= \frac{c\dot{y} - x\dot{y} - y\dot{x}}{2}$; which added to $y\dot{x}$, the Fluxion of the Area AEQ, gives $\frac{c\dot{y} + y\dot{x} - x\dot{y}}{2}$ for the Fluxion of the whole Area ACQ deſcribed about the Center of Force. Whence (*by Art.* 228.) the required centripetal Force at Q will be as $\frac{-s\dot{x}\ddot{y}}{\overline{c\dot{y} + y\dot{x} - x\dot{y}}|^3}$. Which Expreſſion is general, let the Curve be of what Kind it will. But in the Caſe above, y being $= \frac{b}{a}\sqrt{2ax \pm x^2}$, we have $\dot{y} = \frac{b\dot{x} \times \overline{a \pm x}}{a\sqrt{2ax \pm x^2}}$, $\ddot{y} = \frac{-ab\dot{x}^2}{\overline{2ax \pm x^2}|^{\frac{3}{2}}}$, and $c\dot{y} + y\dot{x} - x\dot{y} =$

$\frac{b\dot{x} \times \overline{ca + ax \pm cx}}{a\sqrt{2ax \pm x^2}}$; and therefore, by ſubſtituting theſe Values, we get $\frac{-s\dot{x}\ddot{y}}{\overline{c\dot{y} + y\dot{x} - x\dot{y}}|^3} = \frac{a^4 s}{b^2 \times \overline{ca + ax \pm cx}|^3}$ Which, becauſe $\frac{a^4}{b^2}$ is conſtant, will alſo be as $\frac{s}{\overline{ca + ax \pm cx}|^3}$. From whence it follows,

1°. If c be $= \mp a$, or the Center of Force be in the Center of the Section, the Force itſelf will be barely as $(\pm s)$ the Diſtance.

2°. If it be in the Focus, then $ac + ax \pm cx$ becoming $= \mathrm{CQ} \times a$, the Force will be inverſely as the Square of the Diſtance.

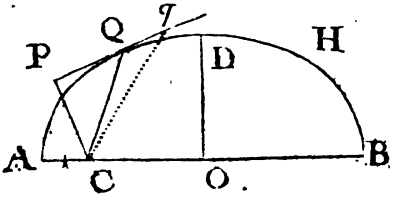
3°. If the given Point be in the Vertex A, the Force will be as $\frac{s}{x^3}$: Which therefore in the Circle (where $x = \frac{s^2}{2a}$) will be as $\frac{1}{s^5}$, or the fifth Power of the Diſtance reciprocally.

4°. Laſtly, if the Point C be at an indefinite Diſtance from the Vertex, or the Force be ſuppoſed to act in the Direction of Lines parallel to the Axis AO; then the Force will be as the Cube of OE inverſely.

PROPOSITION X.

236. *To determine the Ratio of the Velocities of Bodies revolving in different Orbits, about the ſame, or different, Centers; the Orbits themſelves, and the Forces whereby they are deſcribed, being given.*

Let AQH be any Orbit, deſcribed about the Center of Force C, and let the Force itſelf at the principal Vertex A be denoted by F; alſo let r ſtand for the Semiparameter, or the Ray of Curvature at the Vertex, and let CP be perpendicular to the Tangent QP.



Then, the Celerity at A being, always, as $\sqrt{rF}$ (*by Art.* 212.) we have CP : CA :: $\sqrt{rF}$ (the Velocity at A) to $\frac{CA \times \sqrt{rF}}{CP}$, the Velocity at Q (*by Art.* 225.) Which anſwers in all Caſes, let the Values of AC, *r* and *F* be what they will. *Q. E. I.*

COROLLARY I.

237. If the centripetal Force be as the Square of the Diſtance inverſely, or *F* be expounded by $\frac{1}{AC^2}$, the Velocity at Q will become $\frac{AC}{CP} \times \sqrt{\frac{r}{AC^2}}$, or $\frac{\sqrt{r}}{CP}$: Whence the Velocities, in different Orbits, about the ſame Center, are in the ſubduplicate Ratio of the Parameters, and the inverſe Ratio of the Perpendiculars from the Center of Force to the Tangents, conjunctly.

COROLLARY II.

238. Hence, if the Celerity at Q be denoted by Q*q*, and C*q* be drawn; then, Q*q* being as $\frac{\sqrt{r}}{CP}$, it follows that $\sqrt{r}$ is as CP × Q*q*, or as the Triangle QC*q*: There-

fore the Areas deſcribed about a common Center of Force in a given Time, are in the ſubduplicate Ratio of the Parameters.

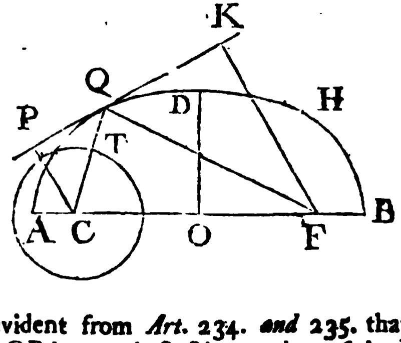
COROLLARY III.

239. Laſtly, ſince the Area of the Curve AQHB *&c.* when an Ellipſe[*], is known to be as $(AO \times OD)$ $AO \times \sqrt{r \times AO}$ (ſuppoſing O to be the Center) if the ſame be apply'd to $\sqrt{r}$, expreſſing the Area deſcribed in a given Part of Time (by the laſt *Corol.*) we ſhall thence have $AO \times \sqrt{AO}$, or $AO^{\frac{3}{2}}$ for the Meaſure of the Time of one whole Revolution. From whence it appears, that the periodic Times, let the Species of the Ellipſes be what they will, are in the ſeſquiplicate Ratio of their principal Axes.

[*] Art. 234.

PROPOSITION XI.

240. *The centripetal Force, tending to a given Point* C, *being as the Square of the Diſtances reciprocally, and the Direction and Velocity of a Body at any Point* Q *being given; to determine the Path in which the Body moves, and the periodic Time, in caſe it returns.*



It is evident from *Art.* 234. *and* 235. that the Trajectory AQB is a conic Section; whereof the Point C is one of the *Foci.*

Let F be the other Focus, and upon the Tangent PQK let fall the Perpendiculars CP and FK, and let CQ and FQ be drawn: Alſo put the ſemi-tranſverſe Axis AO $= a$, the given focal Diſtance CQ $= d$, and the Sine of the Angle of Direction CQP (to the Radius 1) $= m$; and let the given Velocity at Q be to that whereby the Body might revolve in a Circle about the Center C, at that Diſtance, in any given Ratio of n to Unity: Then it will be $n : 1 :: \mathrm{FQ}^{\frac{1}{2}} : \mathrm{AO}^{\frac{1}{2}}$ (by *Art.* 234.) therefore $n^2 : 1^2 :: \mathrm{FQ}\ (2a-d) : \mathrm{AO}\ (a)$; whence AO ($a$) is given $= \frac{d}{2-n^2}$. Moreover, ſince $\mathrm{CP} = m \times \mathrm{CQ}$, and $\mathrm{FK} = m \times \mathrm{FQ}$, we have OD^2 ($= \mathrm{CP} \times \mathrm{FK} = m^2 \times \mathrm{CQ} \times \mathrm{FQ} = \frac{m^2 n^2 d^2}{2-n^2}$; whence the ſemi-conjugate Axis (OD) is given likewiſe.

Laſtly, it will be (by *Art.* 239.) as $\mathrm{CT}^{\frac{3}{2}} : \mathrm{AO}^{\frac{3}{2}} ::$ (P) the periodic Time in any given Circle, whoſe Radius is CT, to $\left(\frac{\mathrm{AO}^{\frac{3}{2}}}{\mathrm{CT}^{\frac{3}{2}}} \times P\right)$ the required Time of one Revolution when the Orbit is an Ellipſis; that is, when n^2 is leſs than 2: For, if n^2 be $= 2$, the Curve (as its Axis $\frac{2d}{2-n^2}$ becomes infinite) will degenerate to a Parabola; and, if n^2 be greater than 2, the Axis being negative, it is then an Hyperbola; whoſe two principal Diameters are equal to $\frac{2d}{n^2-2}$ and $\frac{2mnd}{\sqrt{n^2-2}}$. *Q. E. I.*

COROLLARY.

241. Seeing neither the Value of AO, nor that of the periodic Time, is affected with m, it follows that the principal Axis, and the periodic Time, will remain in-

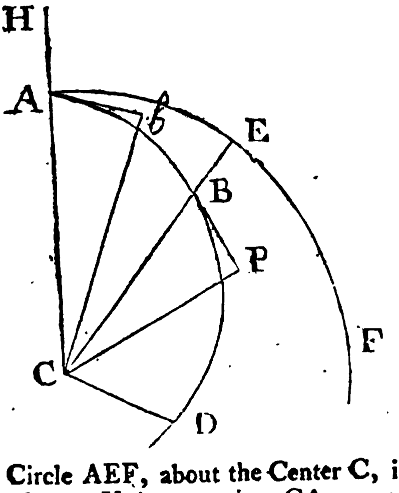
invariable, if the Velocity at Q be the ſame, let the Direction at that Point be what it will.

The ſame Solution may likewiſe be brought out, from *Art.* 238. by firſt finding the *principal Parameter*: For, it is evident that the Area deſcribed by the Body about the Center C, in any given Time, is to the Area deſcribed, in the ſame Time, by another Body revolving in a Circle at the Diſtance CQ, as mn to Unity: Hence, it will be $1^2 : m^2n^2 :: d : (m^2n^2d)$ the Semi-parameter [*]: From which (proceeding as above) we get $a \times m^2n^2d$ $(= OD^2) = m^2 \times \overline{2ad - d^2}$; and conſequently $a = \frac{d}{2 - n^2}$, the ſame as before.

[*] Art. 238.

PROPOSITION XII.

242. *The centripetal Force being as any Power (n) of the Diſtance, and the Direction and Velocity of a Body at any Point* A *being given, to determine the Orbit or Trajectory.*



From the Center of Force C, to any Point B in the required Trajectory ABD, let CB be drawn; join C, A, and let Ab be the given Direction of the Body at the Point A, and Cb perpendicular thereto; alſo let the Velocity at A be to that whereby a Body might deſcribe a Circle AEF, about the Center C, in any given Ratio of p to Unity; putting CA$=a$, and CB$=x$: Then

be

becauſe this laſt Velocity (the centripetal Force being as x^n (or a^n) is rightly defined by $a^{\frac{n+1}{2}}$ *, the Velocity of the Body at A will be truly expreſſed by $pa^{\frac{n+1}{2}}$.

* Art. 214.

Moreover, it is proved in *Art.* 221. *and* 206. that if the Celerity, at any given Diſtance a from the Center, be denoted by c, the Celerity at any other Diſtance x will be truly repreſented by $\sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}$: Whence, $pa^{\frac{n+1}{2}}$ being ſubſtituted for c, we have $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}}$ for the Celerity at B.

But now, to determine the Curve itſelf from hence, let BP be a Tangent to it at B, and CP perpendicular to BP; alſo let CB, produced, meet the Periphery of the Circle in E; putting the Arch AE$=z$, the foreſaid Velocity at B (to ſhorten the Operation) $=v$, and C$b=b$: Then it will be (by *Art.* 225.) $v : c$ (the Velocity at A) $:: b : \text{CP} = \frac{bc}{v}$ Whence BP ($= \sqrt{\text{CB}^2 - \text{CP}^2}$) $= \frac{\sqrt{x^2v^2 - b^2c^2}}{v}$.

Moreover (by *Art.* 35.) we have, as CB : CP :: v : ($\frac{\text{CP}}{\text{CB}} \times v$) the Velocity of the Body at B in a Direction perpendicular to CE; and conſequently, as CB : CE :: $\frac{\text{CP}}{\text{CB}} \times v$ (the ſaid Velocity) to $\frac{\text{CP} \times \text{CE}}{\text{CB}^2} \times v$ the angular Velocity of the Point E (revolving with the Body.) By the ſame *Article*, the Velocity at B in the Di-

Direction CBE will be $\frac{BP}{CB} \times v$: Therefore, the Velocity of E being to the Velocity of B, in the ſaid Direction, as $\frac{CP \times CE}{CB^2}$ to $\frac{BP}{CB}$, the Fluxions of AE (z) and CB (x) muſt conſequently be in that Ratio; that is, $\frac{CP \times CE}{CB^2} : \frac{BP}{CB} :: \dot{z} : \dot{x}$; whence $\dot{z} = \frac{CP \times CE}{CB \times BP} \times \dot{x} = \frac{bc}{v} \times \frac{a}{x} \times \frac{v\dot{x}}{\sqrt{x^2v^2 - b^2c^2}} = \frac{abc\dot{x}}{x\sqrt{x^2v^2 - b^2c^2}} = \frac{ab\dot{x}}{x\sqrt{\frac{x^2v^2}{c^2} - b^2}}$. Which Equation is general, let the Law of the centripetal Force be what it will: But in the Caſe above propoſed, v^2 being $= p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}$, and $c^2 = p^2a^{n+1}$; it becomes $\dot{z} = \frac{abp\dot{x}}{x\sqrt{\overline{p^2 + \frac{2}{n+1}} \times x^2 - p^2b^2 - \frac{2x^{n+3}}{\overline{n+1} \times a^{n+1}}}}$; whoſe Fluent is the Meaſure of the angular Motion; from which, when found, the Orbit may be conſtructed: Becauſe, when AE, or the Angle ACE is given, as well as CB, the Poſition of the Point B is alſo given. But this Value of $\dot{z}$ is indeed too complex to admit of a Fluent in algebraic Terms, or even by circular Arcs and Logarithms, except in certain particular Caſes; as when the Exponent n is equal to 1, — 2, — 3, or — 5; beſides ſome others wherein the Values of p and n are related in a particular Manner. *Q. E. I.*

Co-

COROLLARY I.

243. If the given Velocity at A be ſuch that $p^2 + \frac{2}{n+1} = 0$, or $p = \sqrt{\frac{-2}{n+1}}$ (which is always poſſible when the Value of $n+1$ is negative) our Equation will become $\dot{z} \times \frac{abp\dot{x}}{x\sqrt{-p^2b^2 + \frac{p^2x^{n+3}}{a^{n+1}}}}$: Which, by putting $n+3=m$, *&c.* is reduced to $\dot{z} = \frac{ab\dot{x}}{x\sqrt{-b^2+\frac{x^m}{a^{m-2}}}}$: Whereof the Fluent will be found (by the ſecond Part of this Work (equal to $\pm \frac{2a}{m}$ multiply'd by the Difference of the two circular Arcs, whoſe Secants are $\frac{x^{\frac{1}{2}m}}{ba^{\frac{1}{2}m-1}}$ and $\frac{a}{b}$ to the Radius Unity.) From this Value of the Arch AE the Poſition of the Point B, in the Orbit, is given.

But if the Angle of Direction CA*b* be a right one, the Fluent will become barely $= \pm \frac{2a}{m} \times$ Arch whoſe Secant is $\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}m}}$ (becauſe then $b=a$, and the Arch whoſe Secant is $\frac{a}{b}, = 0$) which therefore when $x^{\frac{m}{2}}$ becomes infinite, will be truly defined by $\pm\frac{1}{2m}\times$ whole Periphery AF, &c. Whence it is evident that the Body muſt either fly intirely off, or fall to the Center C, in a Number of Revolutions expreſſed by $\pm\frac{1}{2m}$; according as the Value of m is poſitive or negative.

Thus, if $n=-2$, and $m=1$, the Body will fly intirely off in half a Revolution: And, if $n=-4$, and $m=-1$, it will fall to the Center in half a Revolution.

COROLLARY II.

244. Moreover, tho' the Fluent expreſſing the Angle at the Center cannot be exhibited in a general Manner yet there are certain Caſes of the Exponent (n) where its reſpective Values may be derived from each other.

For let (as above) $n+3$ be put $=m$, and (to ſhorten the Operation) let CA (a) be taken as Unity: Then our Equation will be transformed to $\dot{x}=\frac{b\dot{x}}{x\sqrt{\overline{1+\frac{2}{\overline{m-2}\cdot p^2}}\times x^2-b^2-\frac{2x^m}{\overline{m-2}\cdot p^2}}}$: Make $y=x^{\frac{m}{2}}$, and it will be farther transformed to $\dot{x}=\frac{2}{m}\times\frac{b\dot{y}}{y\sqrt{\overline{1+\frac{2}{\overline{m-2}\cdot p^2}}\times y^{\frac{4}{m}}-b^2-\frac{2y^2}{\overline{m-2}\cdot p^2}}}$: Put $r=\frac{4}{m}$, and it will become $\dot{x}=\frac{2}{m}\times\frac{b\dot{y}}{y\sqrt{\frac{ry^2}{\overline{r-2}\cdot p^2}-b^2+\overline{1-\frac{r}{\overline{r-2}\cdot p^2}}\times y^r}}$: Laſtly, let $\frac{r}{\overline{r-2}\cdot p^2} = 1 + \frac{2}{\overline{r-2}\cdot q^2}$ (or $1 - \frac{r}{\overline{r-2}\cdot p^2} = - \frac{2}{\overline{r-2}\cdot q^2}$, or $q^2 = \frac{2p^2}{\overline{r-p^2}\times\overline{r-2}}$) and then we ſhall have $\dot{z} = \frac{2}{m} \times \frac{b\dot{y}}{y\sqrt{1+\frac{2}{\overline{r-2}\cdot q^2}\times \overline{y^2-b^2} - \frac{2y^r}{\overline{r-2}\cdot q^2}}}$.

Which Expreſſion (excepting the general Multiplicator $\frac{2}{m}$) being exactly of the ſame Form with the firſt above given, muſt therefore be the Fluxion of the Angle at the Center, when the Index of the Force is $r-3$; for the very ſame Reaſons that the former appears to be the Fluxion thereof when the Index is $m-3$ (or n.)

Hence, if the Fluent of

$$\frac{b\dot{y}}{y\sqrt{1+\frac{2}{\overline{r-2}\cdot q^2}\times \overline{y^2-b^2} - \frac{2y^r}{\overline{r-2}\cdot q^2}}},$$

or the Angle at the Center, when the Exponent is $r-3$ (or $\frac{4}{m}-3 = \frac{4}{n+3}-3$) be denoted by w, the Value of z, (the Meaſure of the ſaid Angle, when the Exponent is $m-3$ (or n) will be truly defined by $\frac{2w}{m}$.

From which we collect that, if the Indices of the Force, in any two Caſes, be repreſented by n and $\frac{4}{n+3}-3$, and the reſpective Diſtances from the Center by x and $x^{\frac{n+3}{2}}$, then the Angles themſelves correſponding to thoſe Diſtances will be every where in the conſtant Ratio of 2 to $n+3$. Therefore, when the Orbit can be conſtructed in the one Caſe, it alſo may in the other, provided the above Equation $q^2 \left(= \frac{2p^2}{r - p^2 \times \overline{r-2}}\right) = \frac{\overline{n+3} \cdot p^2}{2 + \overline{n+1} \cdot p^2}$, for the Relation of the Celerities at A, does not become impoſſible, as it will, ſometimes, when n is a negative Number.

COROLLARY III.

245. If the Body be ſuppoſed to move in a vertical Direction AH; then, putting the Velocity $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}} = 0$, we get x (CH) $= \overline{\tfrac{1}{2} p^2 \times \overline{n+1} + 1}^{\frac{1}{n+1}} \times a =$ the Height to which the Body will aſcend: Hence $\overline{\tfrac{1}{2} p^2 \times \overline{n+1} + 1}^{\frac{1}{n+1}} \times a - a$ ($=$ AH) is the Diſtance through which it muſt freely deſcend to acquire the given Celerity at A: This Diſtance, in caſe of an uniform Force, when $n = 0$, will become $= \tfrac{1}{2} p^2 a$: And, when the Force is inverſely as the Square of the Diſtance, it will then be $= \frac{p^2 a}{2 - p^2}$.

But, when $p = 1$, or the Velocity at A is juſt ſufficient to retain a Body in the Circle AEF, AH becomes $= \overline{\frac{3+n}{2}}^{\frac{1}{n+1}} \times a - a$: Which in the two Caſes aforeſaid will be equal to $\tfrac{1}{2} a$, and a reſpectively; but, infinite, when n is $= -3$.

COROLLARY IV.

246. When the Value of $n+1$ is poſitive, the Velocity at the Center, where $x=0$, will be barely $= \sqrt{p^2 + \frac{2}{n+1} \times a^{n+1}}$; but if the Value of $n+1$ be negative, the Velocity at the Center will be infinite; becauſe, then 0^{n+1} is infinite.

COROLLARY V.

247. Moreover, when $n+1$ is negative and x infinite, the Velocity alſo becomes $= \sqrt{p^2 + \frac{2}{n+1} \times a^{n+1}}$; becauſe then $x^{n+1} = 0$.

Hence, if the centripetal Force be inverſely as ſome Power of the Diſtance greater than the firſt, the Body may aſcend, *ad infinitum*, and have a Velocity always greater than $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1}}$; which is to, $pa^{\frac{n+1}{2}}$, the given Velocity, at A, as $\sqrt{p^2 + \frac{2}{n+1}}$ to p. And this will actually be the Caſe when the Value of $p^2 + \frac{2}{n+1}$ is poſitive, or p^2 greater than $\frac{2}{-n-1}$, but not otherwiſe, the ſquare Root of a negative Quantity being impoſſible.

Thus, if $n = -2$, or the Force be inverſely as the Square of the Diſtance, and p^2, at the ſame time, greater than 2 $\left(\frac{2}{-n-1}\right)$ the Body will not only continue to aſcend *in infinitum*, but have a Velocity always greater than that defined by $\sqrt{p^2-2}$, which is its Limit.

Co-

COROLLARY VI.

248. Hence the leaſt Celerity ſufficient to cauſe the Body to aſcend for ever in a Right-line is given. For, putting $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1}} = 0$, we have $p = \sqrt{\frac{2}{-n-1}}$. Therefore the leaſt Celerity by which the Body might aſcend for ever, is to that whereby it may revolve in a Circle AEF, as $\sqrt{\frac{2}{-n-1}}$ to Unity. From which it appears that, if the Force be inverſely as any Power of the Diſtance greater than the third, a leſs Velocity will cauſe a Body to aſcend *ad infinitum* than would retain it in a Circle.

SCHOLIUM.

249. From the Ratio of the Velocity $\left(\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}}\right)$ wherewith the Body arrives at any Diſtance x from the Center, to that $\left(x^{\frac{n+1}{2}}\right)$ * which it ought to have to revolve in a Circle at the ſame Diſtance, it will not be difficult to determine in what Caſes the Body will be forced to the Center, and in what others it will continue to fly from it *ad infinitum*.

* Art. 214.

For, firſt, if the Angle CA*b* be acute, or the Body from A begins to deſcend, it will continue to do ſo till it actually arrives at the Center, if the former Velocity, during the Deſcent, be not ſomewhere greater than the latter, or the Quotient $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ greater than Unity; becauſe, if it ever begins to aſcend, it muſt have an *Apſe*, as D (where a Right-line drawn from the Center cuts the Orbit at Right-angles) and there the Celerity muſt evidently be greater than that ſufficient to cauſe the Body to revolve in a Circle.

Secondly, but if the Quantity

$$\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}},$$ in the Acceſs of

the Body towards the Center, increaſes ſo as to become greater than Unity, or be every where ſo; then the Velocity at all inferior Diſtances being more than ſufficient to retain a Body in a Circle at any ſuch Diſtance, the Projectile cannot be forced to the Center.

After the ſame Manner, if the Angle CA*b* be obtuſe, or the Body from A begins to aſcend, it will continue to do ſo for ever, when the foreſaid Quantity is always greater than Unity, or, which is the ſame, when the Body, in its Receſs from the Center, has in every Place through which it paſſeth, a Velocity greater than ſufficient to retain it in a Circle at that Diſtance.

It therefore now remains to find in what Laws of the centripetal Force theſe different Caſes obtain: And, firſt, it is eaſy to perceive that when the Value of $n+1$ is poſitive, that of $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ will, by increaſing x, become equal to nothing. Therefore the Body cannot aſcend for ever in this Caſe: Neither can it deſcend to the Center (except in a Right-line) becauſe the foreſaid Quantity, by diminiſhing x, becomes greater than Unity (or any other aſſignable Magnitude.)

But, if the Value of n be betwixt -1, and -3, the ſaid general Expreſſion, taking x infinite, will alſo become infinite, provided the Value of $p^2 + \frac{2}{n+1}$ be poſitive (or p^2 greater than $\frac{2}{-n-1}$). Therefore the

Body, in this Caſe, may aſcend *ad infinitum*, but cannot poſſibly fall to the Center (except in a Right-line) ſince, $\sqrt{-\frac{2}{n+1}}$, the Value of the general Expreſſion, when $x = 0$, is greater than Unity.

Laſtly, if n be expreſſed by any negative Number greater than -3, or the Law of the Force be inverſely as any Power of the Diſtance greater than the third, the two extreme Values of $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ will, *ſtill*, be denoted as in the preceeding Caſe; but here the latter of them, $\sqrt{\frac{-2}{n+1}}$, is leſs than Unity. Therefore the Body muſt, in this Caſe, either aſcend for ever, or be forced to the Center; except in one particular Circumſtance, hereafter to be taken notice of.

Now, from theſe Obſervations we gather,

1°. That, when the centripetal Force is as any Power of the Diſtance directly, or leſs than the firſt Power thereof inverſely, the Orbit will always have an higher and a lower *Apſe*; beyond which the Body cannot aſcend or deſcend.

2°. That, when the centripetal Force is inverſely as any Power of the Diſtance (whole or broken) betwixt the firſt and third, the Orbit will alſo have two *Apſides*, if p be leſs than $\sqrt{-\frac{2}{n+1}}$; but otherwiſe, only one; in which laſt Caſe the Body, after it has paſſed its Apſe, will continue to recede from the Center *in infinitum*.

3°. That when the Force is inverſely as any Power greater than the third, the Orbit can, at moſt, have but one *Apſe*; but, in ſome Caſes, it will have none at all: And it may be worth while to inquire here, under what Reſtrictions of the Velocity (p) this will happen; ſince thereby, beſides being able to know when the Body will be forced to the Center, *&c.* we ſhall fall upon a Circumſtance ſomewhat remarkable and curious.

Now it appears, that, if the Body from A begins to deſcend, it muſt, when it comes to an *Apſe* at D, have a Velocity there greater than is ſufficient to retain it in a Circle; in which Caſe the general Expreſſion

$$\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$$

(ſo often mention'd above) muſt accordingly be greater than Unity. Let it be therefore made equal to Unity, which is the utmoſt Limit thereof, beyond which the Orbit cannot admit of an *Apſe*; putting at the ſame time $\dot{x}$, or its Diviſor

$$\sqrt{p^2 + \frac{2}{n+1} \times x^2 - p^2b^2 - \frac{2x^{n+}}{\overline{n+1}.a^{n+1}}},$$

in the general Equation of the Orbit, equal to nothing (it being always ſo at the *Apſides.*) Then, from theſe two Equations, duly order'd, we ſhall get $x =$

$$\overline{\frac{2+\overline{n+1}.p^2}{n+3}}\Bigg|^{\frac{1}{n+1}} \times a, \text{ and } p^2 \left(= \frac{x^{n+3}}{a^{n+1}}\right) = \overline{\frac{2+\overline{n+1}.p^2}{n+3}}\Bigg|^{\frac{n+3}{n+1}} \times \frac{a^2}{b^2}.$$

Now, it is evident, if the Value of p be greater than is given from the laſt Equation, the Orbit will have an Apſe; but if leſs, it can have none. In the former Caſe, the Body will therefore fly quite off; and in the latter, it will be forced to the Center. But we are now, naturally, led to inquire what will be the Conſequence when the Value of p is neither greater nor leſs, but exactly the ſame as given from the foreſaid Equation: This is the Caſe above hinted at; and here the Body will continue to deſcend for ever in a Spiral, yet never ſo low as to enter within the Circle whoſe Radius CD is $= \overline{\frac{2+\overline{n+1}.p^2}{n+3}}\Bigg|^{\frac{1}{n+1}} \times a$. For, if the contrary were poſſible, the Body, at its Arrival to the Circumference of that Circle, would (becauſe of the foreſaid Equations) not only have a Direction, but alſo Velocity proper to retain it therein; which cannot be, becauſe the Parts of the Orbit on either Side of an Apſe are always ſimilar to each other.

From the ſame Equation, the Value of the Limit will alſo be given when the Angle of Direction CA*b* is obtuſe, or the Body is projected upwards:

For that Equation (as is eaſy to demonſtrate *) admits of two different Roots, or Values of *p*; the one greater, the other leſs, than Unity: Whereof the former, giving CD (*x*) leſs than CA, is to be taken in the preceding Caſe, and the latter (making CD greater than CA) in the preſent. And the Body will, either, continue to aſcend for ever, or come to an *Apſe*, and from thence fall to the Center, according as the given Value of *p* is greater or leſs than that here ſpecified. But if it be neither greater nor leſs, but exactly the ſame, then the Body, tho' it will ſtill continue to aſcend for ever in a Spiral, yet it can never riſe ſo high as the Circumference of the Circle whoſe Radius CD is = $\overline{\frac{2+\overline{n+1}\cdot p^{2}}{n+3}}\Big|^{\frac{1}{n+1}} \times a$, for Reaſons ſimilar to thoſe already delivered, in reſpect to the preceding Caſe.

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* *Mathematical Diſſert. p.* 167.

END OF VOL. I.

THE DOCTRINE AND APPLICATION OF FLUXIONS.

CONTAINING

(Beſides what is common on the Subject)

A Number of New Improvements in the Theory.

AND

The Solution of a Variety of New, and very Intereſting, Problems in different Branches of the MATHEMATICKS.

PART II.

By THOMAS SIMPSON, F. R. S.

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THE DOCTRINE and APPLICATION OF FLUXIONS.

PART the Second.

SECTION I.

The Manner of inveſtigating the FLUXIONS *of* Exponentials, *with* Thoſe *of the Sides and Angles of ſpherical Triangles.*

250. THE Method of deriving the Fluxion of any Power, x^v, of a flowing Quantity, when the Exponent (v) is given or invariable, has been already ſhewn: But, if the Exponent be variable, that Method fails; in which Caſe the Quantity x^v is called an Exponential; whoſe Fluxion is thus determined.

Put $z = x^v$, and let the hyperbolic Logarithm of x be denoted by y; then that of $x^v (z)$ will, by the Nature of Logarithms, be $= vy$; and therefore its Fluxion $=$ $\dot{v}y + v\dot{y}$: But the Fluxion of the Logarithm of $z (= x^v)$ is

* Art. 126. is alſo expreſſed by $\frac{\dot{z}}{z}$ *; whence we have $\frac{\dot{z}}{z} = v\dot{y} + y\dot{v}$,
and conſequently $\dot{z} = zv\dot{y} + zy\dot{v}$: Which Equation, by ſub-
† Art. 126. ſtituting $\frac{\dot{x}}{x}$ for its Equal $\dot{y}$ †, becomes $\dot{z} = zy\dot{v} + \frac{zv\dot{x}}{x} =$
$x^{v}\, y\dot{v} + x^{v} \times \frac{v\dot{x}}{x} = x^{v}\, y\dot{v} + vx^{v-1}\,\dot{x} = \overline{x^{v}\,\dot{v} \times hyp.\ Log.\ x}$
$+ vx^{v-1}\dot{x}$.

The ſame otherwiſe, without introducing the Properties of Logarithms.

251. Let $1+z=x$, and $n+w=v$, ſuppoſing n conſtant and w variable: Then $x^{v} = \overline{1+z}|^{n+w} = \overline{1+z}|^{n} \times \overline{1+z}|^{w} = \overline{1+z}|^{n} \times \overline{1 + wz + \frac{w}{1} \times \frac{w-1}{2} \times z^{2} + \frac{w}{1} \times \frac{w-1}{2} \times \frac{w-2}{3} \times z^{3} + \&c.}$ ‡ $= \overline{1+z}|^{n} \times \overline{1 + wz + \overline{\tfrac{1}{2}w^{2} - \tfrac{1}{2}w} \times z^{2} + \overline{\tfrac{1}{6}w^{3} - \tfrac{1}{2}w^{2} + \tfrac{1}{3}w} \times z^{3} + \&c.}$ whoſe Fluxion, found the common Way, is $n\dot{z} \times \overline{1+z}|^{n-1} \times \overline{1 + wz + \overline{\tfrac{1}{2}w^{2} - \tfrac{1}{2}w} \times z^{2} + \overline{\tfrac{1}{6}w^{3} - \tfrac{1}{2}w^{2} + \tfrac{1}{3}w} \times z^{3}\ \&c.} + \overline{1+z}|^{n} \times \overline{\dot{w}z + w\dot{z} + \overline{w\dot{w} - \tfrac{1}{2}\dot{w}} \times z^{2} + \overline{\tfrac{1}{2}w^{2} - \tfrac{1}{2}w} \times 2z\dot{z} + \overline{\tfrac{1}{2}w^{2}\dot{w} - w\dot{w} + \tfrac{1}{3}\dot{w}} \times z^{3} + \overline{\tfrac{1}{6}w^{3} - \tfrac{1}{2}w^{2} + \tfrac{1}{3}w} \times 3z^{2}\dot{z}\ \&c.}$ which, by ſubſtituting $\dot{x}$ and $\dot{v}$ for their Equals $\dot{z}$ and $\dot{w}$, becomes $n\dot{x} \times \overline{1+z}|^{n-1} \times \overline{1 + wz + \overline{\tfrac{1}{2}w^{2} - \tfrac{1}{2}w} \times z^{2} + \&c.} + \overline{1+z}|^{n} \times \overline{\dot{v}z + w\dot{x} + \overline{w\dot{v} - \tfrac{1}{2}\dot{v}} \times z^{2} + \&c.}$

‡ Art. 99.

But, if w be, now, ſuppoſed to vaniſh, we ſhall have the true Value of the Fluxion when $v=n$; which, in that Circumſtance, appears to be $= n\dot{x} \times \overline{1+z}|^{n-1}$

$+ \overline{1+z}|^{v} \times \overline{z\dot{v} - \frac{1}{2}z^2\dot{v} + \frac{1}{3}z^3\dot{v} - \frac{1}{4}z^4\dot{v}}$ &c. $= v\dot{x} \times x^{v-1}$

$+ \dot{v}x^{v} \times \overline{z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4}$ &c. *Q. E. I.*

It is plain, becauſe the Series, $z - \frac{1}{2}z^2 + \frac{1}{3}z^3$ &c. here brought out, is known to expreſs the Fluent of $\frac{\dot{x}}{1+z}$, or the hyperbolic Logarithm of $1+z$ [*], that the two Concluſions agree exactly with each other: From either of which the following *Rule*, for the Fluxions of Exponentials, is deduced.

252. *To the Fluxion found by the common Rule* (Art. 14.) *conſidering the Exponent as conſtant, add the Quantity ariſing by multiplying the Fluxion of the Exponent, the hyperbolic Logarithm of the Root, and the propoſed Quantity itſelf, continually, together: The Sum will be the Fluxion when the Exponent is variable.*

Thus, for Example, let the Quantity propoſed be $\overline{a^2+z^2}|^{z}$, then the Fluxion thereof will be $z \times 2z\dot{z} \times \overline{a^2+z^2}|^{z-1} + \dot{z} \times \overline{a^2+z^2}|^{z} \times \overline{hyp.\ Log.\ a^2+z^2}$.

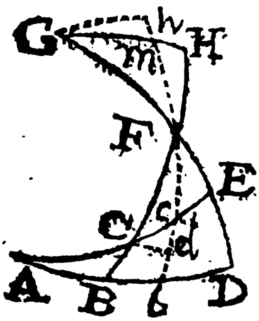
But, if the Root is conſtant, and only the Exponent variable, the *Exponential* will be more ſimple; and its Fluxion will then be had by *barely multiplying the Quantity itſelf by the Product under the Logarithm of the Root and the Fluxion of the Exponent.*

Thus, the Fluxion of a^x will be expreſſed by $a^x \times \dot{x} \times$ *hyp. Log.* a; and that of $\overline{a^2+b^2}|^{nx}$ by $\overline{a^2+b^2}|^{nx} \times n\dot{x} \times$ *hyp. Log.* $\overline{a^2+b^2}$. Theſe Kind of *Exponentials* oftener occur, in Practice, than any other; but, as it is very rare that we meet with *any*, I ſhall therefore proceed now to the other Conſideration propoſed in the Head of this Section; namely, the Method of determining the Fluxions of the Sides and Angles of ſpherical Triangles (a Thing very uſeful in practical Aſtronomy) which I ſhall deliver in the following Propoſitions.

[*] Art. 126.

PROPOSITION I.

253. *To determine the Ratio of the Fluxions of the ſeveral Parts of a right-angled ſpherical Triangle; ſuppoſing the Hypothenuſe, one Leg, or one Angle, to remain conſtant, while the other Parts vary.*



Let A, F, and G be the Poles of the three Great-Circles DEFG, ABD, and ACE; whereof the Poſition of each is ſuppoſed to continue invariable, while another Great-Circle HFCB is conceived to revolve about the Pole F: Whence, if GH be ſuppoſed perpendicular to FH, three variable right-angled Triangles, FGH, FCE, and ABC, will be formed; in the firſt whereof, the Hypothenuſe FG will remain conſtant; in the ſecond, the Leg EF; and in the third, the Angle A.

Let Bb (q) be the Fluxion (or indefinitely ſmall Increment*) of the Baſe AB, or the Angle F; and let Cd meet the Great-Circle bFb, at Right-angles, in d; then it will be (*per Spherics*) as *Sin.* FB (*Rad.*) : *Sin.* FC :: Bb (q) : Cd $= \frac{\textit{Sin.}\,\text{FC}}{\textit{Rad.}} \times q = \frac{\textit{Co-ſ.}\,\text{BC}}{\textit{Rad.}} \times q$:

And, *Tang.* C : *Rad.* :: Cd $\left(\frac{\textit{Co-ſ.}\,\text{BC}}{\textit{Rad.}} \times q\right) : \frac{\textit{Co-ſ.}\,\text{BC}}{\textit{Tang.}\,\text{C}} \times q$ = the Fluxion of BC.

Moreover, *Sin.* C : *Rad.* :: Cd $\left(\frac{\textit{Co-ſ.}\,\text{BC}}{\textit{Rad.}} \times q\right) : \frac{\textit{Co-ſ.}\,\text{BC}}{\textit{Sin.}\,\text{C}} \times q$ = the Fluxion of AC.

* Art. 134.

Laſtly, *Sine* of FB (*Rad.*) : *Sin.* FH (BC) :: B*b* (*q*) : $\frac{\text{Sin. BC}}{\text{Rad.}} \times q$ (=H*m*) = the Fluxion of GH, or its Complement C.

Now, if the ſeveral Quantities, in theſe three Equations for the Triangle ABC, be expounded by their reſpective Equals in the other two Triangles CEF and FGH, we ſhall alſo have

$$\frac{\text{Sin. CF}}{\text{Tang. C}} \times q = -\text{Flux. CF.}$$

$$\frac{\text{Sin. CF}}{\text{Sin. C}} \times q = -\text{Flux. CE.}$$

$$\frac{\text{Co-ſ. CF}}{\text{Rad.}} \times q = \text{Flux. C.}$$

And

$$\frac{\text{Co-ſ. FH}}{\text{Co-tang. GH}} \times q = \text{Flux. FH.}$$

$$\frac{\text{Co-ſ. FH}}{\text{Co-ſ. GH}} \times q = \text{Flux. G.}$$

$$\frac{\text{Sin. FH}}{\text{Rad.}} \times q = -\text{Flux. GH.}$$

Q. E. I.

COROLLARY I.

254. Hence, if, in any right-angled Spherical-Triangle, the Hypothenuſe be denoted by *h*, the two Legs by *L* and *l*, the Angles, reſpectively, adjacent to them by *A* and *a*, we ſhall, by ſubſtituting above, have three Equations for each of the three Caſes. From the Compariſon and Compoſition of which, the three following Tables are deduced; exhibiting all the different Varieties that can poſſibly happen, whether an Angle, a Leg, or the Hypothenuſe be ſuppoſed invariable.

TABLE I.

When one Angle *A* is invariable,

$$\dot{L}=\frac{\textit{Tang. a}}{\textit{Co-ſ. l}}\times\dot{l}=\frac{\textit{Sin. a}}{\textit{Co-ſ. l}}\times\dot{b}=\frac{\textit{Rad.}}{\textit{Sin. l}}\times\dot{a}$$

$$\dot{l}=\frac{\textit{Co-ſ. l}}{\textit{Tang. a}}\times\dot{L}=\frac{\textit{Co-ſ. a}}{R}\times\dot{b}=\frac{\textit{Co-tang. l}}{\textit{Tang. a}}\times\dot{a}$$

$$\dot{b}=\frac{\textit{Co-ſ. l}}{\textit{Sin. a}}\times\dot{L}=\frac{R}{\textit{Co-ſ. a}}\times\dot{l}=\frac{\textit{Co-tang. l}}{\textit{Sin. a}}\times\dot{a}$$

$$\dot{a}=\frac{\textit{Sin. l}}{R}\times\dot{L}=\frac{\textit{Tang. a}}{\textit{Co-tang. l}}\times\dot{l}=\frac{\textit{Sin. a}}{\textit{Co-tang. l}}\times\dot{b}$$

TABLE II.

When one Leg *L* is invariable,

$$\dot{A}=\frac{\textit{Tang. a}}{\textit{Sin. b}}\times\dot{b}=\frac{\textit{Sin. a}}{\textit{Sin. b}}\times\dot{l}=-\frac{\textit{R.}}{\textit{Co-ſ. b}}\times\dot{a}$$

$$\dot{a}=-\frac{\textit{Co-ſ. b}}{R}\times\dot{A}=-\frac{\textit{Sin. a}}{\textit{Tang. b}}\times\dot{l}=-\frac{\textit{Tang. a}}{\textit{Tang. b}}\times\dot{b}$$

$$\dot{b}=\frac{\textit{Sin. b}}{\textit{Tang. a}}\times\dot{A}=\frac{\textit{Co-ſ. a}}{R}\times\dot{l}=-\frac{\textit{Tang. b}}{\textit{Tang. a}}\times\dot{a}$$

$$\dot{l}=\frac{\textit{Sin. b}}{\textit{Tang. a}}\times\dot{A}=\frac{\textit{R.}}{\textit{Co-ſ. a}}\times\dot{b}=-\frac{\textit{Tang. b}}{\textit{Sin. a}}\times\dot{a}$$

TABLE III.

When the Hyp. is invariable,

$$\dot{A}=-\frac{\textit{Co-tang. l}}{\textit{Co-ſ. L}}\times\dot{L}=-\frac{\textit{Co-ſ. l}}{\textit{Co-ſ. L}}\times\dot{a}=\frac{R}{\textit{Sin. L}}\times\dot{l}$$

$$\dot{L}=-\frac{\textit{Co-.ſ. L}}{\textit{Co-tang. l}}\times\dot{A}=\frac{\textit{Sin. l}}{\textit{R.}}\times\dot{a}=-\frac{\textit{Tang. l}}{\textit{Tang. L}}\times\dot{l}$$

Where, and alſo in the two preceding Tables, the Leg *L* is adjacent to the Angle *A*, and the Leg *l* to the Angle *a*.

Co-

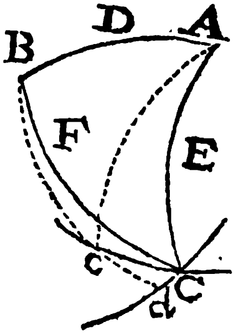
COROLLARY II.

255. From the third original Equation, expreſſing the Fluxion of the Angle *C* (*Vid. Art.* 253.) it appears that the Superficies of any Spherical-Triangle ABC, is proportional to the Exceſs of its three Angles above two Right-Angles. For (BCdb) the Fluxion of the *Triangle* ABC, is = *Sine* BC x Bb, by *Art.* 161.) which being to, $\frac{Sin.\ BC}{Rad.} \times Bb$, the Fluxion of the Angle *C*, above ſpecified, in the conſtant Ratio of Radius to Unity, the Fluents themſelves (properly corrected) muſt therefore be in that Ratio; that is, the Superficies of the Triangle ABC will always be proportional to the Increaſe of the Angle *C*, from its coinciding with *A*, or as the Exceſs of *A* and *C* above two Right-Angles.

PROPOSITION II.

256. *To determine the Ratio of the Fluxions, or the indefinitely ſmall Increments, of the different Parts of an oblique Spherical-Triangle* ABC; *two Sides thereof* AB, AC *being invariable, in Length.*

Let Cc be an indefinitely ſmall Part of the Parallel deſcribed by the Extreme C of the given Side AC, in its Motion about the given Point A; moreover, let Cd be Part of another Parallel, whoſe Pole is the given Point B; let the Great-Circle Bc meet Cd in d; and let the three Sides, AB, AC, and BC, of the Triangle be denoted by *D*, *E*, and *F* reſpectively.



Then (*per Spherics*) we ſhall have

$$R : S.E :: \mathrm{CA}c\ (\dot{A}) : \mathrm{C}c = \frac{S.E.}{R} \times \dot{A};$$

And, $R : S.F :: \mathrm{CB}d\,(\dot{B}) : \mathrm{C}d = \frac{S.F}{R} \times \dot{B}$.

Alſo, $R : S.\,d\mathrm{C}c\ (\mathrm{ACB}) :: \mathrm{C}c : \dot{F} = \frac{S.E \times S.C}{R^2} \times \dot{A}$: But $S.C : S.D :: S.B : S.E$; therefore $S.E \times S.C = S.D \times S.B$, and conſequently $\dot{F}$, alſo, $= \frac{S.D \times S.B}{R^2} \times \dot{A}$.

Again, $R : \text{Co-ſ.}\ d\mathrm{C}c\ (\mathrm{ACB}) :: \mathrm{C}c\ \left(\frac{S.E}{R} \times \dot{A}\right) : \frac{S.E. \times \text{Co-ſ.}\ C}{R^2} \times \dot{A}\ (= \mathrm{C}d) = \frac{S.F}{R} \times \dot{B}$;

Whence $\dot{B} = \frac{S.E \times \text{Co-ſ.}\ C}{R \times S.F} \times \dot{A}$.

Laſtly, $\text{Co-t.}\ c\mathrm{C}d : (C) : R :: \mathrm{C}d\ \left(\frac{S.F}{R} \times \dot{B}\right) : \dot{F} = \frac{S.F}{\text{Co-t.}\ C} \times \dot{B}$.

Whence, by the very ſame Argument (ſubſtituting D for E, and C for B in the two laſt Equations) we likewiſe have $\dot{C} = \frac{S.D \times \text{Co-ſ.}\ B}{R \times S.F} \times \dot{A}$, and $\dot{F}$ $\left(= \frac{S.F}{\text{Co-t.}\ C} \times \dot{B}\right) = \frac{S.F}{\text{Co-t.}\ B} \times \dot{C}$.

Now, from the Equations thus found, it is manifeſt,

1°. $\dot{A} : \dot{F} :: R^2 : S.D \times S.B$ ($::$ Co-ſeca. $D : S.B$)
2°. $\dot{A} : \dot{B} :: R \times S.F : S.E \times \text{Co-ſ.}\ C$
3°. $\dot{A} : \dot{C} :: R \times S.F : S.D \times \text{Co-ſ.}\ B$
4°. $\dot{B} : \dot{F} :: \text{Co-t.}\ C : S.F$
5°. $\dot{C} : \dot{F} :: \text{Co-t.}\ B : S.F$
6°. $\dot{B} : \dot{C} :: \text{Co-t.}\ C : \text{Co-t.}\ B$ ($:: T.B : T.C$) Q.E.I.

257. Theſe Proportions, for the Fluxions of the Parts of a Spherical-Triangle, are very uſeful in various Caſes in *Practical Aſtronomy*; whereof I ſhall here put down one or two Inſtances.

The firſt is; To determine the annual Alteration of the Declination and Right-Aſcenſion of a fixt Star, through the Preceſſion of the Equinox.

Here *A* muſt denote the Pole of the Ecliptic, *B* that of the Equinoctial, and *C* the Place of the Star; and then (by the firſt and fourth Proportions) we have

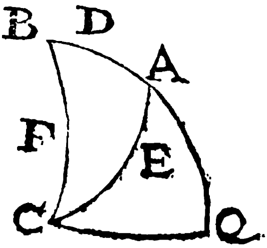
Co-ſeca. D : Sin. B :: $\dot{A} : \dot{F}$; and

S. F : Co-t. C :: $\dot{F} : \dot{B}$;

That is, 1°, As the Co-ſecant of the Obliquity of the Ecliptic is to the Sine of the Star's Right-Aſcenſion from the *ſolſtitial Colure*, ſo is the *Preceſſion* of the Equinox, or Alteration of Longitude, to the Alteration of Declination.

2°. As the Co-ſine of the Star's Declination is to the Co-tangent of its Angle of Poſition, ſo is the Alteration of Declination (found as above) to the Alteration of Right-Aſcenſion correſponding.

The ſecond Example is to find how much the Amplitude, and the Time of the apparent Riſing and Setting of the Sun, or a Star, are affected by Refraction.



In this Caſe *A* muſt denote the Pole of the Equator, and *B* the *Zenith*, and the Side BC muſt be an Arch of 90 Degrees, ſo that the Star *C* may coincide with the Horizon QC: Then, from the very ſame Proportion, we have,

Sin. B : Co-ſeca. D :: $\dot{F} : \dot{A}$,

And, R : Co-t. C :: $\dot{F} : \dot{B}$

But, R : Co-t. C (T. QCA) :: Sin. B (CQ) : Co-tang. D (Tang. QA)

Hence it apppears,

1°. That, as the Co-ſine of the true Amplitude (conſidered independent of Refraction) is to the Tangent of the Pole's Elevation, ſo is the given horizontal Refraction to the Difference of Amplitudes thence ariſing.

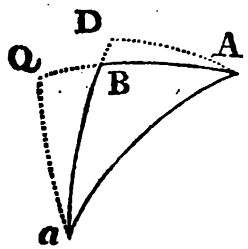
2°. And, that, as the Co-ſine of the true Amplitude is to the Secant of the Pole's Elevation, ſo is the ſaid horizontal Refraction to the Effect thereof in the Time of Riſing, or Setting of the Sun, or Star.

But this laſt Proportion may be otherwiſe expreſſed, without the Amplitude: Thus,

$S.AB \times S.AC \times S.A : R^2 ::$ the horizontal Refraction, to the ſame Effect.

PROPOSITION III.

258. *To determine the ſame as in the preceding Problem; Suppoſing one Side* AB *and one of its adjacent Angles,* B, *to continue invariable.*



If from the End of the given Side, oppoſite to the given Angle, a Perpendicular AD be let fall, that Perpendicular, as well as the Segment BD cut off thereby, will be a conſtant Quantity, while the other Parts of the Triangle A*a*D vary, by the Motion of *a* along the Arch *a*BD. Therefore the Problem is reſolved by *Caſe* 2. of right-angled Triangles. *Vid. Art.* 254.

259. It may not be amiſs to give one Example of the Uſe of this laſt Propoſition: Which ſhall be, in finding the Parallax of a Planet in Longitude and Latitude; that of Altitude being given.

Here *A* muſt ſtand for the Pole of the Ecliptic, *B* the Zenith, and *a* the Planet: Then, if the Hypothenuſe A*a* be denoted by *h*, the Leg. D*a* by *l*, and the given Parallax, in Altitude, by $\dot{l}$, it will appear, from the Place above quoted, that $\dot{A}$ (the Parallax in Long.) will be $= \frac{\text{Sin. } a}{\text{Sin. } b} \times \dot{l} = \frac{\text{Sin. } BaA}{\text{Sin. } Aa} \times \dot{l}$, and $\dot{b}$ (the Parallax in Lat.) $= \frac{\text{Co-ſ. } a}{\text{Rad.}} \times \dot{l} = \frac{\text{Co-ſ. } BaA}{\text{Rad.}} \times \dot{l}$.

If the Planet be in (or very near) the Ecliptic, and *a*Q be ſuppoſed a Portion of the Ecliptic, meeting AB, at Right-Angles, in Q, then (*per Spherics*) $\frac{\text{Sin. } BaA}{\text{Sin. } Aa} \left(\frac{\text{Co-ſ. } BaQ}{\text{Radius}}\right) = \frac{\text{Tang. } Qa}{\text{Tang. } Ba}$; alſo $\frac{\text{Co-ſ. } BaA}{\text{Rad.}} \left(\frac{\text{Sin. } BaQ}{\text{Rad.}}\right) = \frac{\text{Sin. } QB}{\text{Sin. } Ba}$; whence, by ſubſtituting theſe Values above, we ſhall, *in this Caſe*, have $\dot{A} = \frac{\text{Tang. } Qa}{\text{Tang. } Ba} \times \dot{l}$ and $\dot{b} = \frac{\text{Sin. } QB}{\text{Sin. } Ba} \times \dot{l}$; that is, in Words,

As, the Tangent of the Planet's Zenith Diſtance, is to the Tangent of its Longitude from the nonageſimal Degree of the Ecliptic, ſo is the Parallax in Altitude to the Parallax in Longitude.

And, as the Sine of the Zenith Diſtance to the Coſine of the Altitude of the nonageſimal Degree, ſo is the Parallax in Altitude to the Parallax in Latitude.

Becauſe the Parallax in Altitude, the horizontal Parallax (M) being given, is nearly $= \frac{\text{Sin. } Ba}{\text{Rad.}} \times M$, if this Value be ſubſtituted for $\dot{l}$, in the two laſt Equations, we ſhall get $\dot{b} = \frac{\text{Sin. } QB}{\text{Rad.}} \times M$, and $\dot{A} = \frac{\text{Tang. } Qa \times \text{Sin. } Ba}{\text{Rad.} \times \text{Tang. } Ba} \times M = \frac{\text{Sin. } AB \times \text{Sin. } BAa}{\text{Rad.}^2} \times M$.

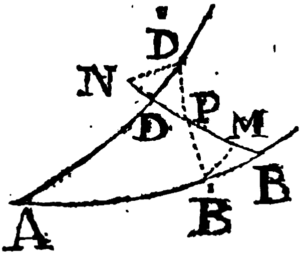
Whence, we have theſe two other Theorems, for finding the required Parallaxes immediately from the horizontal Parallax, without either the Altitude or its *Parallax*.

1. As Radius to the Co-ſine of the Altitude of the *nonageſimal Degree* of the Ecliptic, ſo is the horizontal Parallax to the Parallax in Latitude.

2. And as the Square of Radius to the Rectangle under the Sines of the Altitude of the *nonageſimal Degree* and the Planet's Longitude from thence, ſo is the horizontal Parallax to the Parallax in Longitude.

PROPOSITION IV.

260. *Still, to determine the ſame Thing; ſuppoſing, one Angle* A, *and the Length of its oppoſite Side* BD (*or* $B'D'$) *to remain conſtant.*



Let $B'D'$ (equal to BD) interſect BD in an indefinitely ſmall Angle at *P*, and meet AB and AD in B' and D'; alſo in $B'D'$ produced let there be taken PN = PD' and PM = PB', and let *N*, D', and *M*, B' be joined.

Since, by Hypotheſis, DB = $D'B'$ = MN, if from the firſt and laſt of theſe equal Quantities DM, common, be taken away, there will remain BM = DN.

Moreover, ſince the Triangles BMB' and DND', in their ultimate State, may be conſidered as rectilineal, and right-angled at *M* and *N**, it will therefore be, as

* Art. 134.

BM : BB' :: *Co-ſ. B* : *Radius*

And DN : DD' :: *Co-ſ. D* :: *Radius.*

From whence, the Extremes in both Proportions being the ſame, we have $B\acute{B} : D\acute{D} ::$ *Co-ſ.* D : *Co-ſ.* B: And therefore, if AB be denoted by H and AD by K, it appears that $\dot{H} : \dot{K} ::$ *Co-ſ.* D : *Co-ſ.* B.

Again, *per Spherics*, *Sin.* A : *Sin.* BD (G) :: *Sin.* D : *Sin.* H :: *Flux. Sin.* D : *Flux. Sin.* H; becauſe, the Sines themſelves being in a conſtant Ratio, their Fluxions muſt be in the ſame Ratio: But the Fluxion of the Sine of any Arc, or Angle, is to the Fluxion of the Arc or Angle itſelf, as the Co-ſine to Radius *: Therefore the *Flux. Sin.* D being $= \frac{\text{Co-ſ. } D}{\text{Rad.}} \times \dot{D}$, and *Flux. Sin.* $H = \frac{\text{Co-ſ. } H}{\text{Rad.}} \times \dot{H}$, it follows that, *Sin.* A : *Sin.* G :: *Co-ſ.* $D \times \dot{D}$: *Co-ſ.* $H \times \dot{H}$; or $\dot{D} : \dot{H} ::$ *Sin.* $A \times$ *Co-ſ.* H : *Sin.* $G \times$ *Co-ſ.* D: And, by the very ſame Argument, $\dot{B} : \dot{K} ::$ *Sin.* $A \times$ *Co-ſ.* K : *Sin.* $G \times$ *Co-ſ.* B. Now, by compounding the former of theſe two Proportions with the firſt above given, we get, $\dot{D} : \dot{K} ::$ *Sin.* $A \times$ *Co-ſ.* H : *Sin.* $G \times$ *Co-ſ.* B. And, by compounding this laſt with $\dot{K} : \dot{B} ::$ *Sin.* $G \times$ *Co-ſ.* B : *Sin.* $A \times$ *Co-ſ.* K (that immediately preceding it) we alſo obtain $\dot{D} : \dot{B} ::$ *Co-ſ.* H : *Co-ſ.* K.

Whence, by collecting theſe ſeveral Proportions together, we have the following Table, for all the different Caſes.

$\dot{H} : \dot{K} ::$ *Co-ſ.* D : *Co-ſ.* B

$\dot{D} : \dot{B} ::$ *Co-ſ.* H : *Co-ſ.* K

$\dot{D} : \dot{H} ::$ *Tang.* D : *Tang.* H

$\dot{B} : \dot{K} ::$ *Tang.* B : *Tang.* K

$\dot{K} : \dot{D} ::$ *Sin.* $G \times$ *Co-ſ.* B : *Sin.* $A \times$ *Co-ſ.* H

$\dot{H} : \dot{B} ::$ *Sin.* $G \times$ *Co-ſ.* D : *Sin.* $A \times$ *Co-ſ.* K

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* Art. 142.

It may be obſerved, that the fourth and the laſt are no new Caſes, but only the third and fifth repeated: And that, though the former of the two, laſt named, differs from that found above; yet it is very eaſily deduced from it: For, ſince it appears that $\dot{D} : \dot{H} :: \frac{\text{Sin.}\,A}{\text{Co-ſ.}\,D} : \frac{\text{Sin.}\,G}{\text{Co-ſ.}\,H}$, and becauſe $\text{Sin.}\,A : \text{Sin.}\,G :: \text{Sin.}\,D : \text{Sin.}\,H$, it follows that $\dot{D} : \dot{H} :: \frac{\text{Sin.}\,D}{\text{Co-ſ.}\,D} : \frac{\text{Sin.}\,H}{\text{Co-ſ.}\,H} :: \text{Tang.}\,D : \text{Tang.}\,H$. Q. E. I.

There is yet another Problem, when two Angles remain conſtant; but this, by taking the Triangle formed by the Poles of the three given Circles, is reduced to *Problem* 2.

SECTION II.

Of the Reſolution of fluxional Equations, or the Manner of finding the Relation of the flowing Quantities from that of the Fluxions.

261. WHEN an Equation, expreſſing the Relation of the Fluxions of the two variable Quantities, contains *only* one of thoſe Fluxions with its reſpective flowing Quantity in each Term, the Relation of the Quantities will be obtained by finding the Fluent of every Term; as has been already taught, in *Sect.* VI. *Part* I.

Thus, if $ax^2\dot{x} = y^3\dot{y}$, then will $\frac{ax^3}{3} = \frac{y^4}{4}$.

And, if $x^n y^m \dot{x} = a\dot{y}$; by reducing it firſt to $x^n \dot{x} = ay^{-m}\dot{y}$ (ſo that its variable Quantities may be ſeparated) we have $\frac{x^{n+1}}{n+1} = \frac{ay^{1-m}}{1-m}$.

But, if the given Equation has its indeterminate Quantities and their Fluxions ſo complicated together, that it cannot be brought under the Form there preſcribed, the Taſk will become much more difficult; nor is there any general Method to be given for ſuch Kinds of Equations, whereof there are an infinite Variety.

The Method of Infinite Serieſes (in ſome meaſure explained already, and more fully conſidered hereafter) is indeed very comprehenſive, and may be applied to good Purpoſe in various Caſes; but, being tedious and attended with a Number of Inconveniencies, it is a Method we ought never to have Recourſe to till we have tried what may be, otherways, effected, by help of ſuch particular Rules and Obſervations as we have been able to collect.

Accordingly, I ſhall, here, firſt point out ſome of the moſt proper Ways to be tried, in order, if poſſible, to bring out the Solution without an Infinite Series.

262. *The firſt Method is, by multiplying, or dividing, the given Equation by ſome Power or Product of the Quantities concerned; ſo as to bring it, if poſſible, under the Form of ſuch Fluxions, as, we know, do ariſe, if not from the firſt, yet from the ſecond, or third, of the three general Rules in the direct Method.*

Thus, if the given Equation be $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{x^m \dot{x}}{ay^n}$; then, the whole being multiplied by xy, ſo that the two firſt Terms, $y\dot{x} + x\dot{y}$, may become the (known) Fluxion of the Rectangle xy [*], there ariſes $y\dot{x} + x\dot{y} = \frac{x^{m+1}\dot{x}}{ay^{n-1}}$: But ſtill we are at a Loſs for the Fluent of the laſt Term, unleſs n be taken $= 1$ (ſo that y may vaniſh). In that Caſe we have $xy = \frac{x^{m+2}}{\overline{m+2} \times a}$; expreſſing the Relation of the Fluents when that of the Fluxions is $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{x^m \dot{x}}{ay}$: Which appears to be the only Caſe, of the given Equation, where this Method is of Uſe.

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[*] Art. 10.

Again, let the Equation $\frac{p\dot{x}}{x}+\frac{r\dot{y}}{y}=\frac{x^{m}\dot{x}}{ay^{n}}$ be propoſed.

Here, multiplying by $x^{p}y^{r}$ (where the Exponents are the ſame as the Coefficients of $\frac{\dot{x}}{x}$ and $\frac{\dot{y}}{y}$) we get $px^{p+1}\dot{x}\times y^{r}+x^{p}\times ry^{r-1}\dot{y}=\frac{x^{m+p}\dot{x}}{ay^{n-r}}$; in which the former Part of the Equation is known to expreſs the
* Art. 15. Fluxion of $x^{p}y^{r}$ *. Therefore, when $n=r$, the Relation of the Fluents may be found, and will be expreſſed by $x^{p}y^{r}=\frac{x^{m+p+1}}{\overline{m+p+1}\times a}$: Which, if no Correction by a conſtant Quantity be neceſſary, may be reduced to $y^{r}=\frac{x^{m+1}}{\overline{m+p+1}\times a}$.

The ſame Method may alſo be extended to Fluxions of the higher Orders: Let $\ddot{x}-x\dot{z}^{2}=f\dot{z}^{2}$ (which Equation occurs hereafter, in the Reſolution of a Problem of ſome Difficulty). Then, multiplying by $\dot{x}$, it becomes $\dot{x}\ddot{x}-x\dot{x}\dot{z}^{2}=f\dot{z}^{2}\dot{x}$; where, $\dot{z}$ being conſtant, each Term admits, now, of a perfect Fluent, and we therefore have $\frac{\dot{x}^{2}}{2}-\frac{x^{2}\dot{z}^{2}}{2}=fx\dot{z}^{2}$: From whence, ſuppoſing no Correction neceſſary, $\dot{z}=\frac{\dot{x}}{\sqrt{2fx+xx}}$, and $z=$ *hyp. Log.* $\overline{f+x+\sqrt{2fx+x^{2}}}$ (*by Art.* 126.)

263. *It may happen that the Solution of an Equation will become more eaſy by firſt taking the Fluxion thereof; when, by that means, ſome of the Terms deſtroy each other.*

The following is an Inſtance of it (which, alſo, occurs hereafter). Let $y+\frac{\dot{y}\times\overline{a-x}}{\dot{x}}=x-\frac{y\dot{x}}{\dot{y}}$: Whoſe Fluxion,

ion, making $\dot{x}$ conſtant, is $\dot{y} + \frac{\ddot{y} \times \overline{a-x} - \dot{x}\dot{y}}{\dot{x}} = \dot{x} - \frac{\dot{y}\dot{x}\dot{y} - y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$: Which, by reaſon of the Terms deſtroying one another, is reduced to $\frac{\ddot{y} \times \overline{a-x}}{\dot{x}} = \frac{y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$: Therefore, by expunging $\ddot{y}$, &c. we get $\dot{y}y^{-\frac{1}{2}} = \dot{x} \times \overline{a-x}|^{-\frac{1}{2}}$, and conſequently $2y^{\frac{1}{2}} = -2 \times \overline{a-x}|^{\frac{1}{2}}$ + *ſome conſtant Quantity.*

264. ***Another Method, chiefly applicable to Equations, of the firſt Order of Fluxions, wherein only one of the two variable Quantities (x or y) enters, is, to ſubſtitute for the Ratio of the two Fluxions ($\dot{x}$ and $\dot{y}$): From whence the Value of that Quantity will be had, immediately, in Terms of the ſaid aſſumed Ratio: And then, by taking its Fluxion, that of the other Quantity (and from thence the Quantity itſelf) will become known.***

Thus, let $a\dot{x}\dot{y}^3 = y \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2$ (being the Equation of the Curve that generates the Solid of the *leaſt Reſiſtance*, when the Bulk and greateſt Diameter are given). Then, by putting $\frac{\dot{x}}{\dot{y}} = v$, and ſubſtituting above, we get $av\dot{y}^4 = y \times \overline{v^2\dot{y}^2 + \dot{y}^2}|^2 = y\dot{y}^4 \times \overline{v^2+1}|^2$; and conſequently $y = \frac{av}{\overline{v^2+1}|^2}$: Therefore $\dot{y} = \frac{a\dot{v} - 3av^2\dot{v}}{\overline{vv+1}|^3}$; and conſequently $\dot{x}$ $(= v\dot{y}) = \frac{av\dot{v} - 3av^3\dot{v}}{\overline{vv+1}|^3}$: Whoſe Fluent may be found, from *Art.* 84. or, otherwiſe, thus: Put $w^2 = v^2 + 1$; then $v^2 = w^2 - 1$, and $w\dot{w} = v\dot{v}$; by ſubſtituting which Values there ariſes $\dot{x} = \frac{aw\dot{w} - 3aw\dot{w} \times \overline{w^2-1}}{w^6} = 4a\dot{w}w^{-5} - 3a\dot{w}w^{-3}$; and fore $x = \frac{4aw^{-4}}{-4} - \frac{3aw^{-2}}{-2} = -\frac{a}{w^4} + \frac{3a}{2w^2} = \frac{3aw^2-2a}{2w^4}$ $= \frac{3a\times\overline{v^2+1}-2a}{2\times\overline{vv+1}|^2} = \frac{a\times\overline{3vv+1}}{2\times\overline{vv+1}|^2}$; which, corrected (by taking y, or $v=0$) becomes $x = \frac{a\times\overline{3vv+1}}{2\times\overline{vv+1}|^2} - \frac{a}{2}$. From this Equation, by completing the Square, *&c.* v may be found in Terms of x; whence the correſponding Value of y $\left(=\frac{av}{\overline{vv+1}|^2}\right)$ will alſo be known.

265. *The fourth Method*, which chiefly obtains when one of the indeterminate Quantities and its Fluxion, ariſe but to a ſingle Dimenſion each, *may be thus*:

Let the Value of that Quantity, which is leaſt involved, be firſt ſought, from the fictitious Equation ariſing by neglecting all the Terms in the given Equation, where neither that Quantity, nor its Fluxion, are found: Then, to that Value, let ſome Power, or Powers, of the other Quantity, with unknown Coefficients, be added (according to the Dimenſions of the Terms neglected) and let the Sum be ſubſtituted in the given Equation, as the true Value of the firſt mentioned Quantity: By which means a new Equation will reſult; from whence the aſſumed Coefficients may, ſometimes, be determined.

Ex. Let the given Equation be $cx^2\dot{x} + y\dot{x} = a\dot{y}$.

By neglecting $cx^2\dot{x}$, or feigning $y\dot{x} = a\dot{y}$, we get $\frac{\dot{x}}{a} = \frac{\dot{y}}{y}$: and conſequently $\frac{x}{a}$; = hyp. Log. y — hyp. Log. d [*] = hyp. Log. $\frac{y}{d}$; d being any conſtant Quantity, which the Nature of the Problem may require. Hence $\frac{y}{d}$ = the Number whoſe hyperbolical Logarithm is $\frac{x}{a}$: Which Number, if M be put for (2,71828 *&c.*) the

[*] Art. 126, and 78.

the Number whoſe hyp. Log. is Unity, will be expreſſed by $\overline{M}|^{\frac{x}{a}}$ (ſince it is evident that the hyp. Log. hereof is $\frac{x}{a}\times\text{Log.}\,M=\frac{x}{a}$): Therefore $\frac{y}{d}=\overline{M}|^{\frac{x}{a}}$ and $y=d\times\overline{M}|^{\frac{x}{a}}$. Now, to the Value thus found, let there be added Ax^2+Bx+C, in order to get the true Value; and then, $\dot{y}$ being $=2Ax\dot{x}+B\dot{x}+\frac{d\dot{x}}{a}\times\overline{M}|^{\frac{x}{a}}$ *, we ſhall, by ſubſtituting in the given Equation, have $cx^2\dot{x}+Ax^2\dot{x}+Bx\dot{x}+C\dot{x}+d\dot{x}M^{\frac{x}{a}}=2Aax\dot{x}+Ba\dot{x}+d\dot{x}M^{\frac{x}{a}}$, and conſequently $\overline{c+A}\times x^2\dot{x}+\overline{B-2Aa}\times x\dot{x}+\overline{C-Ba}\times\dot{x}=0$. Whence $A=-c$†, $B=-2ac$, $C=-2aac$; and conſequently $y=-c\times\overline{x^2+2ax+2aa}+dM^{\frac{x}{a}}$. By the very ſame Way, the Value of y, in the Equation $cx^n\dot{x}+y\dot{x}=a\dot{y}$, will come out $=-c\times\overline{x^n+ax^{n-1}+n.\overline{n-1}.a^2x^{n-2}+n.\overline{n-1}.\overline{n-2}.a^3x^{n-3}+\&c.}+dM^{\frac{x}{a}}$.

266. But, what is a little remarkable, in theſe Equations, is, that the *Exponential* $dM^{\frac{x}{a}}$, tho' a variable Quantity, ſhould only ſerve, as it were, to correct the Fluent, or perform the Office of a conſtant Quantity. What I here mean will plainly appear, if it be conſidered, that the Equation $y=-c\times x^2+2ax+2aa$, where the ſaid *Exponential* is wanting, anſwers all the Conditions of the fluxional Equation firſt propoſed; which, upon Trial, will be found; and muſt needs be the

* Art. 252.

† Art. 84.

the Caſe, ſeeing d may be, either, taken Nothing at all, or any Quantity at Pleaſure.

But the Equation $y = -c \times \overline{x^2 + 2ax + 2a^2}$ (when $dM^{\frac{x}{a}}$ is wanting) cannot be corrected, in the uſual Way, ſo as to give $y = 0$, when $x = 0$; ſince, if any other conſtant Quantity, beſides $-2a^2c$ be introduced, the firſt Conditions will not be anſwer'd: The Correction muſt, therefore, be by the Exponential $dM^{\frac{x}{a}}$; and is thus.

Since $y = -cx^2 - 2cax - 2ca^2 + dM^{\frac{x}{a}}$, if y be taken $= 0$ and $x = 0$, then $-2ca^2 + dM^0 = 0$, or $d = 2ca^2$; and ſo the Equation, truly corrected, is $y = -c \times \overline{x^2 + 2ax + 2a^2} + 2a^2cM^{\frac{x}{a}}$.

267. We come now to the laſt Method; namely, that of Infinite Serieſes; which, tho' leſs accurate, is vaſtly more comprehenſive, than *any* yet explained: The Manner of it is thus:

For the Quantity whoſe Value you would find, let an Infinite Series, conſiſting of the Powers of the other Quantity with unknown Coefficients, be aſſumed; which Series, together with its Fluxion, or Fluxions, muſt be ſubſtituted inſtead of their Equals in the given Equation; whence a new Equation will ariſe, from which, by comparing the homologous Terms, the aſſumed Coefficients, and conſequently the Value ſought, will be determined.

Thus, let the Equation $\frac{\dot{x}}{1 + x} = \dot{y}$ (reducible to $\dot{x} - \dot{y} - x\dot{y} = 0$) be propoſed; to find x in Terms of y.
Then, aſſuming $x = Ay + By^2 + Cy^3 + Dy^4 + Ey^5$ &c.
We have $\dot{x} = A\dot{y} + 2By\dot{y} + 3Cy^2\dot{y} + 4Dy^3\dot{y} + 5Ey^4\dot{y} +$ &c.
Which Values being ſubſtituted in $\dot{x} - \dot{y} - x\dot{y} = 0$, we get

$$\left.\begin{array}{l} A\dot{y} + 2By\dot{y} + 3Cy^2\dot{y} + 4Dy^3\dot{y} + \text{\&c.} \\ -\dot{y} - Ay\dot{y} - By^2y - Cy^3\dot{y} - \text{\&c.} \end{array}\right\} = 0.$$

Therefore $A - 1 = 0$, or $A = 1$; $2B - A = 0$, or $B = \frac{A}{2} = \frac{1}{2}$; $3C - B = 0$, or $C = \frac{B}{3} = \frac{1}{2.3}$; $4D - C = 0$, or $D = \frac{C}{4} = \frac{1}{2.3.4}$ &c.

And conſequently $x\ (Ay + By^2 + Cy^3\ \&c.) = y + \frac{y^2}{2} + \frac{y^3}{2.3} + \frac{y^4}{2.3.4} + \frac{y^5}{2.3.4.5} + \&c.$

Again, let it be required to find the Value of y, in the Equation $cx^2\dot{x} + y\dot{x} = a\dot{y}$, or $a\dot{y} - y\dot{x} - cx^2\dot{x} = 0$. Here, aſſuming $y = Ax + Bx^2 + Cx^5 + Dx^4 + Ex^5 + Fx^6$ &c. and proceeding as before, we ſhall have

$$\left.\begin{array}{l} aA\dot{x} + 2aBx\dot{x} + 3aCx^2\dot{x} + 4aDx^3\dot{x} + 5aEx^4\dot{x} + \&c. \\ 0 \quad - \quad Ax\dot{x} - \quad Bx^2\dot{x} - \quad Cx^3\dot{x} - \quad Dx^4\dot{x} - \&c. \\ 0 \qquad 0 \quad - \quad cx^2\dot{x} \end{array}\right\} = 0$$

Whence $A = 0$; $2aB = A = 0$; $3aC = B + c = c$, or $C = \frac{c}{3a}$; $4aD = C = \frac{c}{3a}$, or $D = \frac{c}{3.4a^2}$; $5aE = D = \frac{c}{3.4a^2}$, or $E = \frac{c}{3.4.5a^3}$ &c. and conſequently $y\ (Ax + Bx^2 + Cx^3 + \&c.) = \frac{cx^3}{3a} + \frac{cx^4}{3.4a^2} + \frac{cx^5}{3.4.5a^3} + \frac{cx^6}{3.4.5.6a^4} + \&c.$

268. It appears from this Example, that the Quantity to be found, will not always require all the Terms of the Series $Ax + Bx^2 + Cx^3$ &c. And it may happen, in innumerable Caſes, that the Series to be aſſumed will demand a very different Law from *that* where the Exponents proceed according to the Terms of an arithmetical Progreſſion having Unity for the common Difference. And, indeed, the greateſt Difficulty we have here to encounter, is, to know what Kind of Series, with regard to its Exponents, ought to be aſſumed, ſo as to anſwer the Conditions of the Equation, without introducing more Terms than are actually neceſſary.

The following Rules will be found very uſeful upon this Occaſion: Which, though they may become impracticable in certain particular Caſes, never take in any ſuperfluous Terms.

1°. *Having (if neceſſary) freed your Equation from Fractions and Surds, let the Quantity, whoſe Value is ſought, be ſuppoſed equal to ſome Power of the other Quantity with an unknown Exponent (n); and let that Power, together with its Fluxion, or Fluxions, be ſubſtituted for their (ſuppoſed) Equals in the given Equation.*

2°. *Let the leaſt Exponents of the variable, or indeterminate, Quantity, in the new Equation, thence ariſing, be put equal to each other: Whence the Value of the unknown Exponent n will be found.*

3°. *Subſtitute the Value of n, ſo found, in all the Exponents where n is concerned; and then take the Difference between one of the equal* ones, *above mentioned, and every other Exponent, of the variable Quantity, in the whole Equation.*

4°. *To theſe Differences, write down all the leaſt Numbers that can be compoſed out of them, by continual Addition, either to themſelves, or to one another; till you have, by that means, got, in the whole, as many different Terms, as you would have the required Series continued to.*

5°. *Laſtly, let each of thoſe Terms be increaſed by the Value of n (found by Rule 2.) and you will then have the Exponents of the Series to be aſſumed.*

EXAMPLE I,

269. *Let the Value of x, in the Equation* $a^2\dot{x}^2 + x^2\dot{z}^2 - a^2\dot{z}^2 = 0$, *be required.*

Firſt, by writing z^n for x, and $nz^{n-1}\dot{z}$ for $\dot{x}$, the Indices of z will be $2n-2$, $2n$, and 0 (which are determined by Inſpection, without regarding the Coefficients) whereof the two leaſt ($2n-2$ and 0) being put equal to each other, we here find $n=1$: Therefore, the Exponents being 0, 2, 0, the Differences (according to *Rule* 3.) are alſo 0, 2; from whence, by adding 2 continually, we get 0, 2, 4, 6, 8 &c. which (being each in-

increaſed by the Value of n) give 1, 3, 5, 7, 9 &c. for the Exponents in this Caſe.

Let, therefore, $x = Az + Bz^3 + Cz^5 + Dz^7 + \&c.$ Then, putting $\dot{z} = 1$, in order to facilitate the Operation, we ſhall have $\dot{x} = A + 3Bz^2 + 5Cz^4 + 7Dz^6 + \&c.$ which two Values being ſquared, and ſubſtituted in the given Equation, it will become

$$\left.\begin{array}{llllll} a^2A^2 & + 6a^2ABz^2 & + 10a^2ACz^4 & + 14a^2ADz^6 & + \&c. \\ & & + 9a^2B^2z^4 & + 30a^2BCz^6 & + \&c. \\ & + A^2z^2 & + 2ABz^4 & + 2ACz^6 & + \&c. \\ - a^2 & & & + B^2z^6 & + \&c. \end{array}\right\} = 0$$

Whence, $a^2A^2 = a^2$, and therefore $A = 1$; $6a^2B = -A$, and therefore $B = -\frac{1}{6a^2} = -\frac{1}{2.3a^2}$; $10a^2AC = -9a^2B^2 - 2AB = -B \times \overline{9a^2B + 2A} = -B \times -\frac{3}{2} + 2 = -\frac{B}{2} = \frac{1}{2.3.2a^2}$, and therefore $C = \frac{1}{2.3.4.5a^4}$; $14AD = -30a^2 \times -\frac{1}{6a^2} \times \frac{1}{120a^4} - 2 \times \frac{1}{120a^4} - \frac{1}{36a^4} = \frac{1}{24a^4} - \frac{1}{60a^4} - \frac{1}{36a^4} = -\frac{1}{360a^4}$, and therefore $D = -\frac{1}{14.360a^6} = -\frac{1}{2.3.4.5.6.7a^6}$; and, conſequently, $x = z - \frac{z^3}{2.3a^2} + \frac{z^5}{2.3.4.5a^4} - \frac{z^7}{2.3.4.5.6.7a^6}$ &c.

EXAMPLE II.

270. *Let the given Equation be* $a^2x\ddot{y} - 2a^2\dot{x}\dot{y} + ax\dot{x}^2 + x^3\ddot{y} = 0$; *to find* y.

Here, ſubſtituting x^n for y, the Exponents will be $n-1$, $n-1$, 1, and $n+1$; where, making $n-1=1$, we get $n = 2$: Whence, the Differences being 0, 2, the Series to be aſſumed for y will be $Ax^2 + Bx^4 + Cx^6 + Dx^8 + Ex^{10} +$ &c. From which, making $\dot{x} = 1$, we have $\dot{y} = 2Ax + 4Bx^3 + 6Cx^5 + 8Dx^7$ &c. and

$$\ddot{y} = 2A + 12Bx^2 + 30Cx^4 + 56Dx^6$$

And, theſe Values being ſubſtituted, the Equation becomes

$$\left.\begin{array}{l} 2a^2Ax + 12a^2Bx^3 + 30a^2Cx^5 + 56a^2Dx^7 + \&c. \\ -4a^2Ax - 8a^2Bx^3 - 12a^2Cx^5 - 16a^2Dx^7 + \&c. \\ +ax \quad + 2Ax^3 \quad + 12Bx^5 \quad + 30Cx^7 \quad + \&c. \end{array}\right\} = 0$$

Therefore $A = -\dfrac{1}{2a}$; $B = -\dfrac{2A}{4a^3} = -\dfrac{1}{4a^3}$; $C = -\dfrac{12B}{18a^2} = \dfrac{1}{6a^5}$; $D = -\dfrac{30C}{40a^2} = -\dfrac{1}{8a^7}$ &c. and ſo $y = \dfrac{x^2}{2a} - \dfrac{x^4}{4a^3} + \dfrac{x^6}{6a^5} - \dfrac{x^8}{8a^7} + \dfrac{x^{10}}{10a^9} -$ &c. Which Series is known to expreſs the Fluent of $\dfrac{ax\dot{x}}{a^2+x^2}$, or, $\frac{1}{2}a \times$ hyp. Log. $\dfrac{a^2+x^2}{aa}$: Conſequently y is alſo $= \frac{1}{2}a \times$ hyp. Log. $\dfrac{a^2+x^2}{a^2}$. In this manner, it comes to paſs, *that*, though we are obliged, in very complicated Caſes, to have recourſe to Infinite Serieſes, we are ſometimes able, at laſt, to give the Solution in finite Terms, or, at leaſt, by help of Logarithms, Sines and Tangents: Which will always happen when the Series can be ſummed, or is found to agree with that ariſing from ſome known Quantity.

271. Sometimes it happens, in Equations involving the higher Orders of Fluxions, that the Exponents, mention'd in *Rule* 2. whereof the leaſt ought to be made equal to each other, are ſo expreſſed, as to render ſuch an Equality impoſſible. When this is the Caſe, the Value of n, and the firſt Term of the required Series, can *only* be determined from the Nature of the Problem to which the Equation belongs. We know, in-

indeed, from the Equation itſelf, that n muſt be either equal to Nothing, or to ſome poſitive Integer, leſs than that expreſſing the Order of the higheſt Fluxion in the Equation: Becauſe the Term that has the leaſt Exponent, and which therefore cannot be compared with any other (being always affected by two or more of the Factors, n, $n-1$, $n-2$, &c. will then (one of thoſe Factors being $=0$) vaniſh intirely out of the Equation; which, thereby, is render'd poſſible.

When n and A are known, the reſt of the Terms will be found in the common Way, as in

EXAMPLE III.

Where the Equation propoſed is $y\dot{x}^2 + ax\dot{y} - a^2\ddot{y} = 0$; *to find* y.

By ſuppoſing $\dot{x}=1$, and writing x^n for y, nx^{n-1} for $\dot{y}$, and $n \times \overline{n-1} \times x^{n-2}$ for $\ddot{y}$, we get $x^n + nax^{n-1} - n \times \overline{n-1} \times a^2x^{n-2}$: But it is plain that no two of the Indices of x can, *here*, be equal: The Value of n muſt therefore be either $=0$, or Unity (in both which Caſes the Term $-n \times \overline{n-1} \times a^2x^{n-2}$ vaniſhes) but I ſhall take the latter Value, and ſuppoſe the firſt Term of the Series to be Ax; then, the Differences of the foreſaid Exponents being 1 and 2, the Law of the Series will be expreſſed by 1, 2, 3, 4 &c. Whence, aſſuming $y = Ax + Bx^2 + Cx^3 + Dx^4$ &c. and proceeding as in the former Examples, y will be found $=$ A *into* $x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \frac{x^4}{8a^3} + \frac{x^5}{24a^4} + \frac{x^6}{90a^5}$ &c. or $=$ A *into* $x + \frac{x^2}{2a} + \frac{2x^3}{2.3a^2} + \frac{3x^4}{2.3.4a^3} + \frac{5x^5}{2.3.4.5a^4} + \frac{8x^6}{2.3.4.5.6a^5} +$ &c. where the Law of Continuation is manifeſt, the Coefficient of every Numerator being compoſed by the Addition of the two preceding ones.

272. It will be proper to obſerve here, that, in Equations like the two laſt propoſed, where the higher Orders of Fluxions are concerned, the Series expreſſing the Relation of the two Quantities muſt always be found in Terms of the Quantity flowing uniformly. And, that, if the Number of Dimenſions of the Fluxion of the ſaid Quantity, after Subſtitution, be not the ſame in every Term, the Equation itſelf, put down to be reſolved, is abſurd and impoſſible; and ſuch as never can ariſe in the Solution of any Problem. In all proper Equations the Number of fluxional Points (ſuppoſing the Powers of the Fluxions to be wrote without Indices) will be the ſame in every Term.

EXAMPLE IV.

273. *Where let the given Equation be* $a^3\dot{y} - ay^2\dot{x} + x^2y\dot{y} = x^3\dot{x}$; *to find* y.

By proceeding as uſual the Indices will here be $n-1$, $2n$, $2n+1$ and 3; whereof the leaſt (which can be no other than $n-1$ and 3) being compared, n will be given $=4$: And the Differences will therefore be 0, 5, 6; to which the Double of the Second and the Sum of the ſecond and third, *&c.* being put down, and then every Term increaſed by 4, there ariſes 4, 9, 10, 14, 15, 16, 19 *&c.* for the Exponents of the Series to be aſſumed for y.

Let therefore $y = Ax^4 + Bx^9 + Cx^{10} + Dx^{14}$ *&c.* then, making $\dot{x} = 1$, $\dot{y}$ is $= 4Ax^3 + 9Bx^8 + 10Cx^9 + 14Dx$ $+$ *&c.*

And, by ſubſtituting theſe Values above, we have

$$\left.\begin{array}{l} 4a^3Ax^3 + 9a^3Bx^8 + 10a^3Cx^9 + 14a^3Dx^{13} + \&c. \\ -x^3 \quad - aA^2x^8 \quad + 4A^2x^9 \quad - 2aABx^{13} + \&c. \end{array}\right\} = 0$$

Whence $A = \dfrac{1}{4a^3}$, $B = \dfrac{1}{144a^8}$, *&c.*

And* $y = \dfrac{x^4}{4a^3} + \dfrac{x^9}{144a^8} - \dfrac{x^{10}}{40a^9} + \dfrac{x^{14}}{4032a^{13}}$ *&c.*

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* *If for* y, *the Series* $Ax^4+Bx^5+Cx^6+Dx^7$ *&c. whoſe Exponents are in arithmetical Progreſſion, had been aſſumed, according to the Method of ſome very good Authors, no leſs than ſeven ſuperfluous Terms muſt have been introduced to obtain the four above given.*

274. Before I quit this Subject, it may not be amiſs to ſubjoin the following Remarks.

1°. If the indeterminate Quantities are great in reſpect to the given ones, a deſcending Series will, in moſt Caſes (where it is practicable) converge better than an aſcending one. To obtain ſuch a Series, compare the greateſt Exponents, mention'd in *Rule* 2 inſtead of the leaſt, and proceed according to the third and fourth Rules *, whence a Series of Numbers will be found; which, being ſucceſſively ſubtracted from the Value of n, you will have the Exponents of a deſcending Series.

Thus, let the common-algebraic Equation $a^3x+ax^3-a^2y-y^4=0$ be propounded; to find y, when x is great in compariſon of a.

Then, proceeding as uſual, the Exponents of the four Terms of the Equation will be 1, 3, n, $4n$; whereof the two greateſt ($4n$ and 3) being made equal, we get $n=\frac{3}{4}$; therefore the Differences are 0, 2 and $2\frac{1}{4}$; and $n=\frac{3}{4}$; therefore the Differences are 0, 2 and $2\frac{1}{4}$; and the Numbers to be ſubtracted from n, are 0, 2, $\frac{9}{4}$, 4, $\frac{17}{4}$, &c. Conſequently the Series to be aſſumed for y is $Ax^{\frac{3}{4}}+Bx^{-\frac{5}{4}}+Cx^{-\frac{6}{4}}+Dx^{-\frac{13}{4}}+$ &c. From whence

$$y \text{ will be found} = a^{\frac{1}{4}}x^{\frac{3}{4}}+\frac{a^{\frac{9}{4}}}{4x^{\frac{5}{4}}}-\frac{a^{\frac{10}{4}}}{4x^{\frac{6}{4}}}-\frac{3a^{\frac{17}{4}}}{32x^{\frac{13}{4}}} \text{ \&c.}$$

2°. But, if the Quantity (x) in whoſe Terms the other is to be expreſſed, be neither much greater nor much ſmaller than the given Quantity (a), it will be proper to ſubſtitute for the Exceſs, or Defect, of the ſaid Quantity (x) above, or below, ſome given Quantity; ſo that, having, by this means, exterminated x, the Series ariſing from the new Equation (wherein the ſaid Exceſs, or Defect, is the converging Quantity) will have a due Rate of Convergency.

The Uſe of this is ſo obvious that it needs no Example, or farther Explanation.

3°. Laſtly, it will be proper to obſerve, that, if the Equation for the Value of A, ariſing from the firſt Column of homologous Terms, admits of two or more, equal

* Art. 268.

equal Roots (which is a Caſe that may, perhaps, never happen in practice) all the foregoing Precepts will be inſufficient; unleſs the Equation alſo admits of ſome other Root, beſides the equal ones, whereby A may be more commodiouſly expreſſed. To determine the Exponents, in that particular Caſe, divide each of the Differences mention'd in *Rule* 3. by the Number of the equal Roots; and then proceed as uſual. The Reaſons of which, as well as of the Rules themſelves, I have long ago given elſewhere, and have not Room to repeat them here.

SCHOLIUM.

275. Although the Buſineſs of reverting *Serieſes* is not a Branch of the Doctrine of Fluxions, but, more properly, belongs to common *Algebra*; yet, as it is often uſeful where Fluxions are concerned, and falls under the general Rules illuſtrated in the foregoing Pages, I ſhall here add an Example or two on that Head.

Let, then, $ax+bx^2+cx^3+dx^4+ex^5$ *&c.* $=y$; to revert the Series, or, to find x in an Infinite Series expreſſed in the Powers of y.

Here, by writing y^n for x, the Indices of the Powers of y, in the Equation, will be n, $2n$, $3n$, *&c.* and 1; therefore $n=1$. and the Differences are 0, 1, 2, 3, 4, 5, *&c.* and ſo the Series to be aſſumed, in this Caſe, is $Ay+By^2+Cy^3+Dy^4$ *&c.* Which being involved and ſubſtituted for the reſpective Powers of x (neglecting, every where, all ſuch Powers of x and y as exceed the higheſt you would have the Series carry'd to) there ariſes

$$\left.\begin{array}{lllll}
aAy & +aBy^2 & +aCy^3 & +aDy^4 & \&c. \\
* & +bA^2y^2 & +2bABy^3 & +2bACy^4 & \big\}\ \&c. \\
 & & & +bB^2y^4 & \big\}\ \&c. \\
* & * & +cA^3y^3 & +3cA^2By^4 & \&c. \\
* & * & * & +dA^4y^4 & \&c.
\end{array}\right\}=y$$

Whence, by comparing the homologous Terms, $A = \frac{1}{a}$; $B = -\frac{b}{a^3}$; $C \left(= -\frac{2bAB + cA^3}{a} \right) = \frac{2bb - ac}{a^5}$; $D \left(= -\frac{2bAC + bB^2 + 3cA^2B + dA^4}{a} \right) = \frac{5abc - 5b^3 - a^2d}{a^7}$, &c. and conſequently $x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{2bb - ac}{a^5} \times y^3 - \frac{5b^3 - 5abc + a^2d}{a^7} \times y^4$ &c.

For an Inſtance of the Uſe of this Concluſion, let $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ &c. $= y$: Then, a being, in this Caſe, $= 1$, $b = -\frac{1}{2}$, $c = \frac{1}{3}$, $d = -\frac{1}{4}$, &c. we ſhall, by ſubſtituting theſe Values, have $x = y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24}$ &c. From whence, when y is given, x will alſo be given; provided the Value of y be ſufficiently ſmall*.

Example 2. Let there be given $ax + by + cx^2 + dxy + ey^2 + fx^3 + gx^2y + hxy^2 + iy^3 + kx^4 + lx^3y$ &c. $= 0$; to find y.

By aſſuming $y = Ax + Bx^2 + Cx^3 + Dx^4$ &c. and proceeding as above, A will be found $= -\frac{a}{b}$, $B = -\frac{c + dA + eA^2}{b}$, $C = -\frac{dB + 2eAB + f + gA + hA^2 + iA^3}{b}$, $D = -\frac{dC + 2eAC + eB^2 + gB + 2hAB + 3iA^2B + k + lA + mA^2 + nA^3 + pA^4}{b}$, &c.

* Art. 92.

Example 3. Laſtly, let $x^m + bx^{m+p} + cx^{m+2p} + dx^{m+3p} + \&c. = z.$

Here, in order to determine the Form of the Series to be aſſumed, let z^n be wrote for x in the given Equation, according to the uſual Method; and then the Exponents, ſuppoſing z tranſpoſed, will be 1, nm, $nm + np$, $nm + 2np$, $nm + 3np$, &c. reſpectively; whereof the two leaſt (1 and nm) being made equal to each other, n is found $= \frac{1}{m}$; and the Differences are $\frac{p}{m}$, $\frac{2p}{m}$, $\frac{3p}{m}$, &c. Whence the Series to be aſſumed for x is $z^{\frac{1}{m}} + Bz^{\frac{1+p}{m}} + Cz^{\frac{1+2p}{m}} + Dz^{\frac{1+3p}{m}} + \&c.$ (for it is evident, by Inſpection, that the Coefficient (A) of the firſt Term muſt here be an Unit.) This Series being therefore raiſed to the ſeveral Powers of x, in the given Equation, *by Art.* 108. and the Coefficients of the homologous Terms in the new Equation compared together, it will be found that, $B = -\frac{b}{m}$, $C = \frac{\overline{1+m+2p}\times bb - 2mc}{2m^2}$,

$$D = -\frac{\overline{2m^2 + 9mp + 9p^2 + 3m + 6p + 1} \times b^3}{6m^3} + \frac{\overline{1+m+3p} \times bc}{m^2} - \frac{d}{m}, \&c.$$

From the general Value of x, found above, innumerable Theorems, for reverting particular Forms of Serieſes, may be deduced.

Thus, if $x + bx^2 + cx^3 + dx^4, \&c. = z$; then ($m$ being $= 1$ and $p = 1$) x is $= z - bz^2 + \overline{2bb - c} \times z^3 - \overline{5b^3 - 5bc + d} \times z^4$ &c.

And, if $x + bx^3 + cx^5 + dx^7 + \&c. = z$; ($m$ being $= 1$, and $p = 2$) $x = z - bz^3 + \overline{3bb - c} \times z^5 - \overline{12b^3 - 8cb + d} \times z^7$ &c.

Alſo, if $x^{\frac{1}{2}} + bx^{\frac{3}{2}} + cx^{\frac{5}{2}} + dx^{\frac{7}{2}}$ &c. $= z$; then (m being $= \frac{1}{2}$ and $p = 1$) $x = z^2 - 2bz^4 + \overline{7bb - 2c} \times z^6 - \overline{30b^3 - 18bc + 2d} \times z^8$ &c. &c.

276. It may be obſerved that, in all theſe Forms of Serieſes, the firſt Term is without a Coefficient (which renders the Concluſion much more ſimple.) Therefore, when the Series to be reverted has a Co-efficient in its firſt Term, the whole Equation muſt be firſt of all divided thereby: Thus, if the Equation was $3x - 6x^2 + 8x^3 - 13x^4$ &c. $= y$; by dividing the whole by 3 it will become $x - 2x^2 + \frac{8x^3}{3} - \frac{13x^4}{3}$ &c. $= \frac{1}{3}y$: Where, putting $z = \frac{1}{3}y$, we have, *by Form.* 1. $x = z + 2z^2 + \frac{16}{3}z^3$ &c. $= \frac{y}{3} + \frac{2y^2}{9} + \frac{16y^3}{81}$ &c.

SECTION III.

Of the Compariſon of Fluents, or the Manner of finding one Fluent from another.

277. WE have, already, pointed out the moſt remarkable Forms of Fluxions whoſe Fluents are explicable in finite Terms *; and alſo ſhewn the Uſe of Infinite Serieſes in approximating the Values of ſuch Fluents as do not come under any of thoſe Forms †: But this laſt Method (as is before hinted) being troubleſome, and attended with many Obſtacles; Mathematicians have therefore invented, and ſhewn, the Way of deriving one Fluent from another: Which is of good Advantage when the Fluent

* Art. 77. 78. 83. 84. and 85.

† Art. 99.

ſought can be referred to one, like thoſe in *Art.* 126 and 142. expreſſing the Logarithm of a Number, or the Arch of a Circle; ſince the Trouble of an infinite Series is, then, avoided.

As the Subject here propoſed is of ſuch a Nature, that it would be very tedious and difficult, if not altogether impracticable, to lay down Rules and Precepts for all the various Caſes; I ſhall deliver, what I have to offer thereon, by way of *Problems*; beginning with ſome very eaſy ones, for the Sake of the *young Proficient*.

PROB. I.

278. *The Fluent of* $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ *being given (by Art.* 126.) *'tis propoſed to find, from thence, the Fluent of* $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$.

Let both the Numerator and Denominator of $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$, be multiply'd by *x*, ſo that the Quantity without the Vinculum, in the Fluxion, $\frac{x^3\dot{x}}{\sqrt{a^2x^2+x^4}}$, thus transformed, may become ſome conſtant Part of the Fluxion of the higheſt Term under the Vinculum: Which Part, in this Caſe, being $\frac{1}{4}$, let $\frac{1}{4}$ of the Fluxion of the firſt Term under the Vinculum (or $\frac{1}{2}a^2x\dot{x}$) be therefore added to the Numerator, in order to have the *Whole*, $\frac{\frac{1}{2}a^2x\dot{x}+x^3\dot{x}}{\sqrt{a^2x^2+x^4}}$, a *complete* Fluxion; and then the Fluent thereof, by the common Rule*, will be $\frac{1}{2}\sqrt{a^2x^2+x^4}=\frac{1}{2}x\sqrt{a^2+x^2}$: But, from *this*, we are now to deduct the Fluent of the Quantity $\frac{\frac{1}{2}a^2x\dot{x}}{\sqrt{a^2x^2+x^4}}$ $\left(=\frac{\frac{1}{2}a^2\dot{x}}{\sqrt{a^2+x^2}}\right)$ that was added: Which Fluent, as

that

* Art. 77.

that of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ is given $=$ *hyp. Log.* $x+\sqrt{a^2+x^2}$ *, * Art. 126. will be $= \frac{1}{2}a^2 \times$ *hyp. Log.* $x+\sqrt{a^2+x^2}$; and conſequently the Fluent ſought $= \frac{1}{2}x\sqrt{a^2+x^2}-\frac{1}{2}a^2 \times$ *hyp. Log.* $x+\sqrt{a^2+x^2}$. *Q. E. I.*

PROB. II.

279. *Let it be propoſed to find the Fluent of* $\frac{x^2\dot{x}}{\sqrt{a^2-x^2}}$, *from that of* $\frac{\dot{x}}{\sqrt{a^2-x^2}}$; *given by* Art. 142.

By proceeding as above, and adding $-\frac{1}{2}a^2x\dot{x}$ to the Numerator, we have $-\frac{\frac{1}{2}a^2x\dot{x}-x^3\dot{x}}{\sqrt{a^2x^2-x^4}}$; whereof the Fluent, by the *common Rule*, is $-\frac{1}{2}\sqrt{a^2x^2-x^4}$ $(= -\frac{1}{2}x\sqrt{a^2-x^2}$: From which deducting the Fluent of $-\frac{\frac{1}{2}a^2x\dot{x}}{\sqrt{a^2x^2-x^4}}$, or $-\frac{\frac{1}{2}a^2\dot{x}}{\sqrt{a^2-x^2}}$ (given $= -\frac{1}{2}a^2 \times$ Arc (A) whoſe Radius is Unity and Sine $=\frac{x}{a}$ †) there comes out $\frac{1}{2}a^2A-\frac{1}{2}x\sqrt{a^2-x^2}$. † Art. 142.

Q. E. I.

280. In the ſame Manner, if the Power without the Vinculum, in the Expreſſion whoſe Fluent is ſought, exceeds *that* in the other Expreſſion given, by the Exponent under the Vinculum, or by any *Multiple* of it, the required Fluent may be determined, by one, or by ſeveral Operations, according to the Value of the ſaid *Multiple*.

Thus, if the Fluent of $\frac{x^4\dot{x}}{\sqrt{a^2-x^2}}$ was ſought; then, becauſe the Index of x, without the *Vinculum*, exceeds *that*

that in $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$ by twice the Exponent under the *Vinculum*, the required Fluent may be had from that of $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$, at two Operations; by the firſt whereof, we have already found the Fluent of $\frac{x^2\dot{x}}{\sqrt{a^2 - x^2}}$ to be $= \frac{1}{2}a^2A - \frac{1}{2}x\sqrt{a^2 - x^2}$: Whence, putting this Value $= B$, and proceeding as before, we alſo get $-\frac{1}{4}\sqrt{a^2x^6 - x^8} + \frac{3}{4}a^2B = -\frac{1}{4}x^3\sqrt{a^2 - x^2} - \frac{3a^2x}{8}\sqrt{a^2 - x^2} + \frac{3a^4A}{8} = \frac{3a^4A - \overline{2xx + 3aa} \times x\sqrt{a^2 - x^2}}{8} =$ the true Fluent of $\frac{x^4\dot{x}}{\sqrt{a^2 - x^2}}$.

P R O B. III.

281. *Suppoſing the Fluent of* $\overline{a + cz^n}\rvert^m \times z^{pn-1}\dot{z}$ *to be given* $= A$, *to find the Fluent of* $\overline{a + cz^n}\rvert^m \times z^{pn+n-1}\dot{z} = \dot{B}$ (*where the Exponent of* z, *without the* Vinculum *is increaſed by the Exponent under the* Vinculum).

Let the Part affected by the *Vinculum* be multiplied by z^{mq}, and the Part without be divided by the ſame Quantity; then our Fluxion will be transformed to $\overline{az^q + cz^{n+q}}\rvert^m \times z^{pn+n-mq-1}\dot{z} = \dot{B}$: Where let q be, now, ſo taken that the Exponent $(n+q)$ of the higheſt Power of z under the *Vinculum* may be equal to $(pn+n-mq)$ that of the Power without the *Vinculum* $+ 1$; that is, let $q = \frac{pn}{m+1}$: Then (*by Art.* 77.) if the firſt Term under the *Vinculum* was conſtant, the Fluent of the ſaid Expreſſion, or its Equal $\overline{az^q+cz^{n+q}}|^m \times z^{n+q-1}\dot{z}$, would be had $= \dfrac{\overline{az^q+cz^{n+q}}|^{m+1}}{\overline{m+1}\times\overline{n+q}\times c}$. But the Fluxion hereof, ſuppoſing both Terms to be variable (as they actually are) is $\overline{az^q+cz^{n+q}}|^m \times z^{n+q-1}\dot{z} + \dfrac{qa}{\overline{n+q}\times c}\times\overline{az^q+cz^{n+q}}|^m\times z^{q-1}\dot{z}$ (by the common Rule.) Therefore $\dfrac{\overline{az^q+cz^{n+q}}|^{m+1}}{\overline{m+1}\times\overline{n+q}\times c} - \dfrac{qa}{\overline{n+q}\times c}\times$ *Flu.* *of* $\overline{az^q+cz^{n+q}}|^m\times z^{q-1}\dot{z} = B$; that is, $\dfrac{\overline{a+cz^n}|^{m+1}\times z^{qm+q}}{\overline{m+1}\times\overline{n+q}\times c} - \dfrac{qa}{\overline{n+q}\times c}\times$ *Flu.* $\overline{a+cz^n}|^m\times z^{mq+q-1}\dot{z} = B$; or, by ſubſtituting for q, $\dfrac{\overline{a+cz^n}|^{m+1}\times z^{pn}}{\overline{m+p+1}\times nc} - \dfrac{pa}{\overline{m+p+1}\times c}\times$ *Flu.* $\overline{a+cz^n}|^m\times z^{pn-1}\dot{z} = B$: But the *Flu.* *of* $\overline{a+cz^n}|^m\times z^{pn-1}\dot{z}$ is given $= A$; therefore, laſtly, $\dfrac{\overline{a+cz^n}|^{m+1}\times z^{pn}}{\overline{m+p+1}\times nc} - \dfrac{paA}{\overline{m+p+1}\times c} = B$. *Q. E. I.*

282. If the Quantity under the *Vinculum* be a Multinomial, $a+cz^n+dz^{2n}+ez^{3n}$ &c. Then, ſince the Fluxion of $\overline{a+cz^n+dz^{2n}+ez^{3n}\ \&c.}|^{m+1}\times z^{pn}$ is $\overline{m+1}\times\overline{ncz^{n-1}\dot{z}+2ndz^{2n-1}\dot{z}+3nez^{3n-1}\dot{z}\ \&c.}\times$

$\overline{a+cz^{n}+dz^{2n}\ \&c.}|^{m}\times z^{pn}+\overline{a+cz^{n}+dz^{2n}\ \&c.}|^{m+1}$
$\times\, pnz^{pn-1}\dot{z}=$
$$\left\{\begin{array}{l}\overline{m+1}\times ncz^{pn+n-1}\dot{z}+\overline{m+1}\times 2ndz^{pn+2n-1}\dot{z}\ \&c.\\ pnaz^{pn-1}\dot{z}+pncz^{pn+n-1}\dot{z}+pndz^{pn+2n-1}\dot{z}\ \&c.\end{array}\right\}$$
$\times\ \overline{a+cz^{n}+dz^{2n}\ \&c.}|^{m}$, it is evident, that, if the Fluents of $z^{pn-1}\dot{z}$, $z^{pn+n-1}\dot{z}$, $z^{pn+2n-1}\dot{z}$ &c. drawn into the general Multiplicator $\overline{a+cz^{n}+dz^{2n}\ \&c.}|^{m}$, be denoted by A, B, C, D, &c, the Fluent of the Whole-Quantity exhibited above (which Fluent is $\overline{a+cz^{n}+dz^{2n}+ez^{3n}\ \&c.}|^{m+1}\times z^{pn}$) will alſo be expreſſed by $pnaA+\overline{p+m+1}\times ncB+\overline{p+2m+2}\times ndC+\overline{p+3m+3}\times neD$ &c. Therefore, if there be given as many of the Fluents A, B, C, D &c. as there are Terms in $a+cz^{n}+dz^{2n}+ez^{3n}$ &c. *minus* one, that other Fluent, be it which it will, will alſo be given from hence. Thus if $d=0$, $e=0$, &c. and the Value of A be given, we ſhall have $\overline{a+cz^{n}}|^{m+1}\times z^{pn}=pnaA+\overline{p+m+1}\times ncB$; and conſequently $B=\dfrac{\overline{a+cz^{n}}|^{m+1}\times z^{pn}}{\overline{p+m+1}\times nc}-\dfrac{paA}{\overline{p+m+1}\times c}$, the very ſame as before.

PROB. IV.

283. *The Fluent of* $\overline{a+cz^{n}}|^{m}\times z^{pn-1}\dot{z}$ *being given (as in the preceding Problem) to determine, from thence, the Fluent of* $\overline{a+cz^{n}}|^{m}\times z^{pn+vn-1}\dot{z}$; *ſuppoſing* v *to denote a whole poſitive Number.*

Let $\overline{a+cz^n}\big|^{m+1}$ be denoted by M; alſo put $p+1 = p'$, $p'+1\ (p+2) = p''$, $p''+1\ (p+3) = p'''$ &c. and let the Fluents of $\overline{a+cz^n}\big|^m \times z^{pn-1}\dot{z}$, $\overline{a+cz^n}\big|^m \times z^{p'n-1}\dot{z}$, $\overline{a+cz^n}\big|^m \times z^{p''n-1}\dot{z}$, $\overline{a+cz^n}\big|^m \times z^{p'''n-1}\dot{z}$, &c. be repreſented by A, B, C, D, &c. reſpectively. Then, ſince $\dfrac{Mz^{pn}}{\overline{m+p+1}\times nc} - \dfrac{paA}{\overline{m+p+1}\times c} = B$ (*by the preceding Prob.*) it follows, from the very ſame Argument, that

$$\frac{Mz^{p'n}}{\overline{m+p'+1}\times nc} - \frac{p'aB}{\overline{m+p'+1}\times c} = C$$

$$\frac{Mz^{p''n}}{\overline{m+p''+1}\times nc} - \frac{p''aC}{\overline{m+p''+1}\times c} = D$$

&c. &c.

Hence, by writing the Value of B in the ſecond Equation, we have $\dfrac{Mz^{p'n}}{\overline{m+p'+1}\times nc} - \dfrac{p'aMz^{pn}}{\overline{m+p'+1}\times\overline{m+p+1}\times ncc}$ $+ \dfrac{pp'a^2A}{\overline{m+p+1}\times\overline{m+p'+1}\times c^2} = C$. In the ſame Manner, by ſubſtituting this Value for C in the 3d Equation, we get

$$\frac{Mz^{p''n}}{\overline{m+p''+1}\times nc} - \frac{p''aMz^{p'n}}{\overline{m+p''+1}\times\overline{m+p'+1}\times nc^2} +$$

$$-\frac{\overset{\prime\prime}{p}\,\overset{\prime}{p}a^2Mz^{pn}}{\overline{m+\overset{\prime\prime}{p}+1}\times\overline{m+\overset{\prime}{p}+1}\times\overline{m+p+1}\times nc^3}-$$

$$\frac{\overset{\prime\prime\prime}{p}\overset{\prime\prime}{p}\overset{\prime}{p}a^3A}{\overline{m+p+1}\times\overline{m+\overset{\prime}{p}+1}\times\overline{m+\overset{\prime\prime}{p}+1}\times c^3}=D.$$

Where the Law of Continuation is manifeſt; and from whence it appears that the Value of any of the Quantities *B*, *C*, *D*, *E*, &c. or the Fluent expreſſed in a general Manner, will be $\frac{Mz^{qn}}{\overline{m+q+1}\times nc}-$

$$\frac{qaMz^{\overline{q-1}\times n}}{\overline{m+q+1}\times\overline{m+q}\times ncc}+\frac{q\times\overline{q-1}\times a^2Mz^{\overline{q-2}\times n}}{\overline{m+q+1}\times\overline{m+q}\times\overline{m+q-1}\times nc^3}$$

$$(v)\pm\frac{p\times\overline{p+1}\times\overline{p+2}\times\overline{p+3}\,(v)\times a^vA}{\overline{m+p+1}\times\overline{m+p+2}\times\overline{m+p+3}\,(v)\times c^v};\ \text{or,}$$

$$\frac{\overline{a+cz^n}|^{m+1}\times z^{pn}}{\overline{s+1}\times nc}\times\overline{z^{vn-n}-\frac{qaz^{vn-2n}}{sc}+\frac{q\cdot\overline{q-1}\times a^2z^{vn-3n}}{s.\overline{s-1}\times c^2}}$$

$$\overline{-\frac{q.\overline{q-1}.\overline{q-2}\times a^3z^{vn-4n}}{s.\overline{s-1}.\overline{s-2}\times c^3}-(v)}\pm\frac{p}{t}\times\frac{p+1}{t+1}\times\frac{p+2}{t+2}$$

$\frac{p+3}{t+3}(v)\times\frac{a^vA}{c^v}$: Where, $A=$ Fluent of $\overline{a+cz^n}|^m$ $\times z^{pn-1}\dot{z}$, $q=p+v-1$, $s=q+m$, $t=p+m+1$; and where the Sign of the laſt Term (in which *A* is found) muſt be taken + or − according as *v* is an even or odd Number: Note, alſo, that the Parentheſis *(v)* is put to expreſs the Number of Terms, or Factors, to which the Series, or Product, preceding it, is to be continued. The like Notation is to be underſtood in other Caſes of the ſame Kind, when they hereafter occur.

The ſame otherwiſe.

284. Let $q=p+v-1$, and let $\overline{a+cz^n}^{\,m+1}\times\overline{Rz^{qn}+Sz^{qn-n}+Tz^{qn-2n}\ldots\ldots+\Delta z^{pn}}+\beta A$, be aſſumed for the Fluent ſought: Then, by taking the Fluxion thereof, you will have $\overline{m+1}\times ncz^{n-1}\dot{z}\times\overline{a+cz^n}^{\,m}\times\overline{Rz^{qn}+Sz^{qn-n}\ldots\ldots+\Delta z^{pn}}+\overline{a+cz^n}^{\,m+1}\times\overline{qn\dot{z}Rz^{qn-1}+\overline{qn-n}\times\dot{z}Sz^{qn-n-1}\ldots\ldots+pn\dot{z}\Delta z^{pn-1}}+\beta\times\overline{a+cz^n}^{\,m}\times z^{pn-1}\dot{z}$; which muſt be $=\overline{a+cz^n}^{\,m}\times z^{pn+vn-1}\dot{z}$ (or $\overline{a+cz^n}^{\,m}\times z^{qn+n-1}\dot{z}$) the Fluxion propoſed: Whence, dividing the whole Equation by $\overline{a+cz^n}^{\,m}\times z^{n-1}\dot{z}$, and tranſpoſing, there comes out

$$\left.\begin{array}{l}\overline{m+1}\times nc\times\overline{Rz^{qn}+Sz^{qn-n}+Tz^{qn-2n}\ldots\ldots+\Delta z^{pn}}\\ \overline{a+cz^n}\times qnRz^{qn-n}+\overline{qn-n}\times Sz^{qn-2n}\ldots\ldots+pn\Delta z^{pn-n}\\ \qquad -z^{qn}\qquad\quad *\qquad\qquad *\qquad\qquad +\beta z^{pn-n}\end{array}\right\}=0$$

Which, reduced, and the homologous Terms united, becomes

$$\left.\begin{array}{r}\overline{m+q+1}\times ncR\\ -1\end{array}\right\}\times z^{qn}+\left.\begin{array}{r}\overline{m+q}\times ncS\\ +qnaR\end{array}\right\}\times z^{qn-n}+$$

$$\left.\begin{array}{r}\overline{m+q-1}\times ncT\\ +\overline{qn-n}\times aS\end{array}\right\}\times z^{qn-2n}\ldots\ldots\ldots+\left.\begin{array}{r}pna\Delta\\ +\beta\end{array}\right\}\times z^{pn-n}$$

$=0$: Where, by making $\overline{m+q+1}\times ncR-1=0$, $\overline{m+q}\times ncS+qnaR=0$, &c. we have $R=\dfrac{1}{\overline{m+q+1}\times cn}$, $S=-\dfrac{qaR}{\overline{m+q}\times c}$, $T=-\dfrac{\overline{q-1}\times aS}{\overline{m+q-1}\times c}$; or (putting

$m+q=s$) $R=\frac{1}{\overline{s+1}\times nc}$, $S=-\frac{qaR}{sc}=-\frac{qa}{\overline{s+1}\times snc^2}$,

$T=-\frac{\overline{q-1}\times aS}{\overline{s-1}\times c}=\frac{q\times\overline{q-1}\times a^2}{\overline{s+1}\times s\times\overline{s-1}\times nc^3}$, &c.

Where, becauſe the Exponent of the firſt Term of the Equation is qn ($pn+vn-n$) and that of the laſt Term (in which Δ and β are concerned) $= pn$, it follows that the Number of Coefficients to be taken as above (whereof Δ is the laſt) is expreſſed by v: From which laſt, the Value of β is given $= -pna\Delta$.

But, from the Law of the ſaid Coefficients, R, S, ...Δ, it appears that the Value of Δ (whoſe Place from the Beginning is denoted by v) will be $= \pm$ $\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{q-v+2}}{\overline{s+1}.s.\overline{s-1}\ldots\ldots\overline{s-v+2}}\times\frac{a^{v-1}}{nc^v}=\pm$ $\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{p+1}}{\overline{s+1}.s.\overline{s-1}\ldots\ldots\overline{p+m+1}}\times\frac{a^{v-1}}{nc^v}$: And therefore β $(=-pna\Delta)=\pm\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{p+1}.p}{\overline{s+1}.s.\overline{s-1}\ldots\ldots\overline{p+m+1}}$ $\times\frac{a^v}{c^v}=\pm\frac{p.\overline{p+1}.\overline{p+2}.\overline{p+3}\,(v)}{t.\overline{t+1}.\overline{t+2}.\overline{t+3}\,(v)}\times\frac{a^v}{c^v}$ (putting $p+m+1=t$, as before.) Now, if the ſeveral Values of R, S. T .... and β, thus found, be ſubſtituted in the aſſumed Expreſſion, you will have the very ſame Concluſion as in the preceding Article.

Corollary I.

285. Since q is $=p+v-1$, the Fluent $\overline{a+cz^n}|^{m+1}\times$ $\overline{Rz^{qn}+Sz^{qn-n}\ldots\ldots+\Delta z^{pn}}+\beta A$, given above, may be expreſſed by $N\times\overline{Rz^{vn-n}+Sz^{vn-2n}+Tz^{vn-3n}}$ $(v)+\beta A$; where $N=\overline{a+cz^n}|^{m+1}\times z^{pn}$, $R=$

$\frac{1}{m+p+v \times nc}$, $S = -\frac{\overline{p+v-1}.aR}{m+p+v-1.c}$, $T = -\frac{\overline{p+v-2}.aS}{m+p+v-2.c}$: And, where the Coefficient (β) of the given Fluent (A) will always be expreſſed by the laſt of the Quantities R, S, T . . . Δ, multiplied by $-pna$: This is evident, becauſe it is found that $\beta = -pna\,\Delta$. And the ſame thing will alſo appear from the ſeveral particular Caſes (*in Art.* 283.) for the Values of B, C and D: In each of which the Coëfficient of the laſt Term (where A is concerned) is to *that* of the Term immediately preceding it, in the conſtant Ratio of pa to $\frac{1}{n}$, or of pna to Unity.

COROLLARY II.

286. If the Value of c be negative, the general Fluent (*in Art.* 283.) when $a+cz^n = 0$ (provided $m+1$, n, and p be poſitive) will become barely $= \pm\frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}\,(v) \times \frac{a^v A}{c^v}$; becauſe, in this Circumſtance, all the Terms multiplied by $\overline{a+cz^n}|^{m+1}$ intirely vaniſh. If, therefore, b be wrote for $-c$ (to render the Expreſſion more commodious) we ſhall have $\frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}\,(v) \times \frac{a^v A}{b^v}$ for the true Fluent of $\overline{a-bz^n}|^m \times z^{pn+vn-1}\dot{z}$, generated while bz^n, from Nothing, becomes $= a$: Where A denotes the Fluent of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$, generated in the ſame time; and where $t =$

$t = p + m + 1$. Hence it follows that the Fluent of $\overline{a - bz^n}|^m \times z^{pn-1}\dot{z} \times \overline{e + fz^n + gz^{2n} + hz^{3n}}$ &c. (where $e, f, g,$ are any given Quantities) will be $= A \times \overline{e + \frac{paf}{tb} + \frac{p.\overline{p+1}.a^2g}{t.\overline{t+1}.b^2} + \frac{p.\overline{p+1}.\overline{p+2}.a^3h}{t.\overline{t+1}.\overline{t+2}.b^3}} +$ &c. in the forementioned Circumſtance.

PROB. V.

287. *The Fluent* (A) of $\overline{a + cz^n}|^m \times z^{pn-1}\dot{z}$ *being given, to find the Fluent of* $\overline{a + cz^n}|^{m+r} \times z^{pn-1}\dot{z}$; *ſuppoſing r to denote a whole poſitive Number.*

Since $\overline{a+cz^n}|^{m+1} = \overline{a+cz^n}|^m \times \overline{a+cz^n}$, it is evident that $\overline{a+cz^n}|^{m+1} \times z^{pn-1}\dot{z} = \overline{a+cz^n}|^m \times az^{pn-1}\dot{z} + \overline{a+cz^n}|^m \times cz^{pn+n-1}\dot{z}$: Whoſe Fluent (*by Prop.* 3.) is $aA + \frac{\overline{a+cz^n}|^{m+1} \times z^{pn}}{\overline{m+p+1} \times n} - \frac{paA}{m+p+1} = \frac{\overline{a+cz^n}|^{m+1} \times z^{pn}}{\overline{p+m+1} \times n} + \frac{\overline{m+1} \times aA}{p+m+1}$. In like Manner, if this Fluent, of $\overline{a+cz^n}|^{m+1} \times z^{pn-1}\dot{z}$, be denoted by B, that of $\overline{a+cz^n}|^{m+2} \times z^{pn-1}\dot{z}$ by C, &c. it will appear that $\frac{\overline{a+cz^n}|^{m+2} \times z^{pn}}{\overline{p+m+2} \times n} + \frac{\overline{m+2} \times aB}{p+m+2} = C$; $\frac{\overline{a+cz^n}|^{m+3} \times z^{pn}}{\overline{p+m+3} \times n} + \frac{\overline{m+3} \times aC}{p+m+3} = D$, &c. Whence, by ſubſtituting theſe Values, one by one, as in the preceding Problem, and putting $\mathcal{Q} = a + cz^n$, we get $C = \frac{\mathcal{Q}^{m+2} z^{pn}}{\overline{p+m+2}\,.\,n} + \frac{\overline{m+2} \times a\mathcal{Q}^{m+1} z^{pn}}{\overline{p+m+2}\;\overline{p+m+1}\,.\,n} + \frac{\overline{m+2}\,.\,\overline{m+1} \times a^2 A}{\overline{p+m+2}\,.\,\overline{p+m+1}}$; $D = \frac{\mathcal{Q}^{m+3} z^{pn}}{\overline{p+m+3}\,.\,n} + \frac{\overline{m+3} \times a\mathcal{Q}^{m+2} z^{pn}}{\overline{p+m+3}\,.\,\overline{p+m+2}\,.\,n} + \frac{\overline{m+3}\,.\,\overline{m+2} \times a^2\, \mathcal{Q}^{m+1}\, z^{pn}}{\overline{p+m+3}\,.\,\overline{p+m+2}\,.\,\overline{p+m+1}\,.\,n} + \frac{\overline{m+3}\,.\,\overline{m+2}\,.\,\overline{m+1}\,.\,a^3 A}{\overline{p+m+3}\,.\,\overline{p+m+2}\,.\,\overline{p+m+1}}$, &c. Whence it is evident, by Inſpection, that the Fluent of $\overline{a+cz^n}\big|^{m+r} \times z^{pn-1}\,\dot{z}$, expreſſed in a general Manner, will be $\frac{\mathcal{Q}^{m+r}\, z^{pn}}{\overline{p+m+r}\,.\,n} + \frac{\overline{m+r} \times a\mathcal{Q}^{m+r-1}\,.\,z^{pn}}{\overline{p+m+r} \times \overline{p+m+r-1}\,.\,n}$ &c. Which, by putting $m+r=f$, $p+m+r=g$, and making $\mathcal{Q}^{m+1} \times z^{pn}$ a general Multiplicator, will be reduced to $\mathcal{Q}^{m+1} \times z^{pn} \times \overline{\frac{\mathcal{Q}^{r-1}}{gn} + \frac{f \times a\mathcal{Q}^{r-2}}{g\,.\,\overline{g-1}\,.\,n} + \frac{f\,.\,\overline{f-1} \times a^2\mathcal{Q}^{r-3}}{g\,.\,\overline{g-1}\,.\,\overline{g-2}\,.\,n}\ (r) +}$, $\frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} \times \frac{m+3}{p+m+3}\ (r)\ a^r A$; where it appears (from the foregoing Values of B, C, and D) that the Coefficient of A is always equal to the laſt Term of the preceding Series, multiplied by $\overline{m+1} \times na$ (inſtead of $\mathcal{Q}^{m+1}\, z^{pn}$). $\mathcal{Q}.\ E.\ I.$

COROLLARY.

288. If c be negative, ſo that $\mathcal{Q}$, or its Equal, $a + cz^n$, may become equal to Nothing, the Fluent will,

in that Circumſtance, be barely $\pm \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} \times \frac{m+3}{p+m+3}(r) \times a^{r}A$; provided the Values of $m+r$, p, and n are poſitive: Or, if c, p, and n be poſitive, and $m+r+p$ negative, the ſame Expreſſion will exhibit the true Value of the whole Fluent, generated while z, from Nothing, becomes infinite.

PROB. VI.

289. *The ſame being given as in the preceding Problems; it is propoſed to find the Fluent of* $\overline{a+cz^{n}}|^{m-r} \times z^{pn-1}\dot{z}$.

If $-r$ be wrote inſtead of r, in the laſt Article, we ſhall have $m-r=f$, $p+m-r=g$, and $Q^{m+1}z^{pn} \times \frac{Q^{-r-1}}{gn} + \frac{f \times aQ^{-r-2}}{g\cdot\overline{g-1}\cdot n}(-r) + \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2}(-r) \times a^{-r}A$, expreſſing the required Fluent in this Caſe.

But $\frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2}$ &c. continued to $-r$ Factors, ſignifies the ſame thing as the Product continued downwards, or the contrary way, to r Factors, according to the ſame Law: And therefore is $= \frac{p+m}{m} \times \frac{p+m-1}{m-1} \times \frac{p+m-2}{m-2}(r)$. After the ſame Manner we have $\frac{Q^{-r-1}}{gn} + \frac{f \times aQ^{-r-2}}{g\cdot\overline{g-1}\cdot n}(-r) =$

$$-\frac{Q^{-r}}{\overline{f+1}\cdot na} - \frac{\overline{g+1}\cdot Q^{-r+1}}{\overline{f+1}\cdot\overline{f+2}\cdot na^{2}} - \frac{\overline{g+1}\cdot\overline{g+2}\cdot Q^{-r+2}}{\overline{f+1}\cdot\overline{f+2}\cdot\overline{f+3}\cdot na^{3}}(r)$$

(r) and conſequently the Fluent itſelf $= \mathcal{Q}^{m+r} z^{pn} \times$

$$\overline{-\frac{\mathcal{Q}^{-r}}{\overline{f+1}.na} - \frac{\overline{g+1}.\mathcal{Q}^{1-r}}{\overline{f+1}.\overline{f+2}.na^2} - \frac{\overline{g+1}.\overline{g+2}.\mathcal{Q}^{2-r}}{\overline{f+1}.\overline{f+2}.\overline{f+3}.na^3}\ (r)} + \frac{p+m}{m} \times \frac{p+m-1}{m-1} \times \frac{p+m-2}{m-2}\ (r) \times \frac{A}{a^r}.$$ Q. E. I.

COROLLARY.

290. It appears from hence that the Coefficient of *A*, the given Fluent, will always be equal to *that* of the laſt Term of the preceding Series, multiplied by $\overline{p+m} \times n$: For, ſeeing the Coefficient of the ſaid laſt Term (whoſe Diſtance from the firſt, incluſive, is denoted by *r*) muſt be $\frac{\overline{g+1}.\overline{g+2}.\overline{g+3}\ldots\overline{g+r-1}}{\overline{f+1}.\overline{f+2}.\overline{f+3}\ldots\overline{f+r}} \times \frac{1}{na^r}$ (by the Law of the Series) where $f+r=m$ and $g+r-1=p+m-1$ (as appears from above) it follows, by inverting the Order of both Progreſſions, that $\frac{\overline{p+m-1}.\overline{p+m-2}\ (r-1)}{m.\overline{m-1}.\overline{m-2}\ (r)} \times \frac{1}{na^r}$ will alſo expreſs the ſame Coefficient: Which, multiplied by $\overline{p+m} \times n$, gives $\frac{\overline{p+m}.\overline{p+m-1}.\overline{p+m-2}\ (r)}{m.\overline{m-1}.\overline{m-2}\ (r)}$ $\frac{1}{a^r}$, the very Coefficient of *A*, above determined. The Uſe of this Concluſion will be ſeen in what follows.

PROB. VII.

291. *The ſame being,* ſtill, *given; to find the Fluent of*

$$\overline{a+cz^n}\big|^m \times z^{pn-vn-1}\dot{z}.$$

By proceeding as in the laſt Problem, the required Fluent of $\overline{a+cz^n}\big|^m \times z^{pn-vn-1}$ is derived from *that* of $\overline{a+cz^n}\big|^m \times z^{pn+vn-1}\dot{z}$ *(given by Prob.* 4.) and comes out

$$= \mathcal{Q}^{m+1} z^{pn} \times \overline{\frac{z^{-vn}}{\overline{q+1}.na} - \frac{\overline{s+2}.cz^{n-vn}}{\overline{q+1}.\overline{q+2}.na^2} + \frac{\overline{s+2}.\overline{s+3}.c^2z^{2n-vn}}{\overline{q+1}.\overline{q+2}.\overline{q+3}.na^3}\,(v)} \pm \frac{t-1}{p-1} \times \frac{t-2}{p-2} \times \frac{t-3}{p-3}\,(v) \times \frac{c^vA}{a^v}:$$

Where, $\mathcal{Q} = a+cz^n$, $q = p-v-1$, $s = m+q$, $t = p+m+1$: And where, the Coefficient of A is equal to that of the laſt of the preceding Terms, multiplied by $-\overline{m+p} \times nc$. If the Manner of deducing the required Fluent, in this, and the laſt, Problem, ſhould not appear ſufficiently plain and ſatisfactory to the Beginner; the ſame Concluſions may be, otherwiſe, brought out; by finding A, in Terms of B, C, or D, from the ſeveral particular Equations in *Art.* 283. or, by aſſuming a deſcending Series, inſtead of an aſcending one. *Vid. Art.* 284.

PROB. VIII.

292. *The ſame being,* ſtill, given; *to find the Fluent of*

$$\overline{a+cz^n}\big|^{m+r} \times z^{pn+vn-1}\dot{z}.$$

Let the Fluent of $\overline{a+cz^n}\big|^m \times z^{pn+vn-1}\dot{z}$ *(given by Prob.* 4.) be denoted by B, and that required by F: Then,

Then; if $p+v$ be put $=p'$, the Value of F (the Fluent of $\overline{a+cz^n}|^{m+r} \times z^{p'n-1}\dot{z}$) will be given from that of B (the Fluent of $\overline{a+cz^n}|^{m} \times z^{p'n-1}\dot{z}$) by writing B for A and p' for p, in *Art.* 287. Whence we get

$$F=Q^{m+1}\dot{z}^{p'n}\times\frac{Q^{r-1}}{gn}+\frac{faQ^{r-2}}{g\cdot\overline{g-1}.n}+\frac{f.\overline{f-1}.a^2Q^{r-3}}{g\cdot\overline{g-1}\cdot\overline{g-2}.n}\ (r)+\frac{m+1}{p'+m+1}\times\frac{m+2}{p'+m+2}\times\frac{m+3}{p'+m+3}\ (r)\times a^r B:$$

Where $p'=p+v$, $f=m+r$, $g\ (=p'+m+r)=p+m+v+r$, and $Q=a+cz^n$.

Which Fluent, by ſubſtituting the Value of B (in *Prob.* 4.) becomes

$$F=Q^{m+1}\dot{z}^{p'n}\times\frac{Q^{r-1}}{gn}+\frac{faQ^{r-2}}{g\cdot\overline{g-1}.n}+\frac{f.\overline{f-1}.a^2Q^{r-3}}{g\cdot\overline{g-1}\cdot\overline{g-2}.n}\ (r)+\frac{m+1}{p'+m+1}\times\frac{m+2}{p'+m+2}\ (r)$$

$$\times a^r\times Q^{m+1}z^{pn}\times\frac{z^{vn-n}}{\overline{s+1}.nc}-\frac{qaz^{vn-2n}}{\overline{s+1}.snc^2}+\frac{q.\overline{q-1}.a^2z^{vn-3n}}{\overline{s+1}.s.\overline{s-1}.nc^3}$$

$$(v)\pm\frac{m+1}{p'+m+1}\times\frac{m+2}{p'+m+2}\ (r)\times a^r\times\frac{p}{t}\times\frac{\overline{p+1}}{t+1}$$

$$(v)\times\frac{a^vA}{c^v}:$$

Where $q=p+v-1$, $s=m+q=m+p+v-1$, and $t=p+m+1$; and where the Sign of the laſt Term is + or − according as v is an even or odd Number. *Q. E. I.*

291. If the laſt Term of the firſt Series, excluſive of the general Multiplicator $\mathcal{Q}^{m+1} z^{p'n}$, be denoted by β, the Multiplicator, $\frac{m+1}{p'+m+1} \times \frac{m+2}{p'+m+2} (r) \times a^{r}$, to the ſecond Series will be $= \overline{m+1} \times na\beta$ *; and therefore the firſt Term of this Series, including its Multiplicators, is $= \frac{\overline{m+1}.a\beta\mathcal{Q}^{m+1} z^{pn+vn}}{\overline{s+1}.cz^{n}}$: Which, if R be put to denote the laſt Term $\beta\mathcal{Q}^{m+1} z^{pn+vn}$ of the firſt Series (with its Multiplicator) will be expounded by $\frac{\overline{m+1}.aR}{\overline{s+1}.cz^{n}}$. Hence it follows, that the Fluent of $\overline{a+cz^{n}}|^{m+r} \times z^{pn+vn-1}\dot{z}$, given above, will alſo be truly expreſſed by

$$\frac{\mathcal{Q}^{m+r} \times z^{pn+vn}}{gn} + \frac{f}{g-1} \times \frac{aH}{\mathcal{Q}} + \frac{f-1}{g-2} \times \frac{aI}{\mathcal{Q}} + \frac{f-2}{g-3} \times \frac{aK}{\mathcal{Q}} (r) + \frac{\overline{m+1}}{s+1} \times \frac{aR}{cz^{n}} - \frac{q}{s} \times \frac{aS}{cz^{n}} - \frac{q-1}{s-1} \times \frac{aT}{cz^{n}} - \frac{q-2}{s-2} \times \frac{aV}{cz^{n}} (v) \pm \frac{\overline{m+1}.\overline{m+2}.\overline{m+3}\,(r) \times p.\overline{p+1}.\overline{p+2}\,(v)}{\overline{p'+m+1}.\overline{p'+m+2}\,(r) \times t.\overline{t+1}.\overline{t+2}\,(v)} \times \frac{a^{v+r}A}{c^{v}}:$$

Where H, I, K, L . . . . . R, S, T, V, &c. repreſent the Terms immediately preceding thoſe where they ſtand, under their proper Signs: R being the laſt Term of the firſt Series; alſo $f = m + r$, $g = m + r + p + v$, $q = p + v - 1$, $s = m + q$, $t = m + p + 1$, and $\mathcal{Q} = a + cz^{n}$.

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* Art. 287.

COROLLARY II.

293. Since the Diviſor, $\overline{\acute{p}+m+1}.\overline{\acute{p}+m+2}\ (r) \times t.t+1.t\ (v)$, of the laſt Term of the Fluent (by ſubſtituting for t and $\acute{p}$ &c.) is $= \overline{p+m+1}.\overline{p+m+2}\ (v) \times \overline{p+v+m+1}.\overline{p+v+m+2}\ (r)$: Where, the laſt Factor $(p+m+v)$ of the firſt Progreſſion, is leſs by Unity than the firſt Factor of the Second; it is evident that the ſaid ſecond Progreſſion is only a Continuation of the firſt to r *more* Factors: And ſo, the laſt Term of the Fluent, where A is found, is truly expreſſed by $\pm$

$$\frac{p.\overline{p+1}.\overline{p+2}\ (v) \times \overline{m+1}.\overline{m+2}.\overline{m+3}\ (r)}{\overline{m+p+1}.\overline{m+p+2}.\overline{m+p+3}\ (v+r)} \times \frac{a^{v+r}A}{c^{v}}.$$

Hence it follows, that the Fluent of $\overline{a+cx^n}|^{m+r} \times x^{pn+vn-1}\dot{x}$, or that of $\overline{a-bx^n}|^{m+r} \times x^{pn+vn-1}\dot{x}$ (making $c = -b$) will, when $a-bx^n$ becomes equal to Nothing, be barely $=$

$$\frac{p.\overline{p+1}.\overline{p+2}\ (v) \times \overline{m+1}.\overline{m+2}.\overline{m+3}\ (r)}{\overline{m+p+1}.\overline{m+p+2}.\overline{m+p+3}\ (v+r)} \times \frac{a^{v+r}A}{b^{v}}:$$

A being the Fluent of $\overline{a-bx^n}|^{m} \times x^{pn-1}\dot{x}$, in that Circumſtance, v and r whole poſitive Numbers, and p and $m+1$ any poſitive Numbers, either whole or broken.

SCHOLIUM.

294. If the Fluent of $\overline{a+cx^n}|^{m+r} \times x^{pn-1}\dot{x}$ (given by *Prob.* 5.) be denoted by C; then (F) the Fluent of $\overline{a+cx^n}|^{\acute{m}} \times x^{pn+vn-1}\dot{x}$ (where $\acute{m} = m+r$) will be had, from C (by *Prob.* 4.) according to a new Form, dif-

ferent from thoſe already given. And, by following the ſame Method, the Fluents of $\overline{a+cz^{r}}|^{m-r} \times z^{pn+vn-1}\dot{z}$, $\overline{a+cz^{n}}|^{m+r} \times z^{pn-vn-1}\dot{z}$, and $\overline{a+cz^{n}}|^{m-r} \times z^{pn-vn-1}\dot{z}$ may alſo be found, each, according to two different Forms, from a Combination of the correſponding Caſes in the foregoing Problems.

But, as it is extremely tireſome to repeat the ſame thing, again and again, where ſuch a Number of Symbols are neceſſarily concerned, I ſhall here put down one Solution to each Caſe (becauſe of their Uſe) leaving the Proceſs and the other Forms (which contain no new Difficulty) to *Thoſe* who will be at the Trouble to ſet about them.

1°. The Fluent of $\overline{a+cz^{n}}|^{m-r} \times z^{pn+vn-1}\dot{z}$ is $=$

$$-\frac{Q^{m-r+1} \times z^{pn+vn}}{\overline{f+1}.na} + \frac{g+1}{f+2} \times \frac{QH}{a} + \frac{g+2}{f+3} \times \frac{QI}{a}\,(r)$$

$$-\frac{QR}{cz^{n}} - \frac{q}{s} \times \frac{aS}{cz^{n}} - \frac{q-1}{s-1} \times \frac{aT}{cz^{n}} - \frac{q-2}{s-2} \times \frac{aV}{cz^{n}}\,(v)$$

$$+\frac{s+1}{m} \times \frac{s}{m-1} \times \frac{s-1}{m-2}\,(r) \times \frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}\,(v) \times \frac{a^{v-r}A}{\overline{-c}|^{v}}.$$

Where *H*, *I*, *K*, *L*, ... *R*, *S*, *T*, &c. denote the Terms immediately preceding thoſe where they ſtand, under their proper Signs; *R* being the laſt Term of the firſt Series, alſo $Q=a+cz^{n}$, $f=m-r$, $g=p+m+v-r$, $q=p+v-1$, $s=m+p+v-1$, $t=p+m+1$, and $A=$ the given Fluent of $\overline{a+cz^{n}}|^{m} \times z^{pn-1}\dot{z}$.

2°. The Fluent of $\overline{a+cz^n}|^{m+r} \times z^{pn-vn-1}\dot{z}$ is $=$

$$\frac{Q^{m+r+1}\times z^{pn-vn}}{\overline{q+1}\cdot na} - \frac{s+2}{q+2}\times\frac{Hcz^n}{a} - \frac{s+3}{q+3}\times\frac{Icz^n}{a}\ (v)$$

$$-\frac{Rcz^n}{Q} + \frac{f}{g-1}\times\frac{aS}{Q} + \frac{f-1}{g-2}\times\frac{aT}{Q} + \frac{f-2}{g-3}\times\frac{aV}{Q}\ (r)$$

$$+\frac{s+2}{q+1}\times\frac{s+3}{q+2}\times\frac{s+4}{q+3}(v)\times\frac{m+1}{t}\times\frac{m+2}{t+1}\times\frac{m+3}{t+2}(r)\times\frac{\overline{-c}|^{v}A}{a^{v-r}}:$$

Where $q=p-v-1$, $s=m+r+q$, $f=m+r$, $g=p+m+r$, and the reſt as in the preceding Caſe.

3°. The Fluent of $\overline{a+cz^n}|^{m-r} \times z^{pn-vn-1}\dot{z}$ is $=$

$$-\frac{Q^{m-r+1}\times z^{pn-vn}}{\overline{f+1}.na} + \frac{g+1}{f+2}\times\frac{QH}{a} + \frac{g+2}{f+3}\times\frac{QI}{a}\ (r)$$

$$-\frac{s+1}{q+1}\times\frac{QR}{a} - \frac{s+2}{q+2}\times\frac{Scz^n}{a} - \frac{s+3}{q+3}\times\frac{Tcz^n}{a}\ (v).$$

$$+\frac{\overline{t-1}.\overline{t-2}.\overline{t-3}.\overline{t-4}.\overline{t-5}\ (r+v)}{m\overline{m-1}.\overline{m-2}(r)\times\overline{p-1}\,\overline{p-2}.\overline{p-3}\ (v)}\times\frac{\overline{-c}|^{v}A}{a^{r+v}}.$$

In which $f=m-r$, $g=m+p-r-v$, $q=p-v-1$, $s=q+m$, and the reſt as before.

295. From what has been delivered in this Section, the Fluents of various Forms of Fluxions may be exhibited, by means of circular Arcs and Logarithms. For, ſince the Fluents of $\overline{a+cz^n}|^{-1}\times z^{\frac{1}{2}n-1}\dot{z}$, $\overline{a+cz^n}|^{-\frac{1}{2}}\times z^{\frac{1}{2}n-1}\dot{z}$, and $\overline{a+cz^n}|^{-\frac{1}{2}}\times z^{-1}\dot{z}$ (which I call original Ones) are all of them explicable by one, or the other, of theſe two Kinds of Quantities (as will appear farther on) *thoſe* of $\overline{a+cz^n}|^{-1\pm r}\times z^{\frac{1}{2}n\pm vn-1}\dot{z}$ $\overline{a+cz^n}|^{-\frac{1}{2}\pm r}\times z^{\frac{1}{2}n\pm vn-1}\dot{z}$, and $\overline{a+cz^n}|^{-\frac{1}{2}\pm r}\times z^{\pm vn-1}\dot{z}$ will alſo be given from thence, by the foregoing

going Theorems. Whence the moſt uſeful Forms of Fluents in *Cotes's Harmonia Menſurarum* will be obtained, beſides ſome others, more general than any, of the ſame Kind, put down by that ſagacious Author.

Here follow a few Examples of ſome of the moſt uſeful Caſes.

EXAMPLE I.

296. *Let the Fluxion given be* $\frac{z^{2v}\dot{z}}{\sqrt{d^2 \pm z^2}}$ (or $\overline{d^2+z^2}|^{-\frac{1}{2}}$ $\times z^{2v}\dot{z}$) *v being any whole poſitive Number.*

Then, the Fluent of $\overline{d^2 \pm z^2}|^{-\frac{1}{2}} \times \dot{z}$, or $\frac{\dot{z}}{\sqrt{d^2 \pm z^2}}$ being $=$ *hyp. Log.* $\frac{z+\sqrt{d^2+z^2}}{d}$; or, equal to the Arch whoſe Sine is $\frac{z}{d}$ and Radius Unity *; according as the ſecond Term, in $d^2 \pm z^2$, is poſitive or negative; let *A* be, therefore, taken to denote the ſaid Arch, or Logarithm; and let $\overline{d^2 \pm z^2}|^{-\frac{1}{2}} \times \dot{z}$ be compared with $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$ (whoſe Fluent is, all along, ſuppoſed to be given $= A$) and you will have $a = d^2$, $c = \pm 1$, $n = 2$, $m = -\frac{1}{2}$, $2p-1 = 0$, and therefore $p = \frac{1}{2}$: Whence, by ſubſtituting thoſe Values in *Art.* 283. we likewiſe get $q\,(p+v-1) = \frac{2v-1}{2}$, $s\,(m+q) = -1$, $r\,(m+p+1) = 1$; and, conſequently, the Fluent ſought $= \overline{d^2 \pm z^2}|^{\frac{1}{2}} \times \overline{\pm \frac{z^{2v-1}}{2v} - \frac{\overline{2v-1}\,.\,d^2 z^{2v-3}}{2v\,.\,\overline{2v-2}} \pm}$ $\overline{\frac{\overline{2v-1}\,.\,\overline{2v-3}\,.\,d^4 z^{2v-5}}{2v\,.\,\overline{2v-2}\,.\,\overline{2v-4}} - \frac{\overline{2v-1}\,.\,\overline{2v-3}\,.\,\overline{2v-5}\,.\,d^6 z^{2v-7}}{2v\,.\,\overline{2v-2}\,.\,\overline{2v-4}\,.\,\overline{2v-6}}}$

* Art. 126, 142.

$(v) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} (v) \times d^{2v}$: In which the laſt Term is negative, when the given Fluxion is $\frac{z^{2v}\dot{z}}{\sqrt{d^2+z^2}}$, and v, at the ſame time, an odd Number; but in all other Caſes, affirmative.

EXAMPLE II.

297. *Let* $z^{2v}\dot{z}\sqrt{d^2 \pm z^2}$ (or $\overline{d^2 \pm z^2}|^{-\frac{1}{2}+1} \times z^{2v}\dot{z}$) *be propounded.*

Here, denoting the Fluent of $\overline{d^2 \pm z^2}|^{-\frac{1}{2}}\dot{z}$ by A (as above) and comparing $\overline{d^2 \pm z^2}|^{-\frac{1}{2}+1} \times z^{2v}\dot{z}$, with $\overline{a+cz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}$ (*Vid. Prob.* 8.) we have $r = 1$, and the reſt as in the laſt Example: Whence alſo $p\ (p+v) = v+\frac{1}{2}$, $f\ (m+r) = \frac{1}{2}$, $g = v+1$, $Q = d^2 \pm z^2$, and the Fluent itſelf $= \frac{z^{2v+1}\sqrt{d^2 \pm z^2}}{2v+2} \pm \frac{d^2R}{2vz} \mp \frac{\overline{2v-1}.d^2S}{\overline{2v-2}.z^2} \mp \frac{\overline{2v-3}.d^2T}{\overline{2v-4}.z^2} (1+v) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} (v) \times \frac{d^{2v+2}A}{2v+2}$* (*R*, *S*, *T*, &c. being the preceding Terms with their Signs) $= \frac{\sqrt{d^2 \pm z^2}}{2v+2} \times \overline{z^{2v+1} \pm \frac{d^2z^{2v-1}}{2v} - \frac{\overline{2v-1}.d^4z^{2v-3}}{2v.\overline{2v-2}} \pm \frac{\overline{2v-1}.\overline{2v-3}.d^6z^{2v-5}}{2v.\overline{2v-2}.\overline{2v-4}}}$ $(v+1) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} (v) \times \frac{d^{2v+2}A}{2v+2}$: Where the Sign of the laſt Term muſt be regulated as in the pre-

* Art. 292.

preceding Example—If the Fluent of $\frac{z^{-vn}\dot{z}}{\sqrt{d^2 \pm z^2}}$, or of $z^{-vn}\dot{z}\sqrt{d^2 \pm z^2}$ (in which the Exponent is negative) be required; the Anſwer will be had in finite Terms, independent of *A*, *by Art.* 85.

EXAMPLE III.

298. *Wherein the Fluxion propoſed is* $\overline{d^n - z^n}|^{-\frac{1}{2}+r} \times z^{\frac{1}{2}n+vn-1}\dot{z}$; *r and v being any whole poſitive Numbers.*

Since the Fluent of $\overline{d^n - z^n}|^{-\frac{1}{2}} \times z^{\frac{1}{2}n-1}\dot{z}$, (as will appear hereafter) is truly expreſſed by $\frac{2}{n} \times$ *Arch*, whoſe Sine is $\frac{z^{\frac{1}{2}n}}{d^{\frac{1}{2}n}}$ and Radius Unity, let this Value be denoted by *A*; and then, by writing d^n for *a*, -1 for *c*, $-\frac{1}{2}$ for *m*, and $\frac{1}{2}$ for *p*, *in Art.* 292. we ſhall have $f\,(m+r) = \frac{2r-1}{2}$, $g\,(m+p+r+v) = r+v$, $q\,(p+v-1) = \frac{2v-1}{2}$, $s\,(m+q) = v-1$, $t\,(p+m+1) = 1$, $Q\,(a+cz^n) = d^n - z^n$, and the Fluent, itſelf, equal to

$$\frac{\overline{Q}|^{r-\frac{1}{2}}\, z^{vn+\frac{1}{2}n}}{\overline{r+v}.n} + \frac{2r-1}{r+v-1} \times \frac{\frac{1}{2}d^nH}{Q} + \frac{2r-3}{r+v-2} \times \frac{\frac{1}{2}d^nI}{Q} + \frac{2r-5}{r+v-3} \times \frac{\frac{1}{2}d^nK}{Q}\,(r) - \frac{\frac{1}{2}d^nR}{vz^n} + \frac{2v-1}{v-1} \times \frac{\frac{1}{2}d^nS}{z^n} + \frac{2v-3}{v-2} \times \frac{\frac{1}{2}d^nT}{z^n}\,(v) + \frac{1.3.5.7\,(r) \times 1.3.5.7\,(v)}{2.4.6.8.10.12\,(r+v)}$$

* Art. 293. * $\times\, d^{rn+vn}A$: In which *H*, *I*, *K* . . . *R*, *S*, *T*, &c. denote the preceding Terms with their Signs; *R* being

the laſt Term of the firſt Series. Hence, becauſe all the Terms, but the laſt, vaniſh, when $\mathcal{Q}=0$, it follows that the whole Fluent of $\overline{d^{n}-z^{n}}^{\,r-\frac{1}{2}} \times z^{vn+\frac{1}{2}n-1}\dot{z}$, generated while z, from Nothing, becomes equal to d, is truly expreſſed by $\frac{1.3.5.7\,(r) \times 1.3.5.7\,(v)}{2.4.6.8.10.12\ \ (r+v)} \times d^{rn+vn}A$, or by $\frac{1.3.5.7\,(r) \times 1.3.5.7\,(v)}{2.4.6.8.10.12\,(r+v)} \times \frac{d^{rn+vn}G}{n}$; G being the Semi-Periphery of the Circle whoſe Radius is Unity.

EXAMPLE IV.

299. *Let it be required to find the* whole *Fluent of* $\frac{\overline{a-bz^{n}}^{\,m} \times z^{pn-1}\dot{z}}{\overline{d+kz^{n}}^{\,\beta}}$, *generated while* bz^{n}, *from Nothing, becomes* $=a$; *that of* $\overline{a-bz^{n}}^{\,m} \times z^{pn-1}\dot{z}$ *being given* $(=A.)$

Here, by expanding $\overline{d+kz^{n}}^{\,-\beta}$, our given Fluxion becomes $= \overline{a-bz^{n}}^{\,m} \times z^{pn-1}\dot{z}$ into $d^{-\beta} \times \overline{1-\frac{\beta kz^{n}}{d}+\frac{\beta.\beta+1.k^{2}z^{2n}}{1.2.d^{2}}-\frac{\beta.\beta+1.\beta+2.k^{3}z^{3n}}{1.2.3.d^{3}}}$. &c. Which Series being compared with $e+fz^{n}+gz^{2n}$ &c. (*Vid. Art.* 286.) we have $e=1$, $f=-\frac{\beta k}{d}$, $g=\frac{\beta.\beta+1.k^{2}}{1.2.d^{2}}$, &c. and conſequently the Fluent ſought (by ſubſtituting theſe Values) equal to $\frac{A}{d^{\beta}}$ into $1-$

$$\frac{p}{t} \times \frac{\beta}{1} \times \frac{ak}{bd} + \frac{p}{t}.\frac{p+1}{t+1} \times \frac{\beta}{1}.\frac{\beta+1}{2} \times \overline{\frac{ak}{bd}}^{\,2} -$$

$\frac{p}{t}\cdot\frac{p+1}{t+1}\cdot\frac{p+2}{t+2}\times\frac{\beta}{1}\cdot\frac{\beta+1}{2}\cdot\frac{\beta+2}{3}\times\overline{\frac{ak}{bd}}\Big|^{3}$ + &c. (t being $= p+m+1$.)

Here the Values of $m+1$, n and p are ſuppoſed poſitive;* and it is requiſite that $1+\frac{ak}{bd}$ ſhould alſo be poſitive; otherwiſe the Fluent will fail. Although the Series brought out above runs on to Infinity, yet it may be ſum'd, in many Caſes: Thus, if the given Fluxion be $\frac{\overline{a-bx^n}|^{-\frac{1}{2}}\times x^{\frac{1}{2}n-1}\dot{x}}{d+kx^n}$; then, the foreſaid Series becoming $1-\frac{1}{2}\times\frac{ak}{bd}+\frac{1}{2}\times\frac{3}{4}\times\overline{\frac{ak}{bd}}\Big|^{2}$ — &c. its Sum will be $\overline{1+\frac{ak}{bd}}\Big|^{-\frac{1}{2}}$: And conſequently $\frac{A}{d}\times\overline{1+\frac{ak}{bd}}\Big|^{-\frac{1}{2}}$ = the Fluent ſought: Where, A (the *whole* Fluent of $\overline{a-bx^n}|^{-\frac{1}{2}}\times x^{\frac{1}{2}n-1}\dot{x}$ being $=\frac{1}{n\sqrt{b}}$ × Semi-Periphery of the Circle whoſe Radius is Unity, the Fluent given above will, therefore, be $=\frac{1}{n\sqrt{bd^2+adk}}$ × by the ſame Semi-Periphery. If the Reader is deſirous to ſee a further Application of the Summation of Serieſes, to the finding of Fluents, I muſt refer him to my *Diſſertations* (where it is handled in a general Manner) having neither Room nor Inclination to treat of it here.

* Art. 286.

SECTION IV.

Of the Tranſformation of Fluxions.

301. BY the Transformation of Fluxions may be underſtood, the reducing *any* fluxional Quantity to a different, or more commodious, Form; according to which Senſe, a great Part of the ſecond Section would properly fall under this Head. But, *what* is here propoſed, and *what* is commonly meant by the Transformation of Fluxions, *is*, the Method of ordering thoſe Kinds of Expreſſions which involve one variable Quantity *only* with its Fluxion; which, yet, are ſo affected by radical Signs, that the Fluent, without an Infinite Series, would be impracticable, were it not for a new Subſtitution, or ſome other Kind of Tranſformation, whereby the given Fluxion is render'd more manageable.

Something of this Sort has been already touch'd upon in *Art.* 83. And in what follows I ſhall farther point out and exemplify the principal Caſes wherein ſuch a *Procedure* will be of Service.

302. *If the Number of Dimenſions of the variable Quantity, without the* Vinculum, *increaſed by Unity, be ſome aliquot Part, or Parts, of the Dimenſions of the ſame Quantity, under the* Vinculum, *the Fluxion will be reduced to a better Form by ſubſtituting for that Power of the variable Quantity, which ariſes by dividing its Exponent, under the* Vinculum, *by the Denominator of the Fraction expreſſing the ſaid* aliquot *Part, or Parts.*

Thus, if the Fluxion propounded be $\frac{z^{\overline{\frac{1}{2}n-1}}\dot{z}}{\sqrt{\overline{c^n \pm z^n}}}$; by ſubſtituting $x = z^{\frac{1}{2}n}$, and taking the Fluxion of both Sides of the Equation, we have $\dot{x} = \frac{1}{2}nz^{\overline{\frac{1}{2}n-1}}\dot{z}$; and therefore $z^{\overline{\frac{1}{2}n-1}}\dot{z} = \frac{\dot{x}}{\frac{1}{2}n}$: Which Value, with that of z^n, being wrote for their Equals, in the given Fluxion, it

will

will be transformed to $\frac{\dot{x}}{\frac{1}{2}n\sqrt{c^{n}\pm x^{2}}}$: Which, putting $a=c^{\frac{1}{2}n}$ (to make the Terms homologous), is alſo expreſſed by $\frac{\dot{x}}{\frac{1}{2}n\sqrt{a^{2}\pm x^{2}}}$: Whereof the Fluent will be given by *Art.* 126. or *Art.* 142. according as the Sign of x^{2} is poſitive or negative.

303. If the Power of the variable Quantity under the *Vinculum* has a Coefficient, it will be beſt to bring that Coefficient without the *Vinculum.*

Ex. 2. Where let the Fluxion given be $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{a+cz^{n}}}$: Which, by bringing c without the *Vinculum*, becomes $\frac{z^{\frac{1}{2}n-1}\dot{z}}{c^{\frac{1}{2}}\sqrt{\frac{a}{c}+z^{n}}}$: From whence, by putting $x=z^{\frac{1}{2}n}$ and proceeding as above, we get $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{a}{c}+x^{2}}}$: Whoſe Fluent, *by Art.* 126. is $\frac{1}{\frac{1}{2}nc^{\frac{1}{2}}}\times$ *hyp. Log.* $x+\sqrt{\frac{a}{c}+x^{2}}$. This, by reſtoring z, becomes $\frac{2}{nc^{\frac{1}{2}}}\times$ *hyp. Log.* $z^{\frac{1}{2}n}+\sqrt{\frac{a}{c}+z^{n}}$. Which, corrected (by ſuppoſing it = 0 when $z=0$) gives, at length, $\frac{2}{nc^{\frac{1}{2}}}\times\overline{\text{hyp. Log. } z^{\frac{1}{2}n}+\sqrt{\frac{a}{c}+z^{n}}-\text{hyp. Log. }\sqrt{\frac{a}{c}}}=\frac{2}{nc^{\frac{1}{2}}}\times$ *hyp. Log.* $\sqrt{\frac{cz^{n}}{a}}+\sqrt{1+\frac{cz^{n}}{a}}$ for the true Fluent of the Quantity propoſed.

But, when c is a negative Quantity, this Fluent fails, becauſe the ſquare Root of c is to be extracted. In this Caſe $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{a}{c}+x^2}}$ muſt be transformed to $\frac{\dot{x}}{\frac{1}{2}n\sqrt{-c}\times\sqrt{\frac{a}{-c}-x^2}}$: And then its Fluent (by *Art.* 142.) will be had $=\frac{1}{\frac{1}{2}n\sqrt{-c}}\times$ the *Arch* of a Circle whoſe Radius is Unity, and Right-Sine $=\frac{x}{\sqrt{\frac{a}{-c}}}=\sqrt{\frac{-cz^n}{a}}$.

Ex. 3. Let the given Fluxion be $\frac{\dot{z}}{z\sqrt{a+cz^n}}$. Which, by bringing c without the *Vinculum*, and putting $x=z^{\frac{1}{2}n}$, is transformed to $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}x\sqrt{\frac{a}{c}+x^2}}$: Whereof the Fluent, by *Art.* 126. is $\frac{1}{n\sqrt{a}}\times$ *hyp. Log.* $\frac{\sqrt{\frac{a}{c}}-\sqrt{\frac{a}{c}+x^2}}{\sqrt{\frac{a}{c}}+\sqrt{\frac{a}{c}+x^2}}=\frac{1}{n\sqrt{a}}\times$ *hyp. Log.* $\frac{\sqrt{a}-\sqrt{a+cz^n}}{\sqrt{a}+\sqrt{a+cz^n}}$. But here, when c is poſitive, the Numerator will be negative; in which Caſe it will be proper to change its Signs, and expreſs the Fluent by $\frac{1}{n\sqrt{a}}\times$ *hyp. Log.* $\frac{\sqrt{a+cz^n}-\sqrt{a}}{\sqrt{a+cz^n}+\sqrt{a}}$. That, ſuch an Alteration of the Signs can make no Difference in the Fluxion, is evident from the Nature of Logarithms; becauſe the Fluxion of the *Log.* of $-x$ $\left(= \frac{-\dot{x}}{-x} = \frac{\dot{x}}{x}\right)$ is the ſame with *that* of the *hyp. Log.* of x. It will be proper to obſerve farther, that, inſtead of the Logarithm above derived, any one of the following, equal, Quantities may be taken; *viz. hyp. Log.* $\frac{\overline{\sqrt{a+cz^n}-\sqrt{a}}|^2}{cz^n}$ (found by multiplying both the Numerator and Denominator of the foreſaid Logarithm by $\sqrt{a+cz^n}-\sqrt{a}$) $= 2 \times$ *hyp. Log.* $\frac{\sqrt{a+cz^n}-\sqrt{a}}{\sqrt{cz^n}}$ (by the Nature of Logarithms) $= 2 \times$ *hyp. Log.* $\frac{\sqrt{cz^n}}{\sqrt{a+cz^n}+\sqrt{a}}$ (by multiplying, equally, by $\sqrt{a+cz^n}+\sqrt{a}$) But, take which of theſe Forms you will, the Fluent fails when a is negative; becauſe the general Multiplicator $\frac{1}{n\sqrt{a}}$ is then impoſſible. In this Caſe the Fluent of $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}} \times x\sqrt{\frac{a}{c}+x^2}}$, or its Equal $\frac{\dot{z}}{z\sqrt{a-cz^n}}$, will be given *by Art.* 142. and is expounded by $\frac{1}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{-a}{c}}}$ $\times A = \frac{2}{n\sqrt{-a}}$; where A denotes the *Arch* whoſe Radius is Unity, and Secant $\frac{x}{\sqrt{\frac{-a}{c}}}$ $\left(= \sqrt{\frac{cz^n}{-a}}\right)$.

In the ſame Manner the Fluent of $\frac{x^{\frac{1}{2}n-1}\dot{x}}{a+cx^n}$, is found $=\frac{1}{n\sqrt{ac}}\times$ *Arch*, whoſe Radius is Unity and Tangent $\sqrt{\frac{cx^n}{a}}$, or equal to $\frac{1}{n\sqrt{-ca}}\times$ *hyp. Log.* $\frac{\sqrt{a}+\sqrt{-cx^n}}{\sqrt{a}-\sqrt{-cx^n}}$, according as the Value of c is affirmative or negative; a being ſuppoſed affirmative.

304. *When the Power, or Powers, of the variable Quantity without the* Vinculum, *or radical Sign, fall, moſtly, in the Denominator, it may be of Uſe to ſubſtitute for the Reciprocal of the ſaid Quantity, or for the Quotient which ariſes by dividing ſome known Quantity, either, by it, or by ſome Compound of it in the Denominator.*

Ex. 1. *Let the propoſed Fluxion be* $\frac{a^3\dot{z}}{z^2\sqrt{a^2+z^2}}$; then, putting $z=\frac{a^2}{x}$, we have $x=\frac{a^2}{z}$, and $\dot{z}=-\frac{a^2\dot{x}}{x^2}$; and conſequently $\frac{a^3\dot{z}}{z^2\sqrt{a^2+z^2}}=\frac{-x\dot{x}}{\sqrt{x^2+a^2}}$: Whereof the Fluent is $-\sqrt{x^2+a^2}=-\sqrt{\frac{a^4}{z^2}+a^2}$.

Ex. 2. *Let the given Fluxion be* $\frac{z\dot{z}}{\overline{a+z}|^3\times\sqrt{a^2+az+z^2}}$.

Here, putting $x=\frac{a^2}{a+z}$, we have $z=\frac{aa-ax}{x}=a\times\frac{a-x}{x}$, $\dot{z}=-\frac{a^2\dot{x}}{x^2}$, $z\dot{z}=-\frac{a^3\dot{x}\times\overline{a-x}}{x^3}$, $\sqrt{a^2+az+z^2}=\frac{a}{x}\sqrt{a^2-ax+x^2}$; and therefore the

Quantity propoſed is transformed to $\frac{x^2\dot{x}-ax\dot{x}}{a^4\sqrt{a^2-ax+x^2}}$: Whoſe Fluent may be found from a Table of Logarithms; as will appear farther on.

305. *If the Fluxion given is affected by two different Surds, and the rational Factor, or the Quantity without the* Vinculum, *be in a conſtant Ratio to the Fluxion of the Quantity under the* Vinculum *of either Surd, or be related to it as in* Art. 83. *the given Fluxion will be reduced to a more ſimple Form, by ſubſtituting for that Surd.*

Ex. 1. *Let* $\frac{z\dot{z}\sqrt{b^2+z^2}}{\sqrt{c^2-z^2}}$ *be propounded.*

Then, putting $x=\sqrt{b^2+z^2}$, we have $z^2=x^2-b^2$, $z\dot{z}=x\dot{x}$, and $\sqrt{c^2-z^2}=\sqrt{c^2+b^2-x^2}=\sqrt{a^2-x^2}$ (by making $a=\sqrt{c^2+b^2}$) Whence $\frac{z\dot{z}\sqrt{b^2+z^2}}{\sqrt{c^2-z^2}}=$

79. $\frac{x^2\dot{x}}{\sqrt{a^2-x^2}}$*.

Or, if x be put $=\sqrt{c^2-z^2}$ (inſtead of $\sqrt{b^2+z^2}$); then $z^2=c^2-x^2$, $z\dot{z}=-x\dot{x}$, $\sqrt{b^2+z^2}=\sqrt{b^2+c^2-x^2}=\sqrt{a^2-x^2}$; and conſequently $\frac{z\dot{z}\sqrt{bb+zz}}{\sqrt{cc-zz}}=-\dot{x}\sqrt{a^2-x^2}$: Whoſe Fluent is given by *Art.* 297. or 131.

Ex. 2. *Let the given Fluxion be* $\overline{a+cz^n}|^m\times\overline{e+fz^n}|^r\times$

83. $z^{pn-1}\dot{z}$; *ſuppoſing p to denote any whole poſitive Number* †.

In this Caſe, let that of the two Quantities, $a+cz^n$ and $e+fz^n$, whoſe Index (m or r) is the moſt complex (which we will ſuppoſe the latter) be put $=x$; then we ſhall have $z^n=\frac{x-e}{f}$; $z^{n-1}\dot{z}=\frac{\dot{x}}{nf}$;

$z^{pn-1}\dot{z}\ (=z^{pn-n}\times z^{n-1}\dot{z})=\frac{\overline{x-e}|^{p-1}}{f^{p-1}}\times\frac{\dot{x}}{nf}$;

$a+cz^{n}=a+\frac{cx-ce}{f}=d+\frac{cx}{f}$ by putting $d=a-\frac{ce}{f}$) and conſequently $\overline{d+\frac{cx}{f}}|^{m}\times\frac{\overline{x-e}|^{p-1}\times x^{r}\dot{x}}{nf^{p}}$ = the Fluxion propoſed: Where, $p-1$ being a whole poſitive Number, the Value of $\overline{x-e}|^{p-1}$ will therefore be expreſſed in finite Terms; whence, if m be alſo a whole poſitive Number, the Fluent itſelf will be had in finite Terms: But, if m and r be the Halves of odd Numbers, then the Fluent will be found (*from Art.* 298 or 294.) by means of circular Arcs and Logarithms.

306. *If the given Expreſſion be affected by two Surds wherein the Powers of the variable Quantity are the ſame, and the rational Quantity, without the* Vinculums, *be related to the Fluxion of either Surd, as in Art.* 83. *it may be of Uſe to ſubſtitute for the Quotient, or Ratio, of the two Quantities under the radical Signs; eſpecially, if the Sum of the ſaid radical Signs, or Exponents (ſuppoſing both Surds to be reduced to the Denominator) is a whole Number.*

Ex. 1. *Let the given Fluxion be* $\frac{z^{2}\dot{z}}{\overline{b^{3}+z^{3}}|^{\frac{2}{3}}\times\overline{c^{3}-z^{3}}|^{\frac{4}{3}}}$.

Then, writing $x=\frac{b^{3}+z^{3}}{c^{3}-z^{3}}$, we have $z^{3}=\frac{c^{3}x-b^{3}}{1+x}$; $3z^{2}\dot{z}=\frac{\overline{c^{3}+b^{3}}\times\dot{x}}{\overline{1+x}|^{2}}$; $\overline{b^{3}+z^{3}}|^{\frac{2}{3}}\times\overline{c^{3}-z^{3}}|^{\frac{4}{3}}\left(=\overline{\frac{b^{3}+z^{3}}{c^{3}-z^{3}}}|^{\frac{2}{3}}\times\overline{c^{3}-z^{3}}|^{2}=x^{\frac{2}{3}}\times\overline{c^{3}-\frac{c^{3}x-b^{3}}{1+x}}|^{2}\right)=\frac{\overline{b^{3}+c^{3}}|^{2}\times x^{\frac{2}{3}}}{\overline{1+x}|^{2}}$;

and conſequently $\frac{z^2\dot{z}}{\overline{b^3+z^3}|^{\frac{2}{3}}\times\overline{c^3-z^3}|^{\frac{4}{3}}} = \frac{x^{-\frac{2}{3}}\dot{x}}{3\times b^3+c^3}$:

Whoſe Fluent is $\frac{x^{\frac{1}{3}}}{b^3+c^3} = \frac{1}{b^3+c^3}\times\sqrt[3]{\frac{b^3+z^3}{c^3-z^3}}$.

Ex. 3. *Let there be given* $\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}|^m\times\overline{e+fz^n}|^r}$

Here, putting $x = \frac{e+fz^n}{a+cz^n}$, you will have $z^n = \frac{ax-e}{f-cx}$; $nz^{n-1}\dot{z} = \frac{\overline{af-ce}\times\dot{x}}{\overline{f-cx}|^2}$; $z^{pn-1}\dot{z}$ $(= z^{pn-n}\times z^{n-1}\dot{z}) = \overline{\frac{ax-e}{f-cx}}|^{p-1}\times\frac{\overline{af-ce}\times\dot{x}}{n\times\overline{f-cx}|^2}$; $\overline{a+cz^n}|^m\times\overline{e+fz^n}|^r$ $(= \overline{a+cz^n}|^{m+r}\times\overline{\frac{e+fz^n}{a+cz^n}}|^r = \overline{a+c\times\frac{ax-e}{f-cx}}|^{r+m}\times x^r) = \overline{\frac{af-ce}{f-cx}}|^{m+r}\times x^r$; and conſequently the Fluxion given $= \frac{\overline{ax-e}|^{p-1}r\times\overline{f-cx}|^{m+r-p-1}\times x^{-r}\dot{x}}{n\times\overline{af-ce}|^{m+r-1}}$.

Where, if $m+r$ be a whole poſitive Number, greater than p (alſo a whole poſitive Number) the Fluent will be truly had in finite Terms; becauſe both the Serieſes for the Values of $\overline{ax-e}|^{p-1}$ and $\overline{f-cx}|^{m+r-p-1}$ do in that Caſe terminate *. But, if r and $m+r-p-1$ be the Halves of whole Numbers, poſitive or negative, then the Fluent will be given by the laſt Section.

* Art. 99.

307. *A Trinomial is reduced to a Binomial by taking away its middle Term; that is, by ſubſtituting for the Sum or Difference of the Power of the variable Quantity in*

in that Term and half its Coefficient; according as the Signs of the two Terms, where the ſaid Quantity is found, are like, or unlike.

Ex. 1. *Let the given Fluxion be* $\frac{\dot z}{\sqrt{b^2+cz+z^2}}$; then, putting $x = z + \frac{1}{2}c$, or $z = x - \frac{1}{2}c$, we have $\dot z = \dot x$, and $\sqrt{b^2+cz+z^2}\ (= \sqrt{b^2+cx-\frac{1}{2}c^2+x^2-cx+\frac{1}{4}c^2}) = \sqrt{b^2-\frac{1}{4}c^2+x^2}$; whence (making $a^2 = b^2 - \frac{1}{4}c^2$) there reſults $\frac{\dot z}{\sqrt{b^2+cz+z^2}} = \frac{\dot x}{\sqrt{a^2+x^2}}$: Whoſe Fluent is given, by *Art.* 126.

Ex. 2. *Let the Fluxion given be* $\frac{fz^{n-1}\dot z}{\sqrt{a+bz^n+cz^{2n}}}$.

Firſt, by bringing c without the *Vinculum*, according to *Art.* 303. we have $\sqrt{a+bz^n+cz^{2n}} = \sqrt{c} \times \sqrt{\frac{a}{c}+\frac{bz^n}{c}+z^{2n}}$: And, by putting $x = z^n + \frac{b}{2c}$, or $z^n = x - \frac{b}{2c}$, we alſo get $z^{n-1}\dot z = \frac{\dot x}{n}$, and $\sqrt{\frac{a}{c}+\frac{bz^n}{c}+z^{2n}}\ \left(= \sqrt{\frac{a}{c}+\frac{bx}{c}-\frac{bb}{2cc}+x^2-\frac{bx}{c}+\frac{bb}{4cc}}\right) = \sqrt{\frac{a}{c}-\frac{bb}{4cc}+x^2}$: Therefore the Fluxion, transformed, is $\frac{f\dot x}{n\sqrt{c}\times\sqrt{\frac{a}{c}-\frac{bb}{4cc}+x^2}}$: Whoſe Fluent is given by *Art.* 126. when c is a poſitive Quantity: But, when c is negative, the Fluxion muſt be expreſſed thus, $\frac{f\dot x}{n\sqrt{-c}\times\sqrt{\frac{a}{-c}+\frac{bb}{4cc}-x^2}}$: Anſwering to *Form* 2. *Art.* 142.

Ex. 3. *Let* $\frac{fz^{n-1}\dot{z}+gz^{2n-1}\dot{z}+hz^{3n-1}\dot{z}+kz^{4n-1}\dot{z}}{\overline{a+bz^{n}+cz^{2n}}|^{m}}$ *be propoſed.*

Then, following the Steps of the laſt Example, $\overline{a+bz^{n}+cz^{2n}}|^{m}$ $\left(=c^{m}\times\overline{\frac{a}{c}+\frac{bz}{c}+z^{2n}}|^{m}\right)$ will be transformed to $c^{m}\times\overline{\frac{a}{c}-\frac{bb}{4cc}+x^{2}}|^{m}$: Moreover, z^{n} being $=x-\frac{b}{2c}=x-d$ (by putting $d=\frac{b}{2c}$) and $z^{n-1}\dot{z}=\frac{\dot{x}}{n}$, we alſo have $z^{2n-1}\left(=z^{n}\times z^{n-1}\dot{z}=\overline{x-d}\times\frac{\dot{x}}{n}\right)=\frac{x\dot{x}-d\dot{x}}{n}$; $z^{3n-1}\dot{z}$ $(=z^{2n}\times z^{n-1}\dot{z})=\overline{x-d}|^{2}\times\frac{\dot{x}}{n}=\frac{x^{2}\dot{x}-2dx\dot{x}+d^{2}\dot{x}}{n}$; &c. &c. From whence, by ſubſtituting theſe ſeveral Values in the given Fluxion, and putting $\frac{a}{c}-\frac{bb}{4cc}=e^{2}$, there comes out

$$\frac{f\dot{x}+g\times\overline{x\dot{x}-d\dot{x}}+h\times\overline{x^{2}\dot{x}-2dx\dot{x}+d^{2}\dot{x}}+\&c.}{nc^{m}\times\overline{ee+xx}|^{m}}:$$

Whoſe Fluent, when the Exponent m is the Half of any *Integer*, poſitive or negative, will be found, by means of circular Arcs and Logarithms, from *Art.* 295.

308. *When the Denominator is a rational Trinomial, or Multinomial (that is, when it is without a* Vinculum) *the beſt Way of proceeding, for the general Part, is, to reſolve the given Fraction into binomial Ones. In order to this, let its Denominator be feigned* = 0; *by means*

means of which Equation, whoſe Roots muſt be found, you will, by ſubtracting each Root from the indeterminate Quantity (x)*, have the binomial Denominators, of the required Fractions into which the given* One *may be reſolved: Whoſe correſponding Numerators, let be denoted* $A\dot x$, $B\dot x$, $C\dot x$ &c. *then, by putting the Sum of the Fractions, thus ariſing, equal to the given Fraction, and reducing the whole Equation to the ſame Denominator, the aſſumed Quantities* A, B, C *&c. by comparing the homologous Terms, will be determined.*

Ex. 1. *Let the given Fraction be* $\frac{\dot x}{x^2+ax+b}$; then, ſeigning $x^2+ax+b=0$, the two Roots of the Equation will be $-\frac{1}{2}a-\sqrt{\frac{1}{4}a^2-b}$, and $-\frac{1}{2}a+\sqrt{\frac{1}{4}a^2-b}$: Which being denoted by p and q, we have $x-p$ and $x-q$ for the two binomial Factors whereby x^2+ax+b may be reſolved, or by whoſe Multiplication ($\overline{x-p}\times\overline{x-q}$) the ſaid Quantity is produced.

Let therefore $\frac{A\dot x}{x-p}+\frac{B\dot x}{x-q}$ be now aſſumed $\left(=\frac{\dot x}{x^2+ax+b}\right)=\frac{\dot x}{\overline{x-p}\times\overline{x-p}}$; then, by reducing the whole Equation to one Denomination &c. we get $\overline{A+B}\times x\dot x-\overline{qA+pB+1}\times\dot x=0$: Whence A is found $=\frac{1}{p-q}$, $B=\frac{1}{q-p}$; and, conſequently,

$$\frac{\dot x}{\overline{p-q}\times\overline{x-p}}+\frac{\dot x}{\overline{q-p}\times\overline{x-q}}=\frac{\dot x}{x^2+ax+b}.$$

Ex. 2. *Let the Quantity propoſed be* $\frac{x^2\dot x}{x^3+ax^2+bx+c}$.

Here, if the binomial Factors whereby x^3+ax^2+bx+c is produced be repreſented by $x-p$, $x-q$, and $x-r$, and there be aſſumed $\frac{A\dot x}{x-p}+\frac{B\dot x}{x-q}+\frac{C\dot x}{x-r}$

$\left(=\frac{x^2\dot{x}}{x^3+ax^2+bx+c}\right)=\frac{x^2\dot{x}}{\overline{x-p}\times\overline{x-q}\times\overline{x-r}}$; then, in this Caſe, we ſhall have $A\times\overline{x-q}\times\overline{x-r}+B\times\overline{x-p}\times\overline{x-r}+C\times\overline{x-p}\times\overline{x-q}-x^2=0$; that is, by Reduction,

$$\left.\begin{array}{r}A\\B\\C\\-1\end{array}\right\}\times x^2-\left.\begin{array}{l}\overline{q+r}\times A\\\overline{p+r}\times B\\\overline{p+q}\times C\end{array}\right\}\times x+\left.\begin{array}{l}qrA\\prB\\pqC\end{array}\right\}=0.$$

Whence $A+B+C=1$, $A\times\overline{q+r}+B\times\overline{p+r}+C\times\overline{p+q}=0$, and $Aqr+Bpr+Cpq=0$. Now, from the firſt of theſe Equations, multiply'd by $p+q$, ſubtract the ſecond, and you will have $A\times\overline{p-r}+B\times\overline{q-r}=p+q$: Alſo, from the firſt, multiply'd by pq, ſubtract the third; then $A\times\overline{pq-rq}+B\times\overline{pq-pr}=pq$: Laſtly, from the former of the two Equations thus ariſing, multiply'd by p, ſubtract the latter, then $A\times\overline{pp-pr-pq+qr}=pp$, that is, $A\times\overline{p-q}\times\overline{p-r}=p^2$; and conſequently $A=\frac{p^2}{\overline{p-q}\times\overline{p-r}}$; Whence, by the very ſame Argument, $B=\frac{q^2}{\overline{q-p}\times\overline{q-r}}$, and $C=\frac{r^2}{\overline{r-p}\times\overline{r-q}}$.

309. After the ſame Manner you may proceed in other Caſes: But there is an Artifice, or Compendium, for more readily determining the aſſumed Quantities A, B, C &c. by which a great deal of Trouble is avoided: And that is, by conſidering the Equation in ſuch Circumſtances of the indeterminate Quantity x, when it becomes moſt ſimple, or when moſt of its Terms vaniſh.

Thus, in the preceding Example, becauſe $A\times\overline{x-q}\times\overline{x-r}+B\times\overline{x-p}\times\overline{x-r}+C\times\overline{x-p}\times\overline{x-q}-x^2$ is $=0$ (in all Circumſtances of x whatever) let x be taken $=p$; then, all the Terms vaniſhing, except the firſt and laſt, we have $A \times \overline{p-q} \times \overline{p-r} - p^2 = 0$; and conſequently $A = \frac{p^2}{\overline{p-q} \times \overline{p-r}}$; *the very ſame as before.*

More univerſally, let the given Fraction be

$$\frac{x^m\dot{x}}{x^n + ax^{n-1} + bx^{n-2} + cx^{n-3}\ \&c.} = \frac{x^m\dot{x}}{\overline{x-p} \times \overline{x-q} \times \overline{x-r} \times \overline{x-s}\ \&c.}$$

(where m and n may repreſent any whole poſitive Numbers whatever, provided the latter be greater than the former.) Then, aſſuming $\frac{A\dot{x}}{x-p} + \frac{B\dot{x}}{x-q} + \frac{C\dot{x}}{x-r} + \frac{D\dot{x}}{x-s}$ &c. $= \frac{x^m\dot{x}}{x^n + ax^{n-1} + bx^{n-2}\ \&c.}$ &c. we ſhall have $A \times \overline{x-q} \times \overline{x-r} \times \overline{x-s}\ \&c. + B \times \overline{x-p} \times \overline{x-r} \times \overline{x-s}\ \&c. + C \times \overline{x-p} \times \overline{x-q} \times \overline{x-s}\ \&c.\ \&c. - x^m = 0$: From whence, by expounding x by p, q, r &c. ſucceſſively, we obtain $A = \frac{p^m}{\overline{p-q} \cdot \overline{p-r} \cdot \overline{p-s}\ \&c.}$, $B = \frac{q^m}{\overline{q-p} \cdot \overline{q-r} \cdot \overline{q-s}\ \&c.}$, $C = \frac{r^m}{\overline{r-p} \cdot \overline{r-q} \cdot \overline{r-s}\ \&c.}$, &c. Whence the Fractions themſelves, whereof theſe Quantities are the Coefficient, or Numerators, will likewiſe be given.

But the Numerators thus found may, ſometimes, be more commodiouſly expreſſed by Help of the given Coefficients, a, b, c, d &c. ſo as to involve only one of the Roots p, q, r &c. in each Fraction. For, ſince $\overline{x-p} \times \overline{x-q} \times \overline{x-r}$ &c. is ſuppoſed, *univerſally*, $= x^n + ax^{n-1} + bx^{n-2} + cx^{n-3}$ &c. if both Sides of the

Equation be divided by $x-p$, we ſhall have $\overline{x-q} \times \overline{x-r} \times \overline{x-s}$ &c. $= \frac{x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \text{ \&c.}}{x-p}$ Which laſt Expreſſion, when x is $= p$, that is, when both the Numerator and the Denominator become equal to Nothing, will, manifeſtly, be equal to ($\overline{p-q} \times \overline{p-r} \times \overline{p-s}$ &c.) the Diviſor of A. Therefore, if the Fluxion of the Numerator be taken and divided by that of the Denominator, and p be wrote inſtead of x (*vid. Page* 155.) we ſhall have $np^{n-1} + \overline{n-1} \times ap^{n-2} + \overline{n-2} \times bp^{n-3}$ &c. $= \overline{p-q} \times \overline{p-r} \times \overline{p-s}$ &c. and therefore $A \left(= \frac{p^m}{\overline{p-q} \, . \, \overline{p-r} \, . \, \overline{p-s} \text{ \&c.}} \right) =$

$$\frac{p^m}{np^{n-1} + \overline{n-1} \, . \, ap^{n-2} + \overline{n-2} \, . \, bp^{n-3} \text{ \&c.}}.$$

By the very ſame Reaſoning $B =$

$$\frac{q^m}{nq^{n-1} + \overline{n-1} \, . \, aq^{n-2} + \overline{n-2} \, . \, bq^{n-3} \text{ \&c:}} \quad C =$$

$$\frac{r^m}{nr^{n-1} + \overline{n-1} \, . \, ar^{n-2} + \overline{n-2} \, . \, br^{n-3} \text{ \&c.}} \quad \text{\&c.}$$

Hence it appears, that, if the Numerator of the given Fraction be divided by the Fluxion of the Denominator (neglecting $\dot{x}$*) and the ſeveral Roots* p, q, r &c. *(found by feigning the Denominator* $= 0$*) be, ſucceſſively, ſubſtituted in the Quotient, inſtead of* x*; I ſay, it is evident, that the Quantities ſo reſulting, divided by* $x-p$, $x-q$, $x-r$ *&c. will be the required, binomial, Fractions into which the propoſed multinomial One may be reſolved.*

310. If ſome of the Roots p, q, r &c. are impoſſible, which is often the Caſe, the Fractions thus found, where the impoſſible Roots are concerned, muſt

be united in Pairs, and ſo reduced to trinomial Ones, in order to take away the *imaginary Terms.*

Thus, let the Fraction propoſed be $\frac{x\dot{x}}{x^3+ax^2+bx+c}$, and let two of the Roots, p and q, of the Equation $x^3+ax^2+bx+c=0$ be impoſſible: Then, $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}+\frac{C\dot{x}}{x-r}$ being $=\frac{x\dot{x}}{x^3+ax^2+bx+c}$, we ſhall, by uniting the *imaginary Terms,* have $\frac{\overline{A+B}\times x\dot{x}-\overline{Aq+Bp}\times\dot{x}}{x^2-\overline{p+q}\times x+pq}+\frac{C\dot{x}}{x-r}$, alſo, $=\frac{x\dot{x}}{x^3+ax^2+bx+c}$; where the impoſſible Quantities deſtroy one another. But, to render this more obvious, let a be taken $=0$, $b=0$, and $c=-1$, ſo that the given Fraction may become $\frac{x\dot{x}}{x^3-1}$; then the three Roots (p, q, r) of the Equation, $x^3-1=0$, will here be $-\frac{1}{2}+\sqrt{\frac{-3}{4}}$, $-\frac{1}{2}-\sqrt{\frac{-3}{4}}$, and 1; whereof the two former are impoſſible. Moreover, by dividing the Numerator (x) by the Fluxion of the Denominator $(3x^2)$ (according to the *Preſcript*) we have $\frac{1}{3x}$; which, by writing p, q, r ſucceſſively, inſtead of x, becomes $\frac{1}{3p}$, $\frac{1}{3q}$ and $\frac{1}{3r}$ for the Values of A, B, and C, reſpectively. Whence $\frac{\overline{A+B}\times x-Aq-Bp}{x^2-\overline{p+q}\times x+pq}+\frac{C}{x-r}\left(=\frac{x}{x^3-1}\right)$ is $=\frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}+\frac{\frac{1}{3}}{x-1}=\frac{1-x}{3x^2+3x+3}+\frac{1}{3x-3}$. But the ſame may be, otherwiſe,

wiſe, inveſtigated, in a more general Manner; by aſſuming $\frac{Px+Q}{x^2+x+1}+\frac{R}{x-1}=\frac{x}{x^3-1}$, and proceeding as in the firſt and ſecond Examples; whence the very ſame Concluſion will be derived.

If the Fraction propoſed be of this Form, *viz.* $\frac{z^{pn-1}\dot{z}}{z^{mn}+az^{mn-n}+bz^{mn-2n}\ \&c.}$, the Method of Reſolution will, *ſtill*, be the ſame: Since, by putting $x=z^n$, the given Expreſſion is reduced to

$$\frac{\frac{1}{n}\times x^{p-1}\dot{x}}{x^m+ax^{m-1}+bx^{m-2}\ \&c.}$$

It may alſo, be proper to obſerve, *that*, in very complicated Caſes, the Application of two, or more, of the ſix foregoing Rules, may become neceſſary. Thus, for Example, if the Fluxion given be $\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}\big|^m\times\overline{e+fz^n+gz^{2n}}}$; by reſolving $\frac{1}{e+fz^n+gz^{2n}}$ into two Binomial Fractions, $\frac{A}{b+z^n}+\frac{B}{k+z^n}$ (*according to Art.* 308.) we ſhall have $\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}\big|^m\times\overline{e+fz^n+gz^{2n}}}$

$$=\frac{Az^{pn-1}\dot{z}}{\overline{a+cz^n}\big|^m\times\overline{b+z^n}}+\frac{Bz^{pn-1}\dot{z}}{\overline{a+cz^n}\big|^m\times\overline{k+z^n}}:$$ Where, if m be a whole poſitive Number, greater than p, the Fluent will be had in finite Terms (*by Art.* 306. *Ex.* 2.)

SECTION V.

The Inveſtigation of Fluents of Rational Fractions, of ſeveral Dimenſions, according to the Forms *in* Cotes's HARMONIA MENSURARUM.

311. AS the Subject here propoſed is a Matter of conſiderable Difficulty, and has exerciſed the Attention of ſome of the moſt celebrated Mathematicians (who, yet, ſeem to have condeſcended very little to the Information of their leſs experienced Readers) I ſhall endeavour to ſet it in the cleareſt Light poſſible: In order to which, it will be requiſite to premiſe the following Lemmas.

LEMMA I.

If the Sine of the Mean of three equi-different Arcs, ſuppoſing Radius Unity, be multiplied by the Double of the Co-ſine of the common Difference, and from the Product, the Sine of the leſſer Extreme be ſubtracted, the Remainder will be the Sine of the greater Extreme.

LEMMA II.

312. *If G be taken to denote the greater, and L the leſſer, of two unequal Arcs, and their Difference be expreſſed by D; then will,*

1. $\dfrac{\text{Sin. } G \times \text{Co-ſ. } D - \text{Sin. } L \times \text{Rad.}}{\text{Sin. } D} = \text{Co-ſ. } G$

2. $\dfrac{\text{Co-ſ. } L \times \text{Rad.} - \text{Co-ſ. } G \times \text{Co-ſ. } D}{\text{Sin. } D} = \text{Sin. } G$

3. $\dfrac{\text{Sin. } G \times \text{Rad.} - \text{Sin. } L \times \text{Co-ſ. } D}{\text{Sin. } D} = \text{Co-ſ. } L.$

The former of theſe two Lemmas may be met with in moſt Authors upon Trigonometry; and the latter is nothing more than a Corollary to the *common* Theorems for finding the Sine and Co-ſine of the Sum and Difference of two given Arcs; for which Reaſons I ſhall not ſtop here to give their Demonſtration.

COROLLARY.

313. If any Arch of the Circle, whoſe Radius is Unity, be denoted by $\mathcal{Q}$, its Sine by s, and its Co-ſine by a; and there be taken $A = 2a$, $B = 2aA - 1$, $C = 2aB - A$, $= D = 2aC - B$, $E = 2aD - C$, $F = 2aE - D$, &c. it follows (*from Lemma* 1.) that,

$Sin.\ 2\mathcal{Q}\ (Sin.\ \mathcal{Q} \times 2a - Sin.\ 0) = 2sa - 0 = sA$

$Sin.\ 3\mathcal{Q}\ (Sin.\ 2\mathcal{Q} \times 2a - Sin.\ \mathcal{Q}) = 2sAa - s = sB$

$Sin.\ 4\mathcal{Q}\ (Sin.\ 3\mathcal{Q} \times 2a - Sin.\ 2\mathcal{Q}) = 2sBa - sA = sC$

$Sin.\ 5\mathcal{Q}\ (Sin.\ 4\mathcal{Q} \times 2a - Sin.\ 3\mathcal{Q}) = 2sCa - sB = sD$

$Sin.\ 6\mathcal{Q}\ (Sin.\ 5\mathcal{Q} \times 2a - Sin.\ 4\mathcal{Q}) = 2sDa - sC = sE$

&c. &c.

LEMMA III.

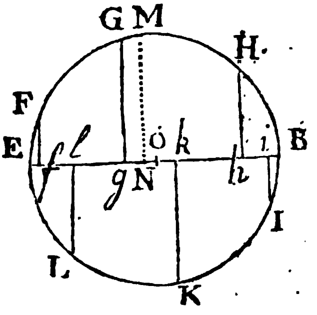
314. *To reſolve the Trinomial* $r^{2n} - 2kr^n x^n + x^{2n}$, *where* n *is any whole Number, into ſimple trinomial Factors.*

Since the firſt Term of the given Quantity $r^{2n} - 2kr^n x^n + x^{2n}$ is diviſible, only, by the Powers of r, and the laſt, only, by thoſe of x; and it appears that r and x are concerned, exactly, alike; let therefore $r^2 - 2arx + x^2$ (where r and x are, *alſo*, alike concerned) be aſſumed for one of the required trinomial Factors, whereby $r^{2n} - 2kr^n x^n + x^{2n}$ may be reſolved: And let $\overline{r^2 - 2arx + x^2} \times \overline{r^8 + Ar^7x + Br^6x^2 + Cr^5x^3 + Dr^4x^4 + Cr^3x^5 + Br^2x^6 + Arx^7 + x^8}$ (where r and x are, *ſtill*, affected alike) be aſſumed $= r \quad - 2kr^5x^5 + x^{10}$ (the Value of n, to render the Operation more perſpicuous, being firſt expreſſed by 5.)

Then, by Multiplication and Tranſpoſition, we ſhall have

$$\left.\begin{array}{l} r^{10}+Ar^{9}x+Br^{8}x^{2}+Cr^{7}x^{3}+Dr^{6}x^{4}+Cr^{5}x^{5}+Br^{4}x^{6}+Ar^{3}x^{7}+r^{2}x^{8}\ldots * \ldots * \\ * -2ar^{9}x-2aAr^{8}x^{2}-2aBr^{7}x^{3}-2aCr^{6}x^{4}-2aDr^{5}x^{5}-2aCr^{4}x^{6}-2aBr^{3}x^{7}-2aAr^{2}x^{8}-2arx^{9}\ldots * \\ * \quad * \quad +r^{8}x^{2}+Ar^{7}x^{3}+Br^{6}x^{4}+Cr^{5}x^{5}+Dr^{4}x^{6}+Cr^{3}x^{7}+Br^{2}x^{8}+Arx^{9}+x^{10} \\ -r^{10}\ldots\ldots\ldots\ldots+2kr^{5}x^{5}\ldots\ldots\ldots\ldots-x^{10} \end{array}\right\}=0$$

Whence, $A=2a$, $B=2Aa-1$, $C=2aB-A$, $D=2aC-B$, and $2C-2aD+2k=0$. But, if Q be taken to denote the Arch (EF) of a Circle EHK, whoſe Radius EO is Unity, and Co-ſine $(Of)=a$; and s be put for (Ff) the Sine of the ſame Arch; then (*by Corol. to Lem.* 1.) $sA=\text{Sin. }2Q$, $sB=\text{Sin. }3Q$, $sC=\text{Sin. }4Q$ *&c.* and conſequently $A=\frac{\text{Sin. }2Q}{s}$, $B=\frac{\text{Sin. }3Q}{s}$, $C=\frac{\text{Sin. }4Q}{s}$, $D=\frac{\text{Sin. }5Q}{s}$ or, $\frac{\text{Sin. }nQ}{s}$). Moreover, becauſe, $2C-2aD+2k=0$, or $D\times a-C\times 1=k$, where (as appears from above) $D\times a-C\times 1=\frac{\text{Sin. }5Q\times\text{Co-ſ. }Q-\text{Sin. }4Q\times\text{Rad.}}{s}=$ *Co-ſ.* $5Q$. (*by Caſe* 1. *Lem.* 2.) we therefore have *Co-ſ.* $5Q$ $(nQ)=k$. Whence this Conſtruction.



Take R to denote the Arch (EM) whoſe Co-ſine (ON) is the given Co-efficient k, and let Q (EF) be taken to EM as 1 to n; then the Co-ſine (Of) of this laſt Arch will be the true Value of a. But this is only one of the Values that a will admit of: For it is well known, that the Co-ſine of any Arch, is alſo the Co-ſine of the ſame Arch increaſed by any Number of Times the whole Periphery (P). Therefore, ſeeing the Co-ſine of nQ (= Co-ſine of R) is likewiſe = Co-ſine $\overline{P+R}$ = Co-ſ. $\overline{2P+R}$ = Coſ. $\overline{3P+R}$ *&c.* it follows that Q (whoſe Co-ſine is a) will be expreſſed by any one of the Arcs, $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$, $\frac{3P+R}{n}$ *&c.* (or by EF, EG, EH, EI,

&c.

&c. ſuppoſing the whole Periphery to be divided ino n equal Parts, from the Point F). Hence, if the Coſines of theſe ſeveral Arcs, expreſſing all the different Values of a, be repreſented by b, c and d, &c. reſpectively, we ſhall have $r^2 - 2brx + x^2$, $r^2 - 2crx + x^2$, $r^2 - 2drx + x^2$, &c. for the ſeveral required Factors; by which $r^{2n} - 2kr^n x^n + x^{2n}$ may be reſolved; and conſequently $\overline{r^2 - 2brx + x^2} \times \overline{r^2 - 2crx + x^2} \times \overline{r^2 - 2drx + x^2}\ (n) = r^{2n} - 2kr^n x^n + x^{2n}$. *Q. E. I.*

Note, If the Sign of the middle Term $2kr^n x^n$ be poſitive, the Diſtance (or Co-ſine) ON muſt be taken on the contrary Side of the Center: But when k is greater than Unity, this Method of Solution fails; ſince no Coſine can be greater than the Radius.

COROLLARY I.

315. If $k = 1$, the Arch R (whoſe Co-ſine is k) being $= 0$, the Values of b, c, d, &c. will be expreſſed by the Co-ſines of the Arcs $\frac{0}{n}$, $\frac{P}{n}$, $\frac{2P}{n}$, $\frac{3P}{n}$ &c. reſpectively: And our general Equation will here become $r^{2n} - 2r^n x^n + x^{2n} = \overline{r^2 - 2brx + x^2} \times \overline{r^2 - 2crx + x^2} \times \overline{r^2 - 2drx + x^2}\ (n)$. From whence, by extracting the Square-Root, on both Sides, we alſo have $r^n \backsim x^n = \overline{r^2 - 2brx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}|^{\frac{1}{2}}\ (n)$.

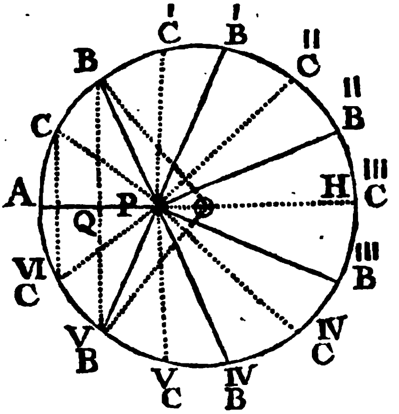
COROLLARY II.

316. But, if $k = -1$ (or the middle Term be $+ 2r^n x^n$) then the Arch R being $= \frac{P}{2}$, the Values of b, c, d, &c. will, *here*, be defined by the Coſines of

the Arcs $\frac{P}{2n}$, $\frac{3P}{2n}$, $\frac{5P}{2n}$, *&c.* and our Equation, by taking the Root, as above, will become $r^n + x^n = \overline{r^2 - 2brx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}|^{\frac{1}{2}}$ (n).

SCHOLIUM.

317. From the two preceding Corollaries, the Demonſtration of that remarkable Property of the Circle given, and applied to finding a vaſt Number of Fluents, in *Cotes's Harmonia Menſurarum*, is very eaſily, and naturally, deduced.



For, let the Periphery of the Circle $AB\overset{\prime}{B}$ *&c.* whoſe Radius is expreſſed by r, be divided into as many equal Parts AB, $B\overset{\prime}{B}$, $\overset{\prime}{B}\overset{\prime\prime}{B}$, *&c.* as there are Units in the given Integer n; ſo that AB, $A\overset{\prime}{B}$, $A\overset{\prime\prime}{B}$, *&c.* may reſpectively exhibit the Values of the foreſaid Arcs $\frac{P}{n}$, $\frac{2P}{n}$, $\frac{3P}{n}$ *&c.* (*vid. Corol.* 1.) Moreover, let OQ be the Co-ſine of the firſt of them; and, in the Radius OA (produced if neceſſary) let there be taken $OP = x$; and let OB, QB, PB, *&c.* *&c.* be drawn: Then, the Co-ſine of the Angle AOB $\left(= \frac{P}{n}\right)$ to the Radius 1, being expreſſed by c (*vid. Corol.* 1.) it will be $1 : c :: r$ (OB) : $OQ = cr$: Whence PB^2 ($= OB^2 + OP^2 - 2OQ \times OP$) $= r^2 + x^2 - 2crx = r^2 - 2crx + x^2$.

By the very ſame Argument $\overset{\prime}{PB}{}^2$ is $= r^2 - 2drx + x^2$, &c. &c. Therefore, becauſe $r^n \backsim x^n = \overline{r^2 - 2brx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2drx + x^2}|^{\frac{1}{2}}$ (n), *by Corol.* 1. it follows that their Equals, $AO^n \backsim OP^n$ and $PA \times PB \times \overset{\prime}{PB} \times \overset{\prime\prime}{PB}$ &c. muſt be equal likewiſe: *Which is the firſt Part of the Theorem above hinted at.*

After the ſame Manner, if the Arcs AC, $\overset{\prime}{AC}$, $\overset{\prime\prime}{AC}$, $\overset{\prime\prime\prime}{AC}$ be taken reſpectively equal to $\frac{P}{2n}$, $\frac{3P}{2n}$, $\frac{5P}{2n}$ &c. it will appear (*from Corol.* 2.) that $AO^n + PO^n$ is $= PC \times \overset{\prime}{PC} \times \overset{\prime\prime}{PC}$ (n) *Which is the latter Part of the ſame Theorem.*

Hence (by the Bye) all the Roots of the Equation $x^n = r^n$ are very readily found: For, ſince $AO^n \backsim PO^n = PA \times PB \times \overset{\prime}{PB}$ &c. where the ſecond Factor and the laſt, the third and the laſt but one, &c. are reſpectively equal to each other, it is evident that $AO^n \backsim PO^n$ ($r^n \backsim x^n$) is alſo $= PA \times PB^2 \times PB^2 \times PB^2 = \overline{r \backsim x} \times \overline{r^2 - 2crx + x^2} \times \overline{r^2 - 2drx + x^2}$ &c. Whence, $x^n \backsim r^n$ being $= 0$, it follows that $\overline{r \backsim x} \times \overline{r^2 - 2crx + x^2}$ &c. is $= 0$: From which, by extracting the Roots out of the Equations $r \backsim x = 0$, $r^2 - 2crx + x^2 = 0$, $r^2 - 2drx + x^2 = 0$ &c. we get r, $r \times \overline{c + \sqrt{c^2 - 1}}$, $r \times \overline{c - \sqrt{c^2 - 1}}$, $r \times \overline{d + \sqrt{d^2 - 1}}$,

&c.

&c. for the ſeveral Roots of the Equation $x^n = r^n$; whereof the firſt, only, is poſſible when n is odd; and the firſt and laſt when n is even.

By the ſame Way of proceeding all the Roots of the Equation, $x^n + r^n = 0$, will alſo be found: For, ſeeing $x^n + r^n$ is $= \overline{r^2 - 2brx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}|^{\frac{1}{2}}$ *&c.* $(= PC \times PC' \times PC''$ *&c.*) where the firſt Factor and the laſt, the ſecond and the laſt but one, *&c.* are reſpectively equal to each other, it is plain that $x^n + r^n$ is likewiſe $= \overline{r^2 - 2brx + x^2} \times \overline{r^2 - 2crx + x^2}$ *&c.* and conſequently $x = r \times \overline{b \pm \sqrt{b^2 - 1}}$ *&c. &c.* Where the Roots are all impoſſible; except the laſt, when their Number (n) is odd.

LEMMA IV.

318. *Suppoſing every thing to remain as in the preceding* Lemma, *and that* k', b', c', d' &c. *denote the Sines of the Arcs* R, $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$ *&c.* (*whoſe Co-ſines are* k, b, c, d, &c.) *then, I ſay, the Fraction*

$$\frac{nk'r^n x^n}{r^{2n} - 2kr^n x^n + x^{2n}} \text{ is equal to } \frac{b'rx}{r^2 - 2brx + x^2} + \frac{c'rx}{r^2 - 2crx + x^2} + \frac{d'rx}{r^2 - 2drx + x^2} \text{ &c.}$$

For, ſince $r^{10} - 2kr^5x^5 + x^{10}$ $(r^{2n} - 2kr^n x^n + x^{2n})$ is $= \overline{r^2 - 2arx + x^2} \times \overline{r^8 + Ar^7x + Br^6x^2 + Cr^5x^3 + Dr^4x^4 + Cr^3x^5 + Br^2x^6 + Arx^7 + x^8}$ (*by the foreſaid* Lemma) and it is alſo proved that $A = \frac{\text{Sin. } 2Q}{s}$, $B = \frac{\text{Sin. } 3Q}{s}$,

$C =$

$C = \frac{Sin.\ 4\mathcal{Q}}{s}$ &c. it is evident, therefore, that $\frac{r^{10}-2kr^5x^5+x^{10}}{r^2-2arx+x^2}$ $(=r^8+Ar^7x+Br^6x^2$ &c.) is $= r^8 + \frac{Sin.\ 2\mathcal{Q}}{s} \times r^7x$ &c. and conſequently $\overline{\frac{s \times r^{10}-2kr^5x^5+x^{10}}{r^2-2arx+x^2}}$ $= Sin.\ \mathcal{Q} \times r^8 + Sin.\ 2\mathcal{Q} \times r^7x + Sin.\ 3\mathcal{Q} \times r^6x^2 + Sin.\ 4\mathcal{Q} \times r^5x^3 + Sin.\ 5\mathcal{Q} \times r^4x^4 + Sin.\ 4\mathcal{Q} \times r^3x^5$ &c. In which Equation, for a and s, let their ſeveral reſpective Values b, c, d, &c. and $\acute{b}$, $\acute{c}$, $\acute{d}$, &c. be, ſucceſſively, ſubſtituted; and let the correſponding Arcs $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$ &c. be repreſented by $\mathcal{Q}$, $\overset{\prime}{\mathcal{Q}}$, $\overset{\prime\prime}{\mathcal{Q}}$, &c. then we ſhall have

$$\acute{b} \times \overline{\frac{r^{10}-2kr^5x^5+x^{10}}{r^2-2brx+x^2}} = Sin.\ \mathcal{Q} \times r^8 + Sin.\ 2\mathcal{Q} \times r^7x \text{ \&c.}$$

$$\acute{c} \times \overline{\frac{r^{10}-2kr^5x^5+x^{10}}{r^2-2crx+x^2}} = Sin.\ \overset{\prime}{\mathcal{Q}} \times r^8 + Sin.\ 2\overset{\prime}{\mathcal{Q}} \times r^7x \text{ \&c.}$$

&c. &c.

Which Equations, added all together, give

$$\overline{r^{10}-2kr^5x^5+x^{10}} \times \overline{\frac{\acute{b}}{r^2-2brx+x^2} + \frac{\acute{c}}{r^2-2crx+x^2} + \frac{\acute{d}}{r^2-2drx+x^2}} \text{ \&c.}$$

$$= \left\{\begin{matrix}\text{Sin. } Q\\ \text{Sin. } Q'\\ \text{Sin. } Q''\\ \text{Sin. } Q'''\\ \text{Sin. } Q''''\end{matrix}\right\} \times r^8 + \left\{\begin{matrix}\text{Sin. } 2Q\\ \text{Sin. } 2Q'\\ \text{Sin. } 2Q''\\ \text{Sin. } 2Q'''\\ \text{Sin. } 2Q''''\end{matrix}\right\} \times r^7 x + \ldots\ldots + \left\{\begin{matrix}\text{Sin. } nQ\\ \text{Sin. } nQ'\\ \text{Sin. } nQ''\\ \text{Sin. } nQ'''\\ \text{Sin. } nQ''''\end{matrix}\right\} \times r^4 x^4 + \text{\&c.}$$

But the Sines of the firſt Column, being thoſe of an arithmetical Progreſſion (whoſe common Difference is $\frac{P}{n}$) by which the whole Periphery is divided into n (5) equal Parts, their Sum will therefore, it is well known, be equal to Nothing; or all the negative ones equal to all the poſitive ones.

The ſame is alſo true with regard to the Sines of the ſecond Column; whoſe Arcs $\frac{2R}{n}$, $\frac{2P+2R}{n}$, $\frac{4P+2R}{n}$ &c. (having $\frac{2P}{n}$ for their common Difference) divide the Periphery (twice taken) into the ſame Number (n) of equal Parts. But the Sines of the middle Column (which is the laſt above exhibited) will not vaniſh, as all the reſt do: For, $n\mathcal{Q}$ being $= R$, $n\overset{\prime}{\mathcal{Q}} = P + R$, $n\overset{\prime\prime}{\mathcal{Q}} = 2P + R$, &c. the common Difference will here be equal to (P) the whole Periphery; and therefore, every Arch terminating in the ſame Point with the firſt, the Circle will, in this Caſe, remain undivided, and the Sine of each be equal to $(\overset{\prime}{k})$ the Sine of the firſt.

Hence, our Equation is reduced to $\overline{r^{10} - 2\overset{\prime}{k}r^5x^5 + x^{10}} \times \overline{\frac{b}{r^2 - 2brx + x^2} + \frac{\overset{\prime}{c}}{r^2 - 2crx + x^2}\ \&c.} = 5\overset{\prime}{k}r^4x^4$; which divided by $r^{10} - 2kr^5x^5 + x^{10}$, and multiplied by rx, gives

$$\frac{\overset{\prime}{b}rx}{r^2 - 2brx + x^2} + \frac{\overset{\prime}{c}rx}{r^2 - 2crx + x^2} + \frac{\overset{\prime}{d}rx}{r^2 - 2drx + x^2}\ \&c. = \frac{5\overset{\prime}{k}r^5x^5}{r^{10} - 2kr^5x^5 + x^{10}} = \frac{nkr^nx^n}{r^{2n} - 2kr^nx^n + x^{2n}}.\quad \mathcal{Q}.E.D.$$

The ſame otherwiſe.

319. Since $r^{2n} - 2kr^nx^n + x^{2n}$ is $= \overline{r^2 - 2brx + x^2} \times \overline{r^2 - 2crx + x^2} \times \overline{r^2 - 2drx + x^2}\ (n)$ by *Lemma* 3. it is evident that, Log. $\overline{r^{2n} - 2kr^nx^n + x^{2n}} =$ Log. $\overline{r^2 - 2brx + x^2}$ + Log. $\overline{r^2 - 2crx + x^2}$ + Log. $\overline{r^2 - 2drx + x^2}\ (n)$. And, as this Equation holds univerſally, let k and x be what they will (which two Quantities may be ſuppoſed to flow independently of each other) let the

Fluxion of the *whole* Equation be taken, making k variable (and x conſtant); which gives $\frac{-2\dot{k}r^n x^n}{r^{2n}-2kr^n x^n+x^{2n}}$ $= \frac{-2\dot{b}rx}{r^2-2brx+x^2} - \frac{2\dot{c}rx}{r^2-2crx+x^2} - \frac{2\dot{d}rx}{r^2-2drx+x^2}$ (n) *. But, k, b, c, d, &c. are the Co-ſines of the Arcs R, $\frac{R}{n}$, $\frac{R+P}{n}$, $\frac{R+2P}{n}$ &c. (whereof the correſponding Sines are $\acute{k}$, $\acute{b}$, $\acute{c}$, &c.) therefore, the Fluxion of the firſt of theſe Arcs being denoted by $\dot{R}$, the Fluxion of each of the reſt will be expreſſed by $\frac{\dot{R}}{n}$: And ſo (the Fluxion of the Co-ſine of an Arch being equal to the Fluxion of the Arch itſelf drawn into its Sine, applied to Radius †) it follows that $\dot{k} = \dot{R}\acute{k}$, $\dot{b} = \frac{\dot{R}}{n} \times \acute{b}$, $\dot{c} = \frac{\dot{R}}{n} \times \acute{c}$, &c. Which Values being ſubſtituted in the foregoing Equation, and the whole divided by $\frac{-2\dot{R}}{n}$, we have $\frac{n\acute{k}r^n x^n}{r^{2n}-2kr^n x^n+x^{2n}} = \frac{\acute{b}rx}{r^2-2brx+x^2} + \frac{\acute{c}rx}{x^2-2crx+x^2} + \frac{\acute{d}rx}{r^2-2drx+x^2}$ (n).

LEMMA V.

320. *To determine the Series, ariſing from the Diviſion of Unity by a Trinomial, $x^2-2arx+r^2$; and to exhibit the Remainder after any given Number (v) of Terms in the Quotient.*

Let $x^{-2}+Arx^{-3}+Br^2x^{-4}+Cr^3x^{-5}+Dr^4x^{-6}$ repreſent the required Quotient continued to 5 Terms

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* Art. 126.

† Art. 142.

(v, to render the Proceſs the more obvious, being firſt expounded by that Number) and let $Er^5x^{-5}+Fr^6x^{-6}$ be the Remainder. Then, becauſe $\frac{1}{x^2-2arx+r^2}$ is $= x^{-2}+Arx^{-3}+Br^2x^{-4}+Cr^3x^{-5}+Dr^4x^{-6}+\frac{Er^5x^{-5}+Fr^6x^{-6}}{x^2-2arx+r^2}$, we ſhall, by reducing the whole Equation to one Denomination, have

$$\left.\begin{array}{l} 1+Arx^{-1}+Br^2x^{-2}+Cr^3x^{-3}+Dr^4x^{-4}+Er^5x^{-5}+Fr^6x^{-6} \\ \ast\ -2arx^{-1}-2aAr^2x^{-2}-2aBr^3x^{-3}-2aCr^4x^{-4}-2aDr^5x^{-5}\ \ast \\ -1\ \ \ast\ \ +r^2x^{-2}+Ar^3x^{-3}+Br^4x^{-4}+Cr^5x^{-5}+Dr^6x^{-6} \end{array}\right\} = 0$$

Whence $A=2a$, $B=2aA-1$, $C=2aB-A$, $D=2aC-B$, $E=2aD-C$, and $F=-D$.

There-

Therefore, if Q be now put for the Arch whoſe Radius is 1 and Co-ſine a, and there be taken $S = \text{Sin.}\ Q$, $\overset{\prime}{S} = \text{Sin.}\ 2Q$, $\overset{\prime\prime}{S} = \text{Sin.}\ 3Q$, &c. we ſhall, alſo, have $A\,(2a) = \frac{\overset{\prime}{S}}{S}$, $B = \frac{\overset{\prime\prime}{S}}{S}$, $C = \frac{\overset{\prime\prime\prime}{S}}{S}$, $D = \frac{\overset{\prime\prime\prime\prime}{S}}{S}$, $E = \frac{\overset{\mathrm{v}}{S}}{S} = \frac{\text{Sin.}\ 6Q}{S}$, $F\ (-D) = -\frac{\text{Sin.}\ 5Q}{S}$ (*by Corol. to Lem.* 1.) And conſequently $\frac{1}{x^2 - 2arx + r^2} =$

$$\frac{Sx^{-2} + \overset{\prime}{S}rx^{-3} + \overset{\prime\prime}{S}r^2x^{-4} + \overset{\prime\prime\prime}{S}r^3x^{-5} + \overset{\prime\prime\prime\prime}{S}r^4x^{-6}}{S} +$$

$$\frac{\text{Sin.}\ 6Q \times r^5x^{-5} - \text{Sin.}\ 5Q \times r^6x^{-6}}{S \times \overline{x^2 - 2arx + r^2}}.$$

Whence, *univerſally*, $\frac{1}{x^2 - 2arx + r^2} =$

$$\frac{Sx^{-2} + \overset{\prime}{S}rx^{-3} + \overset{\prime\prime}{S}r^2x^{-4} + \overset{\prime\prime\prime}{S}r^3x^{-5}\ \text{\&c. (\textit{to v Terms})}}{S} +$$

$$\frac{\text{Sin.}\ \overline{v+1}\,.\,Q \times r^v x^{-v} - \text{Sin.}\ vQ \times r^{v+1} x^{-v-1}}{S \times \overline{x^2 - 2arx + r^2}}.$$

Which laſt Equation (though obvious enough from the preceding one) may be inveſtigated in a general Manner (if required) by aſſuming $x^{-2} + Arx^{-3} + Br^2x^{-4} + Cr^3x^{-5} + \ldots\ldots dr^{v-2}x^{-v} + er^{v-1}x^{-v-1} + \frac{fr^v x^{-v} + gr^{v+1}x^{-v-1}}{x^2 - 2arx + r^2} = \frac{1}{x^2 - 2arx + r^2}$, and proceeding as above: By which Means you will find $A = 2a$, $B = 2aA - 1$, &c. $f = 2ae - d = \frac{\text{Sin.}\ \overline{v+1} \times Q}{S}$, and $g\ (= -e) = -\frac{\text{Sin.}\ vQ}{S}$. And thus may the third

Lemma be made out, if any Objection, or Difficulty, ſhould ariſe about its being general.

COROLLARY.

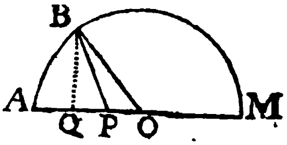
321. If, in the given Trinomial $x^2 - 2arx + r^2$, we ſuppoſe r^2, inſtead of x^2, to be the leading Term whereby the Quotient is produced; then, ſince r and x are affected exactly alike; we ſhall, by writing r for x, and x for r, have $\frac{1}{r^2 - 2axr + x^2} =$

$$\frac{Sr^{-2} + \overset{\prime}{S}xr^{-3} + \overset{\prime\prime}{S}x^2r^{-4}\ (v)}{S} + \frac{\overline{\text{Sin.}\,\overline{v+1} \times \mathcal{Q}} \times x^v r^{-v} - \overline{\text{Sin.}\,v\mathcal{Q}} \times x^{v+1} \times r^{-v-1}}{S \times \overline{r^2 - 2axr + x^2}}.$$

PROB. I.

322. *To find the Fluent of* $\frac{\dot{x}}{rr - 2arx + xx}$, *together with that of* $\frac{x\dot{x}}{rr - 2arx + xx}$.

Let ABM &c. be a Circle whoſe Radius OA (or OM) is r, and let the Angle AOB be ſuch, that its Co-ſine, to the Radius 1, may be equal to a; or ſo, that OQ (ſuppoſing BQ perpendicular to OA) may be $= ar$: Moreover let s denote the Sine of the ſaid Angle AOB, correſponding to the Co-ſine a, and let OP (conſidered as variable by the Motion of P along OA) expreſs the Value of x: Then, PB^2 $(OB^2 + OP^2 - 2OQ \times OP)$ $= rr - 2arx + xx$: And the Fluxion of the Meaſure of the Angle QBP (Radius being Unity) will be repre-ſented



ſented by $\frac{BQ \times Flux.\ QP}{BP^2}$ (*vid. Art.* 142.) or by $\frac{rs \times -\dot{x}}{rr - 2arx + xx}$; and conſequently *that* of *OBP*, by $\frac{rs\dot{x}}{rr - 2arx + xx}$: Whence it is evident that the Fluent of $\frac{\dot{x}}{rr - 2arx + xx}$ (contemporaneous with *x*) is truly expreſſed by $\frac{1}{rs} \times OBP$.

Again, ſince $\frac{x\dot{x}}{rr - 2arx + xx}$ may be transformed to $\frac{-ar\dot{x} + x\dot{x}}{rr - 2arx + xx} + \frac{ar\dot{x}}{rr - 2arx + xx}$; where the Fluent of the former Part is $= \frac{1}{2}$ hyp. Log. $\frac{rr - 2arx + xx}{rr}$ * $= \frac{1}{2}$ *hyp. Log.* $\frac{PB^2}{OB^2} =$ *hyp. Log.* $\frac{PB}{OB}$; and that of the latter Part $= \frac{a}{s} \times OBP$; it appears that the Fluent of $\frac{x\dot{x}}{rr - 2arx + xx}$ is truly expounded by *hyp. Log.* $\frac{PB}{OB}$, $+ \frac{a}{s} \times OBP$. *Q. E. I.*

* Art. 126.

COROLLARY.

323. Since, *PB* : *PO* :: *Sin. BOP* (*s*) : *Sin. OBP* $= \frac{sx}{\sqrt{rr - 2arx + xx}}$; it follows, if the hyperbolical Logarithm of $\frac{\sqrt{r^2 - 2arx + xx}}{r}$, be repreſented by *M*, and the Arch, whoſe Sine is $\frac{sx}{\sqrt{rr - 2arx + xx}}$ and Radius

Unity, by N, that the Fluents of $\frac{\dot{x}}{rr - 2arx + xx}$ and $\frac{x\dot{x}}{rr - 2arx + xx}$ will be expreſſed by $\frac{N}{rs}$ and $M + \frac{aN}{s}$ reſpectively.

PROB. II.

324. *To determine the Fluent of* $\frac{x^{m}\dot{x}}{x^{2} - 2arx + r^{2}}$; *ſuppoſing* m *any whole poſitive Number, and* a *leſs than Unity.*

Let every thing remain as in *Lemma* 5. and then, if the Equation there brought out be multiplied by $x^{m}\dot{x}$, and v at the ſame time be expounded by $m - 1$, we ſhall get

$$\frac{x^{m}\dot{x}}{x^{2} - 2arx + r^{2}} = \frac{Sx^{m-2}\dot{x} + \overset{\prime}{S}rx^{m-3}\dot{x} + \overset{\prime\prime}{S}r^{2}x^{m-4}\dot{x}}{S}$$

$$(m - 1) + \frac{\overline{\text{Sin.}\ mQ} \times r^{m-1}x\dot{x} - \overline{\text{Sin.}\ \overline{m-1} \times Q} \times r^{m}\dot{x}}{S \times \overline{xx - 2arx + rr}}$$

Whoſe Fluent will therefore be given by the preceding Propoſition: For, ſuppoſing the Values of M and N to be as there ſpecified, the Fluent of the laſt Term $\left(\frac{\overline{\text{Sin.}\ mQ} \times r^{m-1}x\dot{x} - \overline{\text{Sin.}\ \overline{m-1} \times Q} \times r^{m}\dot{x}}{S \times \overline{xx - 2arx + rr}}\right)$ will, it is manifeſt [*], be expreſſed by $\frac{1}{S}$ *into* $\overline{\text{Sin.}\ mQ} \times r^{m-1} \times \overline{M + \frac{aN}{S}} - \overline{\text{Sin.}\ \overline{m-1} \times Q} \times r^{m} \times \frac{N}{rS} = \frac{r^{m-1}}{S}$ into $\overline{\text{Sin.}\ mQ} \times M + \frac{\overline{\text{Sin.}\ mQ} \times a - \overline{\text{Sin.}\ \overline{m-1} \times Q}}{S} \times N$ $= \frac{r^{m-1}}{S}$ into $\overline{\text{Sin.}\ mQ} \times M + \overline{\text{Co-ſ.}\ mQ} \times N$ (*by Lem.* 2.

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[*] Art. 323.

Caſe 1.) To which adding the Fluent of the preceding Series, there reſults $\frac{1}{S}\times\overline{\frac{Sx^{m-1}}{m-1}+\frac{\overset{\prime}{S}rx^{m-2}}{m-2}+\frac{\overset{\prime\prime}{S}r^{2}x^{m-3}}{m-3}\,(m-1)}$ $+\frac{r^{m-1}}{S}\times\overline{\overline{\text{Sin. }m\mathcal{Q}}\times M+\overline{\text{Co-ſ. }m\mathcal{Q}}\times N}$. *Q. E. I.*

COROLLARY.

325. Hence, the Fluent of $\frac{-ax^{m}\dot{x}+rx^{m-1}\dot{x}}{xx-2arx+rr}$ may be deduced: For, by writing $m-1$, inſtead of m, the Fluent of $\frac{x^{m-1}\dot{x}}{xx-2arx+rr}$ will be found $=\frac{1}{S}\times$ $\overline{\frac{Sx^{m-2}}{m-2}+\frac{\overset{\prime}{S}rx^{m-3}}{m-3}+\frac{\overset{\prime\prime}{S}r^{2}x^{m-4}}{m-4}\,(m-2)}+\frac{r^{m-2}}{S}\times$ $\overline{\text{Sin. }\overline{m-1}\times\mathcal{Q}\times M+\text{Co-ſ. }\overline{m-1}\times\mathcal{Q}\times N}$: Which Fluent being multiplied by r, and *that* of $\frac{x^{m}\dot{x}}{xx-2arx+rr}$ (given above) by $-a$, we ſhall, when the homologous Terms are united, have $\frac{1}{S}\times\overline{-aS\times\frac{x^{m-1}}{m-1}-\overline{a\overset{\prime}{S}-S}\times\frac{rx^{m-2}}{m-2}-\overline{a\overset{\prime\prime}{S}-\overset{\prime}{S}}\times\frac{r^{2}x^{m-3}}{m-3}\,(m-1)}+\frac{r^{m-1}}{S}$ into $-\overline{\overline{\text{Sin. }m\mathcal{Q}\times a-\text{Sin. }\overline{m-1}\times\mathcal{Q}}\times M-\overline{\text{Co-ſ. }m\mathcal{Q}\times a-\text{Co-ſ. }\overline{m-1}\times\mathcal{Q}}\times N}$, for the true Fluent of the Quantity propounded.

But (*by Caſe* 1. *Lem.* 2.) $\frac{a\overset{\prime}{S}-S}{S}$ ($=\frac{\text{Sin. }2\mathcal{Q}\times a-\text{Sin. }\mathcal{Q}\times\text{Rad.}}{\text{Sin. }\mathcal{Q}}$) $=$ Co-ſ. $2\mathcal{Q}$: Alſo

$$\frac{a\overset{\prime\prime}{S}-\overset{\prime}{S}}{S}\left(\frac{\text{Sin. }3\mathcal{Q}\times a-\text{Sin. }2\mathcal{Q}\times\text{Rad.}}{S}\right)=\text{Co-ſ. }3\mathcal{Q}$$

&c. And, by *Caſe* 2. of the ſame *Lemma*,

$$\frac{\text{Co-ſ. }\overline{m-1}\times\mathcal{Q}-\overline{\text{Co-ſ. }m\mathcal{Q}}\times a}{S}=\text{Sin. }m\mathcal{Q}:$$

Whence, by ſubſtituting theſe Values, our Fluent is reduced to

$$-\text{Co-ſ. }\mathcal{Q}\times\frac{x^{m-1}}{m-1}-\text{Co-ſ. }2\mathcal{Q}\times\frac{rx^{m-2}}{m-2}-\text{Co-ſ. }3\mathcal{Q}\times\frac{r^2x^{m-3}}{m-3}-\text{Co-ſ. }4\mathcal{Q}\times\frac{r^3x^{m-4}}{m-4}\;(m-1)-r^{m-1}\times\overline{\overline{\text{Co-ſ. }m\mathcal{Q}}\times M-\overline{\text{Sin. }M\mathcal{Q}}\times N}.$$

PROB. III.

326. *To determine the Fluent of* $\frac{x^{-m}\dot{x}}{r^2-2arx+x^2}$; *under the Reſtrictions ſpecified in the preceding Problem.*

If the Equation in *Art.* 321. be multiply'd by $x^{-m}\dot{x}$, and v at the ſame time be expounded by m, we ſhall have

$$\frac{x^{-m}\dot{x}}{r^2-2arx+x^2}=\frac{Sr^{-2}x^{-m}\dot{x}+\overset{\prime}{S}r^{-3}x^{1-m}\dot{x}+\overset{\prime\prime}{S}r^{-4}x^{2-m}\dot{x}}{S}\;(m)+\frac{r^{-m-2}}{S}\times\frac{\overline{\text{Sin. }\overline{m+1}\times\mathcal{Q}}\times r\dot{x}-\overline{\text{Sin. }m\mathcal{Q}}\times x\dot{x}}{r^2-2arx+x^2}:$$

Where, the Fluent of the laſt Term being

$$\frac{r^{-m-1}}{S}\times\overline{\text{Sin. }\overline{m+1\times\mathcal{Q}}\times\frac{N}{S}-\overline{\text{Sin. }m\mathcal{Q}}\times\overline{M+\frac{aN}{S}}}\,{}^{*}=$$ * Art. 323.

$$\frac{r^{-m-1}}{S}\text{ into }-\overline{\text{Sin. }m\mathcal{Q}}\times M+$$

$$\frac{\text{Sin.}\,\overline{m+1\times Q} - \text{Sin.}\,mQ \times a}{S} \times N = \frac{r^{-m-1}}{S} \times$$

$$\overline{-\,\overline{\text{Sin.}\,mQ} \times M + \overline{\text{Co-ſ.}\,mQ} \times N}$$ *(by Caſe 3. Lem. 2.)* it follows that the Fluent of the whole Expreſſion, or the Quantity ſought, will be truly expreſſed by

$$\frac{1}{S} \times \frac{Sr^{-2}x^{1-m}}{1-m} + \frac{\overset{\prime}{S}r^{-3}x^{2-m}}{2-m} + \&c.$$ or its Equal

$$\frac{-1}{S} \times \frac{Sx^{1-m}}{m-1.r^{2}} + \frac{\overset{\prime}{S}x^{2-m}}{m-2.r^{3}} + \frac{\overset{\prime\prime}{S}x^{3-m}}{m-3.r^{4}}\ (m) +$$

$$\frac{1}{Sr^{m+1}} \times \overline{\overline{\text{Co-ſin.}\,mQ} \times N - \overline{\text{Sin.}\,mQ} \times M}.$$

PROB IV.

327. *To find the Fluent of* $\frac{x^{m-1}\dot{x}}{r^{n}+x^{n}}$; *m and n being any whole poſitive Numbers, whereof the former does not exceed the latter.*

Let *b*, *c*, *d*, *&c.* denote the Co-ſines of the Arcs $\frac{360^\circ}{2n}$, $\frac{3\times 360^\circ}{2n}$, $\frac{5\times 360^\circ}{2n}$, *&c.* (Radius being Unity) Then *(by Corol. 2. Lem. 3.)* we ſhall have $r^{n} + x^{n} = \overline{rr-2brx+xx}|^{\frac{1}{2}} \times \overline{rr-2crx+xx}|^{\frac{1}{2}} \times \overline{rr-2drx+xx}|^{\frac{1}{2}}$ *(n)*. Whence *Log.* $\overline{r^{n}+x^{n}} = \frac{1}{2}$ *Log.* $\overline{rr-2brx+xx} + \frac{1}{2}$ *Log.* $\overline{rr-2crx+xx} + \frac{1}{2}$ *Log.* $\overline{rr-2drx+xx}$ *(n)* and, conſequently, by taking the Fluxion, on both Sides, $\frac{nx^{n-1}\dot{x}}{r^{n}+x^{n}} = \frac{x\dot{x}-br\dot{x}}{xx-2brx+rr} + \frac{x\dot{x}-cr\dot{x}}{xx-2crx+rr} + \frac{x\dot{x}-dr\dot{x}}{xx-2drx+rr}$ * *(n)*; which laſt Equation, multiply'd by $\frac{x}{\dot{x}}$, gives $\frac{nx^{n}}{r^{n}+x^{n}} = \frac{xx-brx}{xx-2brx+rr} + \frac{xx-crx}{xx-2crx+rr}$

* Art. 126.

$+ \frac{xx - drx}{xx - 2drx + rr}$ (n). Let each Side hereof be now ſubtracted from n (or, which comes to the ſame thing, let $\frac{nx^n}{r^n + x^n}$ be taken from n, and each of the (n) Terms on the other Side, from Unity) then we ſhall have $\frac{nr^n}{r^n + x^n} = \frac{-brx + rr}{xx - 2brx + rr} + \frac{-crx + rr}{xx - 2crx + rr}$ $+ \frac{-drx + rr}{xx - 2drx + rr}$ (n): Which multiply'd by $\frac{x^{m-1}\dot{x}}{r}$, gives $\frac{nr^{n-1} \times x^{m-1}\dot{x}}{r^n + x^n} = \frac{-bx^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2brx + rr} +$ $\frac{-cx^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2crx + rr}$ *&c.*

But now, to determine the Fluent hereof, let the ſeveral Arcs $\left(\frac{180^\circ}{n}, \frac{3 \times 180^\circ}{n}, \frac{5 \times 180^\circ}{n}\right.$ *&c.*) above ſpecified, be denoted by $\mathcal{Q}, \overset{\prime}{\mathcal{Q}}, \overset{\prime\prime}{\mathcal{Q}}, \overset{\prime\prime\prime}{\mathcal{Q}}$, *&c.* reſpectively; alſo let $N, \overset{\prime}{N}, \overset{\prime\prime}{N}$, *&c.* expreſs the Meaſures of the Angles whoſe Sines are $\frac{x \times Sin.\, \mathcal{Q}}{\sqrt{rr - 2brx + xx}}$, $\frac{x \times Sin.\, \overset{\prime}{\mathcal{Q}}}{\sqrt{xx - 2crx + rr}}$, $\frac{x \times Sin.\, \overset{\prime\prime}{\mathcal{Q}}}{\sqrt{xx - 2drx + rr}}$ *&c.* and $M, \overset{\prime}{M}, \overset{\prime\prime}{M}$, *&c.* the hyperbolic Logarithms of $\frac{\sqrt{xx - 2brx + rr}}{r}$, $\frac{\sqrt{xx - 2crx + rr}}{r}$, $\frac{\sqrt{xx - 2drx + rr}}{r}$ *&c.* Then (*by Corol. to Prob.* 2.) the Fluent of the firſt Term, $\frac{-bx^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2brx + rr}$ (expounding a by b) comes out

$$-\text{Co-ſ.}\,Q\times\frac{x^{m-1}}{m-1}-\text{Co-ſ.}\,2Q\times\frac{rx^{m-2}}{m-2}-\text{Co-ſ.}\,3Q\times\frac{r^2x^{m-3}}{m-3}\ (m-1)+r^{m-1}\text{ into }\overline{\text{Sin. }mQ\times N-\text{Co-ſ. }mQ\times M}.$$

In the ſame Manner, by writing c for a, $\acute{Q}$ for Q, $\acute{M}$ for M, and $\acute{N}$ for N) the Fluent of the ſecond Term, $\frac{-cx^m\dot{x}+rx^{m-1}\dot{x}}{xx-2crx+rr}$, is found $=-\text{Co-ſ.}\,\acute{Q}\times\frac{x^{m-1}}{m-1}$ $-\text{Co-ſ.}\,2\acute{Q}\times\frac{rx^{m-2}}{m-2}$ &c. &c.

Therefore the Fluent of the whole Expreſſion, by collecting the homologous Terms, appears to be

$$-\left\{\begin{matrix}\text{Co-ſ. }Q\\ \text{Co-ſ. }Q'\\ \text{Co-ſ. }Q''\\ \text{Co-ſ. }Q'''\\ \text{\&c.}\end{matrix}\right\}\times\frac{x^{m-1}}{m-1}-\left\{\begin{matrix}\text{Co-ſ. }2Q\\ \text{Co-ſ. }2Q'\\ \text{Co-ſ. }2Q''\\ \text{Co-ſ. }2Q'''\\ \text{\&c.}\end{matrix}\right\}\times\frac{rx^{m-2}}{m-2}-\left\{\begin{matrix}\text{Co-ſ. }3Q\\ \text{Co-ſ. }3Q'\\ \text{Co-ſ. }3Q''\\ \text{Co-ſ. }3Q'''\\ \text{\&c.}\end{matrix}\right\}\times\frac{r^2x^{m-3}}{m-3}\ \text{\&c.}$$

$$+\, r^{n-1}\times\left\{\begin{array}{l} \overline{\text{Sin.}\, m\mathcal{Q}}\times N-\overline{\text{Co-ſ.}\, m\mathcal{Q}}\times M \\ \overline{\text{Sin.}\, m\overset{\prime}{\mathcal{Q}}}\times \overset{\prime}{N}-\overline{\text{Co-ſ.}\, m\overset{\prime}{\mathcal{Q}}}\times \overset{\prime}{M} \\ \overline{\text{Sin.}\, m\overset{\prime\prime}{\mathcal{Q}}}\times \overset{\prime\prime}{N}-\overline{\text{Co-ſ.}\, m\overset{\prime\prime}{\mathcal{Q}}}\times \overset{\prime\prime}{M} \\ \overline{\text{Sin.}\, m\overset{\prime\prime\prime}{\mathcal{Q}}}\times \overset{\prime\prime\prime}{N}-\overline{\text{Co-ſ.}\, m\overset{\prime\prime\prime}{\mathcal{Q}}}\times \overset{\prime\prime\prime}{M} \\ \quad\&c. \qquad\qquad\qquad \&c. \end{array}\right.$$

But the Co-ſines of the firſt Column being thoſe of an arithmetical Progreſſion $\left(\frac{180^\circ}{n}, \frac{3\times 180^\circ}{n}, \frac{5\times 180^\circ}{n}\right.$ &c.) whoſe common Difference is $\frac{360^\circ}{n}$, whereby the whole Periphery is divided into n equal Parts (*vid. Art.* 317.) they will therefore deſtroy one another; ſince it is well known that, if the Periphery of any Circle be divided into any Number (n) of equal Parts, the negative Sines and Co-ſines will be equal to the poſitive ones; which is ſelf-evident when their Number is even.

Hence the Co-ſines in the ſecond and third Columns, &c. will alſo deſtroy one another (*vid. Art.* 318.) But *thoſe* of the laſt Column of all, as well as the Sines, having unequal Multiplicators, muſt remain as above, and that Column, *alone*, (drawn into r^{n-1}) will be the true Fluent of $\frac{nr^{n-1}\times x^{m-1}\dot{x}}{r^n+x^n}$. Whence, putting $m\mathcal{Q}$ $\left(= m\times\frac{180^\circ}{n}\right) = R$, and dividing by nr^{n-1}, we ſhall (becauſe $\overset{\prime}{\mathcal{Q}} = 3\mathcal{Q}$, $\overset{\prime\prime}{\mathcal{Q}} = 5\mathcal{Q}$, $\overset{\prime\prime\prime}{\mathcal{Q}} = 7\mathcal{Q}$ &c.) have

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} \overline{\text{Sin. } R} \times N - \overline{\text{Co-ſ. } R} \times M \\ \overline{\text{Sin. } 3R} \times N' - \overline{\text{Co-ſ. } 3R} \times M' \\ \overline{\text{Sin. } 5R} \times N'' - \overline{\text{Co-ſ. } 5R} \times M'' \\ \overline{\text{Sin. } 7R} \times N''' - \overline{\text{Co-ſ. } 7R} \times M''' \\ \quad \text{\&c. (to } n \text{ Lines.)} \end{array} \right\} = \text{Fluent of } \frac{x^{m-1}\dot{x}}{r^n + x^n}$$

Q. E. I.

COROLLARY.

328. Since the firſt and the laſt, the ſecond and the laſt but one, &c. of the foregoing Quantities $x^2 - 2brx + rr$, $xx - 2crx + rr$, $xx - 2drx + rr$ &c. are reſpectively equal to each other (*vid. Art.* 317.) the correſponding Fluents, found above, will likewiſe be equal: And therefore the Fluent of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ will, *alſo*, be expreſſed by

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} \overline{\text{Sin. } R} \times 2N - \overline{\text{Co-ſ. } R} \times 2M \\ \overline{\text{Sin. } 3R} \times 2N' - \overline{\text{Co-ſ. } 3R} \times 2M' \\ \overline{\text{Sin. } 5R} \times 2N'' - \overline{\text{Co-ſ. } 5R} \times 2M'' \\ \qquad \text{\&c.} \qquad\qquad\qquad \text{\&c.} \end{array} \right.$$

The Number of Lines to be thus taken being $= \frac{1}{2} n$, when n is even; but, otherwiſe, $= \frac{n+1}{2}$; in which laſt Caſe, the Logarithm, &c. in the laſt Line, muſt be taken only once, inſtead of twice; being that of $\frac{r+x}{r}$ (*vid. Art.* 317.)

PROB. V.

329. *To find the Fluent of* $\frac{x^{m-1}\dot{x}}{r^n - x^n}$; m *and* n *being as in the preceding Problem.*

If *b*, *c*, *d*, &c. be taken to denote the Co-ſines of the Arcs $\frac{0}{n}$, $\frac{360^\circ}{n}$, $\frac{2 \times 360^\circ}{n}$ &c. to *n* Terms, it will appear (*from Corol.* 1. *to Lem.* 3.) that $r^n - x^n$ is $= \overline{rr - 2brx + xx}|^{\frac{1}{2}} \times \overline{rr - 2crx + xx}|^{\frac{1}{2}} \times \overline{rr - 2drx + xx}|^{\frac{1}{2}}$ (*a*). From whence, by following the Method of the laſt Problem, we alſo have $\frac{nr^{n-1} \times x^{m-1}\dot{x}}{r^n - x^n} =$ $\frac{-bx^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2brx + rr} + \frac{-cx^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2crx + rr}$ &c.
Which Fluxion having exactly the ſame *Form* with *that* in the preceding Problem, its Fluent will alſo be expreſſed in the very ſame Manner; that is, by

$$r^{m-1} \times \begin{cases} \text{Sin. } m\mathcal{Q} \times N - \text{Co-ſ. } m\mathcal{Q} \times M \\ \text{Sin. } m\acute{\mathcal{Q}} \times \acute{N} - \text{Co-ſ. } m\acute{\mathcal{Q}} \times \acute{M} \\ \text{Sin. } m\acute{\acute{\mathcal{Q}}} \times \acute{\acute{N}} - \text{Co-ſ. } m\acute{\acute{\mathcal{Q}}} \times \acute{\acute{M}} \\ \text{(\&c. to } n \text{ Lines.)} \end{cases}$$

Only $\mathcal{Q}$, $\acute{\mathcal{Q}}$, $\acute{\acute{\mathcal{Q}}}$ &c. muſt here ſtand for $\frac{0}{n}$, $\frac{360^\circ}{n}$, $\frac{2 \times 360^\circ}{n}$, $\frac{3 \times 360^\circ}{n}$ &c. (inſtead of $\frac{180^\circ}{n}$, $\frac{3 \times 180^\circ}{n}$, $\frac{5 \times 180^\circ}{n}$ &c.)

Therefore, ſince the multiple Arcs $m\mathcal{Q}$, $m\acute{\mathcal{Q}}$, $m\acute{\acute{\mathcal{Q}}}$ &c. are, in this Caſe, equal to 0, $m \times \frac{360^\circ}{n}$, $2m \times \frac{360^\circ}{n}$, $3m \times \frac{360^\circ}{n}$ &c. (whereof the Sine of the firſt is $= 0$, and its Co-ſine = Unity) we ſhall, by putting $R = m \times \frac{360^\circ}{n}$, and dividing the foreſaid Fluent by nr^{n-1}, have

$$\frac{r^{m-n}}{n} \times \left\{\begin{array}{l} * \ldots\ldots - \ldots\ldots M \\ \mathit{Sin.}\ R \times \acute{N} - \mathit{Co\text{-}ſ.}\ R \times \acute{M} \\ \mathit{Sin.} 2R \times \acute{\acute{N}} - \mathit{Co\text{-}ſ.} 2R \times \acute{\acute{M}} \\ \mathit{Sin.} 3R \times \acute{\acute{\acute{N}}} - \mathit{Co\text{-}ſ.} 3R \times \acute{\acute{\acute{M}}} \\ \quad (\textit{\&c. to n Lines.}) \end{array}\right\} = \text{Fluent of } \frac{x^{m-1}\dot{x}}{r^n - x^n}$$

Q. E. I.

COROLLARY.

330. Since, in the Fluent here given, the ſecond Line and the laſt, the third and the laſt but one, &c. are reſpectively equal (*vid. Art.* 317.) the ſame may alſo be exhibited, thus;

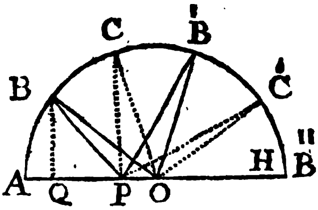
$$\frac{r^{m-n}}{n} \times \left\{\begin{array}{l} * \ldots\ldots - \ldots\ldots M \\ \mathit{Sin.}\ R \times 2\acute{N} - \mathit{Co\text{-}ſ.}\ R \times 2\acute{M} \\ \mathit{Sin.}\ 2R \times 2\acute{\acute{N}} - \mathit{Co\text{-}ſ.}\ 2R \times 2\acute{\acute{M}} \end{array}\right.$$

(&c. to $\frac{n+1}{2}$ Lines.)

SCHOLIUM.

331. If the Semi-Periphery ABCH of the Circle whoſe Diameter AH is $2r$, be divided into as many

equal Parts AB, BC, CB', $B'C'$ &c. as there are Units in n (ſo that $AB = \frac{180^\circ}{n} = \mathcal{Q}$, $AB' = 3 \times \frac{180^\circ}{n} = \mathcal{Q}'$ &c. *vid. Art.* 317. and 327.) and in the Radius OA (produced, if neceſſary) there be taken OP$=x$, and PB, OB &c. be drawn, it will appear (from the ſaid Articles, and from *Prop.* 1.) that the Quantities $\sqrt{r^2-2brx+xx}$, $\sqrt{r^2-2crx+x^2}$ &c. in the former of the two preceding Problems, will here be expounded by PB, PB' &c. reſpectively: From whence it is alſo plain, that the Meaſures N, N' &c. of the Angles whoſe Sines are $\frac{x \times Sin.\, \mathcal{Q}}{\sqrt{r^2-2brx+x^2}}$, $\frac{x \times Sin.\, \mathcal{Q}'}{\sqrt{r^2-2crx+x^2}}$ &c. * will here be expounded by OBP, $OB'P$, &c. &c.



Therefore the Fluent of $\frac{x^{m-1}\dot{x}}{r^n+x^n}$, given in the Corollary to the foreſaid Propoſition, may be thus exhibited,

$$\frac{r^{m-n}}{n} \times \begin{cases} \overline{Sin.\ R} \times 2\,(OBP) - \overline{Co\text{-}ſ.\ R} \times 2\,(OA:PB) \\ \overline{Sin.\, 3R} \times 2\,(OB'P) - \overline{Co\text{-}ſ.\ R} \times 2\,(OA:PB') \\ \quad \&c. \qquad\qquad\qquad\qquad \&c. \end{cases}$$

Where the Arch R is $\left(= m \times \frac{180^\circ}{n}\right) = m \times AB$, and where $(OA:PB)$ is put (after the Manner of *Cotes*) to expreſs the hyperbolical Logarithm of $\frac{PB}{OA}$. It is alſo to be obſerved, *that*, when the laſt of the Points B,

* Art. 322, and 323.

$\overset{\prime}{B}$, $\overset{\prime\prime}{B}$ *&c.* falls upon *H* (which will always happen when *n* is an odd Number) the Angle, in the laſt Line of the Fluent, will vaniſh, and the correſponding Logarithm (which is that of $\frac{PH}{AO}$) muſt then be taken, inſtead of twice, only once.

In the very ſame Manner it will appear, *that*, the Arcs $\overset{\prime}{Q}$, $\overset{\prime\prime}{Q}$ *&c.* in the ſecond Caſe, where the Fluent of $\frac{x^{m-1}\dot{x}}{r^n - x^n}$ is ſought, will be, reſpectively, expounded by *AC*, $A\overset{\prime}{C}$ *&c.* alſo the correſponding Angles $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$ *&c.* by *OCP*, $O\overset{\prime}{C}P$ *&c.* and the Fluent itſelf by

$$\frac{r^{m-n}}{n}\times\begin{cases} * \;\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\; (OA:PC)\\ \overline{\text{Sin. } R}\times 2\,(OCP) - \overline{\text{Co-ſ. } R}\times 2\,(OA:PC)\\ \overline{\text{Sin.}2R}\times 2\,(O\overset{\prime}{C}P) - \overline{\text{Co-ſ.}2R}\times 2\,(OA:P\overset{\prime}{C})\\ \text{\&c.} \qquad\qquad\qquad\qquad \text{\&c.}\end{cases}$$

Where the Arch $R\ (= m\times\frac{360^\circ}{n}) = m\times AC$; and where, as well as in the preceding Caſe, all the Arcs, Sines and Co-ſines are ſuppoſed to have Unity for their Radius.

332. From the Fluents of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ and $\frac{x^{m-1}\dot{x}}{r^n - x^n}$, thus given, *thoſe* of $\frac{x^{vn+m-1}\dot{x}}{r^n + x^n}$, $\frac{x^{-vn+m-1}\dot{x}}{r^n + x^n}$, $\frac{x^{vn+m-1}\dot{x}}{r^n - x^n}$, and $\frac{x^{-vn+m-1}\dot{x}}{r^n - x^n}$, where *v* denotes any whole Number, may be very eaſily deduced; either from *Art.* 283. and 291. or (more readily) by dividing the Numerator by the Denominator, and continuing the

Quotient to as many Terms as there are Units in v*. By which means, if p be put $= vn + m$, $q = vn - m$, and the Fluents of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ and $\frac{x^{m-1}\dot{x}}{r^n - x^n}$ be denoted by V and W reſpectively, the Fluents, in the four Caſes ſpecified above, will be expreſſed by

$$\frac{x^{p-n}}{p-n} - \frac{r^n x^{p-2n}}{p-2n} + \frac{r^{2n} x^{p-3n}}{p-3n} (v) \pm r^{vn} V,$$

$$\frac{x^{-q}}{-qr^n} - \frac{x^{n-q}}{\overline{n-q}.r^{2n}} + \frac{x^{2n-q}}{\overline{2n-q}.r^{3n}} (v) \pm \frac{V}{r^{vn}},$$

$$-\frac{x^{p-n}}{p-n} - \frac{r^n x^{p-2n}}{p-2n} - \frac{r^{2n} x^{p-3n}}{p-3n} (v) + r^{vn} W,$$

and, $\frac{x^{-q}}{qr^n} + \frac{x^{n-q}}{\overline{n-q}\,.\,r^{2n}} + \frac{x^{2n-q}}{\overline{2n-q}\,.\,r^{3n}} (v) + \frac{W}{r^{vn}}$,

reſpectively.

Moreover, from the ſame Fluents, *thoſe* of $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e+fzq}$, and $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e-fz^q}$ will likewiſe become known:

For (having transformed the Fluxions here propoſed to $\frac{1}{e} \times \frac{z^{\frac{m}{n}q-1}\dot{z}}{1+\frac{fzq}{e}}$, *&c.*) let $\frac{fz^q}{e}$ be put $= x^n$, or $x = \overline{\frac{fz^q}{e}}\Big|^{\frac{1}{n}}$; then will $z^{\frac{m}{n}q} = \overline{\frac{e}{f}}\Big|^{\frac{m}{n}} \times x^m$, and conſequently $\frac{mq}{n} \times z^{\frac{m}{n}q-1}\dot{z} = \overline{\frac{e}{f}}\Big|^{\frac{m}{n}} \times mx^{m-1}\dot{x}$.

* Art. 150.

Whence $z^{\frac{m}{n}q-1}\dot{z} = \frac{n}{q} \times \left.\overline{\frac{e}{f}}\right|^{\frac{m}{n}} \times x^{m-1}\dot{x}$, and $1 \pm \overline{\frac{fz^q}{e}} = 1 \pm z^n$; and therefore $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e \pm fz^q}$ $\left(= \frac{1}{e} \times \frac{n}{q} \times \left.\overline{\frac{e}{f}}\right|^{\frac{m}{n}} \times \frac{x^{m-1}\dot{x}}{1 \pm x^n}\right) = \frac{n}{qe} \times \left.\overline{\frac{e}{f}}\right|^{\frac{m}{n}} \times \frac{x^{m-1}\dot{x}}{1 \pm x^n}$: Whoſe Fluent is given, by *Prob.* 4. or 5. But, r being here $= 1$, the general Multiplicator $\frac{r^{m-n}}{n}$, there given, will be barely $= \frac{1}{n}$: Which, drawn into $\frac{n}{qe} \times \left.\overline{\frac{e}{f}}\right|^{\frac{m}{n}}$, gives $\frac{1}{qe} \times \left.\overline{\frac{e}{f}}\right|^{\frac{m}{n}}$, for the general Multiplicator in this Caſe.

One thing more, though well known to Mathematicians, it may be proper here to take notice of; and that relates to the Sines and Co-ſines of the fore-mention'd Arcs, R, $2R$, $3R$, *&c. &c.* (multiplying the ſeveral Angles and Ratios) ſome of which Arcs do frequently exceed the whole Periphery: When this happens to be the Caſe, the Periphery, or 360°, muſt be ſubtracted as often as poſſible, and the Sine and Co-ſine of the Remainder be taken. If the Remainder be greater than 180°, the Sine, falling in the lower Semi-Circle, will be negative; if, between 90° and 270°, the Co-ſine, falling beyond the Center, will be negative.

PROB. VI.

333. *To find the Fluent of* $\frac{x^{n+m-1}\dot{x}}{r^{2n} - 2kr^nx^n + x^{2n}}$; *where* n *and* m *denote any whole poſitive Numbers, and where the given Expreſſion cannot be reſolved into two Binomials* (k *being leſs than Unity.* Art. 308. and 310.)

Let R be the Arch whoſe Co-ſine is $\acute{k}$ and Radius Unity, and let k be the Sine of the ſame Arch; moreover, let the Arcs $\frac{R}{n}$, $\frac{R+360^\circ}{n}$, $\frac{R+2\times 360^\circ}{n}$, $\frac{R+3\times 360^\circ}{n}$ &c. be denoted by $\mathcal{Q}$, $\acute{\mathcal{Q}}$, $\overset{\prime\prime}{\mathcal{Q}}$, $\overset{\prime\prime\prime}{\mathcal{Q}}$, $\overset{\prime\prime\prime\prime}{\mathcal{Q}}$ &c. and let $\acute{b}$, $\acute{c}$, $\acute{d}$ &c. and b, c, d &c. expreſs the Sines, and the Co-ſines of the ſame Arcs reſpectively.

Then will $\frac{n\acute{k}r^n x^n}{r^{2n}-2kr^n x^n+x^{2n}} = \frac{\acute{b}rx}{r^2-2brx+x^2} + \frac{\acute{c}rx}{r^2-2crx+x^2} + \frac{\acute{d}rx}{r^2-2drx+x^2}$ &c. (n) *by Lemma* 4.) From whence, multiplying the whole Equation by $\frac{x^{m-1}\dot{x}}{n\acute{k}r^n}$ we have $\frac{x^{n+m-1}\dot{x}}{r^{2n}-2kr^n x^n+x^{2n}} = \frac{1}{n\acute{k}r^{n-1}}$ into $\frac{\acute{b}x^m\dot{x}}{r^2-2brx+x^2} + \frac{\acute{c}x^m\dot{x}}{r^2-2crx+x^2} + \frac{\acute{d}x^m\dot{x}}{r^2-2drx+x^2}$ &c.

Now, the Fluent of the firſt Term hereof $\frac{\acute{b}x^m\dot{x}}{r^2-2brx+x^2}$, (if M be put for the hyp. Log. of $\frac{\sqrt{x^2-2brx+x^2}}{r}$, and N for the Arch whoſe Radius is Unity, and Sine $\frac{x\times\overline{\text{Sin.}\,\mathcal{Q}}}{\sqrt{r^2-2brx+x^2}}$) will appear (*from Prop.* 2.) to be $=$ $\overline{\text{Sin.}\,\mathcal{Q}}\times\frac{x^{m-1}}{m-1} + \overline{\text{Sin.}\,2\mathcal{Q}}\times\frac{rx^{m-2}}{m-2} + \overline{\text{Sin.}\,3\mathcal{Q}}\times\frac{r^2x^{m-3}}{m-3}$ &c. $(m-1) + r^{m-1}\times\overline{\text{Sin.}\,m\mathcal{Q}\times M+\text{Co-ſin.}\,m\mathcal{Q}\times N}$.

From whence, if the Arcs whoſe Sines are $\frac{x \times \text{Sin. } Q'}{\sqrt{r^2 - 2crx + x^2}}$, $\frac{x \times \text{Sin. } Q''}{\sqrt{r^2 - 2drx + x^2}}$ &c. be repreſented by M', M'' &c. and the Logarithms whoſe Numbers are $\frac{\sqrt{r^2 - 2crx + x^2}}{r}$, $\frac{\sqrt{r^2 - 2drx + x^2}}{r}$ &c. by N', N'' &c. reſpectively, the Fluent of the whole Expreſſion, omitting the general Multiplicator $\left(\frac{1}{nk'r^{n-1}}\right)$ will be

$$\left.\begin{matrix} \text{Sin. } Q \\ \text{Sin. } Q' \\ \text{Sin. } Q'' \\ \text{Sin. } Q''' \\ \text{\&c.} \end{matrix}\right\} \times \frac{x^{m-1}}{m-1} + \left\{\begin{matrix} \text{Sin. } 2Q \\ \text{Sin. } 2Q' \\ \text{Sin. } 2Q'' \\ \text{Sin. } 2Q''' \\ \text{\&c.} \end{matrix}\right\} \times \frac{rx^{m-2}}{m-2} + \left\{\begin{matrix} \text{Sin. } 3Q \\ \text{Sin. } 3Q' \\ \text{Sin. } 3Q'' \\ \text{Sin. } 3Q''' \\ \text{\&c.} \end{matrix}\right\}$$

$$\times \frac{r^2 x^{m-3}}{m-3} \text{ (\&c. to } m-1 \text{ Terms)}$$

$$+ r^{m-1} \times \left\{\begin{matrix} \overline{\text{Sin. } mQ} \times M + \overline{\text{Co-ſ. } mQ} \times N \\ \overline{\text{Sin. } mQ'} \times M' + \overline{\text{Co-ſ. } mQ'} \times N' \\ \overline{\text{Sin. } mQ''} \times M'' + \overline{\text{Co-ſ. } mQ''} \times N'' \\ \overline{\text{Sin. } mQ'''} \times M''' + \overline{\text{Co-ſ. } mQ'''} \times N''' \\ \text{\&c.} \qquad \text{\&c.} \end{matrix}\right\}$$

But, the Sines of the firſt Column being thoſe of an arithmetical Progreſſion (whoſe common Difference is

$\frac{360°}{n}$) which ariſes by dividing the whole Periphery into n equal Parts, their Sum will, therefore, be equal to Nothing.

Moreover, the Sines of the ſecond Column, having $\frac{2 \times 360°}{n}$ for the common Difference of their reſpective Arcs do, alſo, divide the whole Periphery (twice taken) into n equal Parts, and therefore deſtroy each other.

The ſame is likewiſe true, with regard to the Sines of every other Column (except the laſt of all) when $m-1$ is leſs than n. But, if m be greater than n, the Arcs, in the Column, whoſe Place from the firſt, incluſive, is denoted by n, being expreſſed by $n\mathcal{Q}$, $n\mathcal{Q}'$, $n\mathcal{Q}''$ &c. (or R, $R+360°$, $R+2\times 360°$ &c.) whereof the common Difference is the whole Periphery; the Sines of that Column do not deſtroy one another, but each is equal to that of the firſt Arc R (*Vid. Art.* 314 and 318.) and conſequently their Sum equal to $n \times Sin. R$.

In like Manner, if m be greater than $2n$, the Series, continued to $m-1$ Terms, will take in the Column, where the Arcs are $2n\mathcal{Q}$, $2n\mathcal{Q}'$, $2n\mathcal{Q}''$ &c. (or $2R$, $2R+2\times 360°$, $2R+4\times 360°$ &c.) whereof the Sine of each is, alſo, equal to the Sine of the firſt ($2R$) and therefore their Sum $= n \times Sin.\ 2R$.

Thus, alſo, it will appear that the Sines of the Column whoſe Diſtance from the firſt, incluſive, is $3n$ (when m is greater than $3n$) will be each equal to $Sin.\ 3R$; &c. &c.

Therefore, ſeeing all the Columns do actually vaniſh, except thoſe above ſpecified; whoſe Places from the Beginning are denoted by n, $2n$, $3n$ &c. and whoſe correſponding Terms, or Multiplicators are, therefore, repreſented by $\frac{r^{n-1}x^{m-n}}{m-n}$, $\frac{r^{2n-1}x^{m-2n}}{m-2n}$, $\frac{r^{3n-1}x^{m-3n}}{m-3n}$ &c. it is evident that the whole Expreſſion will be reduced to

$$\overline{\text{Sin.}\,R}\times\frac{nr^{n-1}x^{m-n}}{m-n}+\overline{\text{Sin.}\,2R}\times\frac{nr^{2n-1}x^{m-2n}}{m-2n}$$

$$+\overline{\text{Sin.}\,3R}\times\frac{nr^{3n-1}x^{m-3n}}{m-3n}\ \text{\&c.}$$

$$+\,r^{m-1}\ \text{into}\left\{\begin{array}{l}\text{Sin.}\,mQ\times M+\text{Co-ſ.}\,mQ\times N\\ \text{Sin.}\,m\overset{\prime}{Q}\times\overset{\prime}{M}+\text{Co-ſ.}\,m\overset{\prime}{Q}\times\overset{\prime}{N}\\ \text{Sin.}\,m\overset{\prime\prime}{Q}\times\overset{\prime\prime}{M}+\text{Co-ſ.}\,m\overset{\prime\prime}{Q}\times\overset{\prime\prime}{N}\\ \text{Sin.}\,m\overset{\prime\prime\prime}{Q}\times\overset{\prime\prime\prime}{M}+\text{Co-ſ.}\,m\overset{\prime\prime\prime}{Q}\times\overset{\prime\prime\prime}{N}\\ \quad\text{\&c.}\qquad\qquad\qquad\qquad\text{\&c.}\end{array}\right.$$

Which, multiply'd by $\frac{1}{n\overset{\prime}{k}r^{n-1}}$, the foreſaid, general, Multiplicator, gives $\overline{\text{Sin.}\,R}\times\frac{x^{m-n}}{\overline{m-n}\cdot\overset{\prime}{k}}+\overline{\text{Sin.}\,2R}\times\frac{r^{n}x^{m-2n}}{\overline{m-2n}\cdot\overset{\prime}{k}}+\overline{\text{Sin.}\,3R}\times\frac{r^{2n}x^{m-3n}}{\overline{m-3n}\cdot\overset{\prime}{k}}$ &c.

$$+\frac{r^{m-n}}{n\overset{\prime}{k}}\times\left\{\begin{array}{l}\text{Sin.}\,mQ\times M+\text{Co-ſ.}\,mQ\times N\\ \text{Sin.}\,m\overset{\prime}{Q}\times\overset{\prime}{M}+\text{Co-ſ.}\,m\overset{\prime}{Q}\times\overset{\prime}{N}\\ \text{Sin.}\,m\overset{\prime\prime}{Q}\times\overset{\prime\prime}{M}+\text{Co-ſ.}\,m\overset{\prime\prime}{Q}\times\overset{\prime\prime}{N}\\ \text{Sin.}\,m\overset{\prime\prime\prime}{Q}\times\overset{\prime\prime\prime}{M}+\text{Co-ſ.}\,m\overset{\prime\prime\prime}{Q}\times\overset{\prime\prime\prime}{N}\\ \quad\text{\&c.}\qquad\qquad\qquad\qquad\text{\&c.}\end{array}\right.$$

for the true Fluent of $\frac{x^{n+m-1}\dot{x}}{r^{2n}-2kr^{n}x^{n}+x^{2n}}$: Where the former Part of the Expreſſion muſt be continued to as many Terms as there are Units in $\frac{m-1}{n}$ (the Remainder, if any, being neglected.) *Q. E. I.*

COROLLARY

COROLLARY.

334. If the Quotient ariſing from the Diviſion of m by n (when the former exceeds) be denoted by v, and the Remainder by t; or, which is the ſame, if $vn + t = m$, it is evident the Arcs $m\acute{\mathcal{Q}}$, $m\acute{\acute{\mathcal{Q}}}$, $m\acute{\acute{\acute{\mathcal{Q}}}}$ &c. which are reſpectively equal to $m\mathcal{Q} + m \times \frac{360^\circ}{n}$, $m\mathcal{Q} + 2m \times \frac{360^\circ}{n}$, $m\mathcal{Q} + 3m \times \frac{360^\circ}{n}$, &c. (by Conſtruction) will alſo be equal to $m\mathcal{Q} + v \times 360^\circ + t \times \frac{360^\circ}{n}$, $m\mathcal{Q} + 2v \times 360^\circ + 2t \times \frac{360^\circ}{n}$ &c. whereof the Sines and Co-ſines (omitting $v \times 360^\circ$, $2v \times 360^\circ$ &c. the Multiples of the whole Periphery) are the ſame with thoſe of $m\mathcal{Q} + t \times \frac{360^\circ}{n}$, $m\mathcal{Q} + 2t \times \frac{360^\circ}{n}$ &c. reſpectively.

Therefore, if the Arcs of the Progreſſion, whereof the firſt Term is $m\mathcal{Q}$, and the common Difference $t \times \frac{360^\circ}{n}$, be repreſented by T, $\acute{T}$, $\acute{\acute{T}}$ &c. reſpectively; it follows that the Fluent of $\frac{x^{\overline{n+m-1}}\dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}}$ $\left(\text{or, } \frac{x^{\overline{n+vn+t-1}}\dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}}\right)$ will, *alſo*, be truly expreſſed by

$$\overline{Sin.\ R} \times \frac{x^{\overline{m-n}}}{\overline{m-n}\,.\,\acute{k}} + \overline{Sin.\ 2R} \times \frac{r^n x^{\overline{m-2n}}}{\overline{m-2n}\,.\,\acute{k}} + \overline{Sin.\ 3R} \times \frac{r^{2n} x^{\overline{m-3n}}}{\overline{m-3n}\,.\,\acute{k}} \text{ \&c. } \left(\frac{m-1}{n}\right)$$

$$+\frac{r^{m-n}}{n\acute{k}}\left\{\begin{array}{l}\text{Sin. }T\times M+\text{Co-ſ. }T\times N\\ \text{Sin. }\overset{\prime}{T}\times\overset{\prime}{M}+\text{Co-ſ. }\overset{\prime}{T}\times\overset{\prime}{N}\\ \text{Sin. }\overset{\prime\prime}{T}\times\overset{\prime\prime}{M}+\text{Co-ſ. }\overset{\prime\prime}{T}\times\overset{\prime\prime}{N}\\ \text{Sin. }\overset{\prime\prime\prime}{T}\times\overset{\prime\prime\prime}{M}+\text{Co-ſ. }\overset{\prime\prime\prime}{T}\times\overset{\prime\prime\prime}{N}\\ \quad\&c. \qquad\qquad\qquad \&c.\end{array}\right.$$

In the very ſame Manner the Fluent of $\frac{x^{n+m-1}\dot{x}}{r^{2n}+2kr^nx^n+x^{2n}}$ (where the Sign of the ſecond Term is poſitive) will be exhibited; *if R* be taken to denote the Arch whoſe Co ſine is $-k$; which will, in this Caſe, be greater than a Quadrant.

PROPOSITION VII.

335. *To find the Fluent of* $\frac{x^{n-m-1}\dot{x}}{r^{2n}-2kr^nx^n+x^{2n}}$; *under the Reſtrictions mentioned in the laſt Problem.*

Let every thing remain as before: Then we ſhall have $\frac{x^{n-m-1}\dot{x}}{r^{2n}-2kr^nx^n+x^{2n}}=\frac{1}{n\acute{k}r^{n-1}}$ into $\frac{\acute{b}x^{-m}\dot{x}}{r^2-2krx+x^2}$ $+\frac{cx^{-m}\dot{x}}{r^2-2krx+x^2}$ (*n*) Whereof the Fluent (*by Prob.* 3.) appears to be $\frac{1}{n\acute{k}r^{n-1}}$ into

$$-\left\{\begin{array}{l}\text{Sin. }\mathcal{Q}\\ \text{Sin. }\overset{\prime}{\mathcal{Q}}\\ \text{Sin. }\overset{\prime\prime}{\mathcal{Q}}\\ \quad\&c.\end{array}\right\}\times\frac{x^{1-m}}{m-1.r^2}-\left\{\begin{array}{l}\text{Sin. }2\mathcal{Q}\\ \text{Sin. }2\overset{\prime}{\mathcal{Q}}\\ \text{Sin. }2\overset{\prime\prime}{\mathcal{Q}}\\ \quad\&c.\end{array}\right\}\times\frac{x^{2-m}}{m-2\times r^4}$$

$$-\begin{Bmatrix} \text{Sin. } 3\mathcal{Q} \\ \text{Sin. } 3\mathcal{Q}' \\ \text{Sin. } 3\mathcal{Q}'' \\ \text{\&c.} \end{Bmatrix} \times \frac{x^{3-m}}{\overline{m-3}\cdot r^{4}}\,(m) +$$

$$\frac{1}{r^{m+1}} \times \begin{cases} -\text{Sin. } m\mathcal{Q}\times M + \text{Co-ſ. } m\mathcal{Q}\times N \\ -\text{Sin. } m\mathcal{Q}'\times M' + \text{Co-ſ. } m\mathcal{Q}'\times N' \\ -\text{Sin. } m\mathcal{Q}''\times M'' + \text{Co-ſ. } m\mathcal{Q}''\times N'' \\ \quad\text{\&c.} \qquad\qquad \text{\&c.} \end{cases}$$

Which, by Reaſoning as above, will be reduced to

$$-\overline{\text{Sin. } R} \times \frac{x^{n-m}}{\overline{m-n}\cdot \acute{k}r^{2n}} - \overline{\text{Sin. } 2R} \times \frac{x^{2n-m}}{\overline{m-2n}\cdot \acute{k}r^{3n}}$$

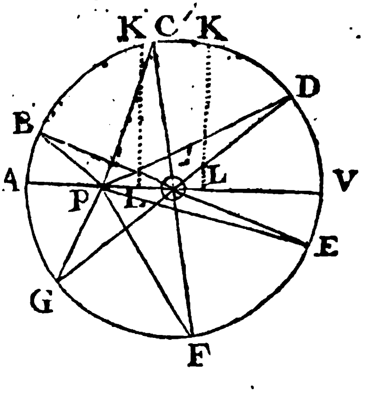
$$-\overline{\text{Sin. } 3R} \times \frac{x^{3n-m}}{\overline{m-3n}\cdot \acute{k}r^{4n}} \left(\text{to } \frac{m}{n} \text{ Terms}\right)$$

$$+\frac{1}{\acute{k}nr^{n+m}} \times \begin{cases} -\text{Sin. } T\times M + \text{Co-ſ. } T\times N \\ -\text{Sin. } T'\times M' + \text{Co-ſ. } T'\times N' \\ -\text{Sin. } T''\times M'' + \text{Co-ſ. } T''\times N'' \\ \quad\text{\&c.} \qquad\qquad \text{\&c.} \end{cases}$$

Q. E. I.

SCHOLIUM.

336. If, from the Center O, of the Circle ABCD, whoſe Radius OA, or OV, is r, there be taken OL equal to k and OP $=x$; and if the Arch AB be to the Arch AK, whoſe Co-ſine is $\pm\, k$, as 1 to n; and each of



of the Arcs BC, CD, DE *&c.* be taken equal to $\frac{360^\circ}{n}$ *&c.* *&c.* Then the Angles R, Q, $\acute{Q}$ *&c.* ſpecified (in the two preceding Problems) being here expounded by AK, AB, AC *&c.* reſpectively, we have $PB = \sqrt{r^2 - 2brx + x^2}$, $PC = \sqrt{r^2 - 2crx + x^2}$ *&c.* (*Vid. Art.* 317. and 323.) Whence, alſo, the Angles N, $\acute{N}$, $\overset{''}{N}$ *&c.* whoſe Sines are $\frac{x \times \text{Sin.}\, Q}{\sqrt{r^2 - 2brx + x^2}}$, $\frac{x \times \text{Sin.}\, \acute{Q}}{\sqrt{r^2 - 2crx + x^2}}$, $\frac{x \times \text{Sin.}\, \overset{''}{Q}}{\sqrt{r^2 - 2drx + x^2}}$ *&c.* will here be equal to B, C, D *&c.* Therefore the Fluents of $\frac{x^{n+m-1}\dot{x}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$, and $\frac{x^{n-m-1}\dot{x}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$ (there given) will, alſo, be truly defined by

$$\frac{x^{m-n}}{m-n} + \frac{\text{Sin.}\, 2R}{\text{Sin.}\, R} \times \frac{r^n x^{m-2n}}{m-2n} + \frac{\text{Sin.}\, 3R}{\text{Sin.}\, R} \times \frac{r_{2n} x^{m-3n}}{m-3n}$$

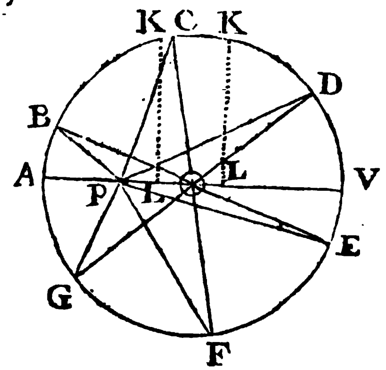
$\left(\text{to } \frac{m-1}{n} \text{ Terms}\right)$

$$+\frac{r^{m-n}}{n \times \text{Sin.}\, R} \times \begin{cases} \text{Sin.}\, T \times (OB : PB) + \text{Co-ſ.}\, T \times (B) \\ \text{Sin.}\, \acute{T} \times (OC : PC) + \text{Co-ſ.}\, \acute{T} \times (C) \\ \text{Sin.}\, \overset{''}{T} \times (OD : PD) + \text{Co-ſ.}\, \overset{''}{T} \times (D) \\ \text{Sin.}\, \overset{'''}{T} \times (OE : PE) + \text{Co-ſ.}\, \overset{'''}{T} \times (-E) \\ \text{Sin.}\, \overset{''''}{T} \times (OF : PF) + \text{Co-ſ.}\, \overset{''''}{T}\ (-F) \\ \quad \&c. \qquad\qquad\qquad \&c. \end{cases}$$

And by $-\frac{x^{n-m}}{m-n\,.\,r^{2n}}-\frac{Sin.\ 2R}{Sin.\ R}\times\frac{x^{2n-m}}{m-2n\,.\,r^{3n}}-$
$-\frac{Sin.\ 3R}{Sin.\ R}\times\frac{x^{3n-m}}{m-3n\,.\,r^{4n}}\left(\frac{m}{n}\right)$

$$+\frac{1}{nr^{n+m}\times Sin.\ R}\times\begin{cases}-Sin.\ T\times(OB:PB)+Co\text{-}ſ.\ T\times(B)\\-Sin.\ \overset{\prime}{T}\times(OC:PC)+Co\text{-}ſ.\ \overset{\prime}{T}\times(C)\\-Sin.\ \overset{\prime\prime}{T}\times(OD:PD)+Co\text{-}ſ.\ \overset{\prime\prime}{T}\times(D)\\-Sin.\ \overset{\prime\prime\prime}{T}\times(OE:PE)+Co\text{-}ſ.\ \overset{\prime\prime\prime}{T}\times(-E)\\-Sin.\ \overset{\prime\prime\prime\prime}{T}\times(OF:PF)+Co\text{-}ſ.\ \overset{\prime\prime\prime\prime}{T}\times(-F)\\ \quad \&c.\end{cases}$$

reſpectively.



Where the Arc *AK* (or *R*) will be greater than a Quadrant when the Sign of *k* is poſitive; but leſs, when negative; and where the Arcs *T*, $\overset{\prime}{T}$, $\overset{\prime\prime}{T}$ &c. denote an arithmetical Progreſſion, whoſe firſt Term (*T*) is equal to $m\times AB$, and whereof the common Difference is equal to $\frac{360°}{n}$ (or *BC*) multiplied by *m*, when *m* is leſs than *n*; but otherwiſe by the Remainder, of *m* divided by *n*.

337. Hence the Fluent of $\frac{z^{q \pm \frac{m}{n} q - 1} \dot{z}}{e \mp f z^q + g z^{2q}}$, where q is any Number, either whole or broken, may be very eaſily deduced: For, having transformed the Denominator to $g \times \overline{\frac{e}{g} \mp \frac{fz^q}{g} + z^{2q}}$, put $\frac{e}{g} = r^{2n}$, $\frac{f}{g} = 2kr^n$, and $z^q = x^n$; and then it will become $= g \times \overline{r^{2n} \mp 2kr^n x^n + x^{2n}}$: Moreover, $z^{q \pm \frac{m}{n} q}$ being $= \overline{x^n}|^{1 \pm \frac{m}{n}} = x^{n \pm m}$, and $\overline{q \pm \frac{m}{n} q} \times z^{q \pm \frac{m}{n} q - 1} \dot{z} = \overline{n \pm m} \times x^{n \pm m - 1} \dot{x}$, the Numerator will be reduced to $\frac{n}{q} \times x^{n \pm m - 1} \dot{x}$: And ſo, we have $\frac{z^{q \pm \frac{m}{n} q - 1} \dot{z}}{e \mp f z^q + g z^{2q}} = \frac{n}{qg} \times \frac{x^{n \pm m - 1} \dot{x}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$: In which $x = z^{\frac{q}{n}}$, $r = \overline{\frac{e}{g}}|^{\frac{1}{2n}}$, and $k \left(= \frac{\frac{1}{2} f}{gr^n}\right) = \frac{\frac{1}{2} f}{\sqrt{eg}}$. But, it may be obſerved, that the Fluent hereof is, only, given when

Art. 333. $\frac{\frac{1}{2} f}{\sqrt{eg}}$ (or its Equal k) is leſs than Unity. Therefore, if $\frac{1}{2} f$ be greater than $\sqrt{eg}$; or if the Values of e and g are unlike, with regard to poſitive and negative, ſo that $\sqrt{eg}$ is impoſſible, the above Solution fails. But, here, the given Trinomial may be reſolved into two Binomials (*by Art.* 310.) and, from thence, the Fluent may be found at two Operations (*by Prob.* 4. *and* 5.)

For, by feigning $e \mp fy + gy^2 = 0$, in order to ſuch a Reſolution, we get $\frac{\pm\frac{1}{2}f + \sqrt{\frac{1}{4}f^2 - eg}}{g}$, and $\frac{\pm\frac{1}{2}f - \sqrt{\frac{1}{4}f^2 - eg}}{g}$ for the Roots of that Equation, or the two firſt Terms of the required Binomials: Which therefore are always poſſible when $\frac{1}{4}f^2 - eg$ is poſitive, or when the foregoing Solution fails.

By denoting the ſaid Roots by H and K, the Trinomial $e \mp fz^q + gz^{2q}$ is reſolved into $g \times \overline{H - z^q} \times \overline{K - z^q}$, from whence $\frac{z^{q \pm \frac{m}{n}q - 1}\dot{z}}{e \mp fz^q + gz^{2q}}$ is reduced to

$$\frac{z^{q \pm \frac{m}{n}q - 1}\dot{z}}{g \times \overline{K - H} \times \overline{H - z^q}} + \frac{z^{q \pm \frac{m}{n}q - 1}\dot{z}}{g \times \overline{H - K} \times \overline{K - z^q}},$$ whoſe Fluent is given by *Art.* 332.

338. By proceeding the ſame Way the Fluent of $\frac{z^{\frac{m}{n}q - 1}\dot{z}}{e + fz^q + gz^{2q} + hz^{3q}}$ may likewiſe be found: For, ſince one, at leaſt, of the three Roots of the Equation $e + fy + gy^2 + hy^3 = 0$, muſt be poſſible, the propoſed Fluxion, if it cannot be reſolved into three Binomials, may, however, be reduced to one Binomial and one Trinomial; and ſo, be brought under the foregoing Forms: But this being a Speculation too much out of the Way of common Uſe to be farther purſued, I ſhall here conclude this Section, with obſerving, that, when *k*, in the original Trinomial, above ſpecified, is neither leſs, nor greater than Unity, the Fluent cannot then be had directly, from either of the preceding Methods; but muſt be found by Compariſon from the Fluent of $\frac{x^{n \pm m - 1}\dot{x}}{r^n \pm x^n}$. *Vid. Art.* 289.

SECTION VI.

The Manner of inveſtigating Fluents, when Quantities, and their Logarithms; Arcs and their Sines, &c. are involved together: With other Caſes of the like Nature.

PROB. I.

339. *SUPPOSING Q and n to denote given Quantities; it is propoſed to find the Fluent of $x^n\dot{x}Q^x$.*

Let $Q^x \times \overline{Ax^n + Bx^{n-1} + Cx^{n-2}\ \&c.}$ be aſſumed for the Fluent required: Then the Fluxion thereof, which is

* Art. 252. $$Q^x\dot{x} \times \text{hyp. Log. } Q^* \times \overline{Ax^n + Bx^{n-1} + Cx^{n-2}\ \&c.} + Q^x \times \overline{n\dot{x}Ax^{n-1} + \overline{n-1}.B\dot{x}x^{n-2} + \overline{n-2}.C\dot{x}x^{n-3}\ \&c.}$$

muſt conſequently be $= x^n\dot{x}Q^x$: And therefore, by putting m for the *hyp. Log.* of Q, we have

$$\left.\begin{array}{l} mAx^n + mBx^{n-1} + mCx^{n-2} + mDx^{n-3}\ \&c. \\ -x^n + nAx^{n-1} + \overline{n-1}.Bx^{n-2} + \overline{n-2}.Cx^{n-3}\ \&c. \end{array}\right\} = 0$$

Whence, comparing the Coefficients of the homologous Terms, we get $A = \frac{1}{m}$, $B = -\frac{nA}{m} = -\frac{n}{m^2}$, $C = -\frac{\overline{n-1}.B}{m} = \frac{n.\overline{n-1}}{m^3}$ &c. and conſequently $Q^x \times \overline{Ax + Bx^{n-1} + Cx^{n-2} + \&c.} = \frac{Q^x}{m} \times \overline{x^n - \frac{nx^{n-1}}{m} + \frac{n.\overline{n-1}.x^{n-2}}{m^2} - \frac{n.\overline{n-1}.\overline{n-2}.x^{n-3}}{m^3}\ \&c.}$ Which

Series, it is plain, will always terminate when n is a whole poſitive Number. *Q. E. I.*

340. In the preceding Problem the Coefficients A, B, C, *&c.* of the aſſumed Series were taken, in the common Way, as conſtant Quantities; which, becauſe of the general Multiplicator Qx, was ſufficient.

But, in other Caſes, where a proper Multiplicator, to expreſs the mechanical, or logarithmic, *&c.* Part of the required Fluent, cannot readily be known, it will be convenient to aſſume a Series for the *Whole* (independent of any general Multiplicator) wherein the Quantities A, B, C, D, *&c.* muſt be conſidered as variable.

PROB. II.

341. *To find the Fluent of* $z^m x^{n-1}\dot{x}$; z *being the Hyperbolic-Logarithm of* x; *and* m *and* n *any given Numbers.*

Let there be aſſumed $Az^m + Bz^{m-1} + Cz^{m-2} + Dz^{m-3}$ *&c.* $=$ the Fluent of $z^m x^{n-1}\dot{x}$: Then, in Fluxions, we ſhall have

$$\left.\begin{array}{l}\dot{A}z^m + \dot{B}z^{m-1} + \dot{C}z^{m-2} + \dot{D}z^{m-3}\ \&c.\\ +mAz^{m-1}\dot{z} + \overline{m-1}.Bz^{m-2}\dot{z} + \overline{m-2}.Cz^{m-3}\dot{z}\ \&c.\end{array}\right\} = z^m x^{n-1}\dot{x}.$$

But $\dot{z} = \frac{\dot{x}}{x}$, whence, by ordering the Equation, there ariſes

$$\left.\begin{array}{l}\dot{A}\\ -x^{n-1}\dot{x}\end{array}\right\}\times z^m + \left.\begin{array}{c}\dot{B}\\ \frac{mA\dot{x}}{x}\end{array}\right\}\times z^{m-1} + \left.\begin{array}{c}\dot{C}\\ \frac{\overline{m-1}.B\dot{x}}{x}\end{array}\right\}z^{m-2},\ \&c. = 0$$

Now, by making the Coefficients of the like Powers of z, equal to Nothing, we have $\dot{A} = x^{n-1}\dot{x}$, $A = \frac{x^n}{n}$; $\dot{B} = \left(-\frac{mA\dot{x}}{x}\right) = -\frac{mx^{n-1}\dot{x}}{n}$, $B = -\frac{mx^n}{n^2}$;

$\dot{C}\,(= -\frac{\overline{m-1}\,.\,B\dot{x}}{x} =)\ \frac{m\,.\,\overline{m-1}\,.\,x^{n-1}\dot{x}}{n^2}\quad C = \frac{m\,.\,\overline{m-1}\,.\,x^n}{n^3}$ &c. and conſequently the Fluent ſought

$$= \frac{x^n}{n} \text{ into } z^m - \frac{mz^{m-1}}{n} + \frac{m\,.\,\overline{m-1}\,.\,z^{m-2}}{n^2} - \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,z^{m-3}}{n^3} + \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,\overline{m-3}\,.\,z^{m-4}}{n^4}$$

&c. Which, when m is a whole poſitive Number, will terminate in $m+1$ Terms. *Q. E. I.*

P R O B. III.

342. *To find the Fluent of* $z^n\dot{y}$; z *being the Arch of a given Circle, and* y *the Sine correſponding.*

Let there be aſſumed $Az^n + Bz^{n-1} + Cz^{n-2} + Dz^{n-3}$ = Fluent of $z^n\dot{y}$; then, by taking the Fluxion, we ſhall have

$$\left.\begin{matrix} \dot{A}z^n + \dot{B}z^{n-1} + \dot{C}z^{n-2} + \dot{D}z^{n-3} \quad \&c. \\ -z^n\dot{y} + nAz^{n-1}\dot{z} + \overline{n-1}\,.\,Bz^{n-2}\dot{z} \quad \&c. \end{matrix}\right\} = 0$$

Whence, putting $\dot{A} - \dot{y} = 0$, $\dot{B} + nA\dot{z} = 0$, $\dot{C} + \overline{n-1}\,.\,B\dot{z} = 0$, $\dot{D} + \overline{n-2}\,.\,C\dot{z} = 0$, &c. we get $A = y$; $\dot{B} = -ny\dot{z}$, $\dot{C} = -\overline{n-1}\,.\,B\dot{z}$ &c.

But, if a and x be taken to denote the Radius and Co-ſine of the Arch z, it will appear, from *Art.* 142. that $y\dot{z} = -a\dot{x}$ and $x\dot{z} = a\dot{y}$: Therefore $\dot{B} = na\dot{x}$, and $B = nax$; alſo $\dot{C}\ (= -\overline{n-1}\,.\,B\dot{z}) = -n\,.\,\overline{n-1}\,.\,ax\dot{z} = -n\,.\,\overline{n-1}\,.a^2\dot{y}$, and $C = -n.\overline{n-1}.a^2y$; likewiſe $\dot{D}\ (= -\overline{n-2}\,.\,C\dot{z}) = n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,a^2y\dot{z} = -n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,a^3\dot{x}$, and $D = -n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.a^3x$ &

&c. &c. and conſequently $Az^n + Bz^{n-1} + Cz^{n-2}$ &c. $= yz^n + naxz^{n-1} - n.\overline{n-1}.a^2yz^{n-2} - n.\overline{n-1}.\overline{n-2}.a^3xz^{n-3} +$ &c.

$$= \begin{cases} y \times \overline{z^n - n.\overline{n-1}.a^2z^{n-2} + n.\overline{n-1}.\overline{n-2}.\overline{n-3}.a^4z^{n-4}}\ \&c. \\ x \times \overline{naz^{n-1} - n.\overline{n-1}.\overline{n-2}.a^3z^{n-3} + n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.a^5z^{n-5}}\ \&c. \end{cases}$$

Q. E. I.

In the very ſame Manner the Fluent of $z^n\dot{w}$, or $z^n \times -\dot{x}$ (w being the Verſed-Sine of the Arch z) will be found $= -xz^n + nyaz^{n-1} + n.\overline{n-1}.xa^2z^{n-2} - n.\overline{n-1}.\overline{n-2}.ya^3z^{n-3} - n.\overline{n-1}.\overline{n-2}.\overline{n-3}.xa^4z^{n-4}$ + &c.

PROB.

PROB. IV.

343. *The Quantities, x, y and z being the ſame as in the preceding Problem; to find the Fluent of $z^n x^r y^m \dot{y}$.*

By aſſuming $Az^n + Bz^{n-1} + Cz^{n-2} + Dz^{n-3}$ &c. and proceeding as above, we have $\dot{A} = x^r y^m \dot{y}$, $\dot{B} = -nA\dot{z}$, $\dot{C} = -\overline{n-1}\,.\,B\dot{z}$, $\dot{D} = -\overline{n-2}\,.\,C\dot{z}$ &c. or (becauſe $\dot{z} = \frac{a\dot{y}}{x}$) $\dot{B} = -\frac{naA\dot{y}}{x}$, $\dot{C} = -\frac{\overline{n-1}\,.\,aB\dot{y}}{x}$, $\dot{D} = -\frac{\overline{n-2}\,.\,aC\dot{y}}{x}$ &c. Therefore, if the Fluent of $x^r y^m \dot{y}$ (found from *Art.* 142. and 291.) be denoted by Q; *that* of $\frac{Q\dot{y}}{x}$, by R; *that* of $\frac{R\dot{y}}{x}$, by S; *that* of $\frac{S\dot{y}}{x}$, by T &c. it follows that the Fluent of $z^n x^r y^m \dot{y}$ will be truly repreſented by $Qz^n - naRz^{n-1} + n\,.\,\overline{n-1}\,.\,a^2 Sz^{n-2} - n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,a^3 Tz^{n-3}$ &c.

COROLLARY.

344. Since $\dot{y} = -\frac{x\dot{x}}{y} = \frac{x\dot{z}}{a}$ (*Vid. Art.* 142.) it follows that $z^n x^r y^m \dot{y}$ is $= -z^n x^{r+1} y^{m-1} \dot{x} = \frac{z^n x^{r+1} y^m \dot{z}}{a}$: Therefore the Fluents of theſe two laſt Expreſſions are, *alſo*, exhibited in the foregoing Series.

345. As the Values of Q, R, S, &c. in the preceding Articles, are too complex to be purſued in a general Manner, it may not be amiſs to illuſtrate the Method of proceeding by an Example or two.

Let, then, the Fluxion propoſed be $\frac{zy^2\dot{y}}{x}$: Where n being $=1$, $m=2$, and $r=-1$, we have ˙ $= \frac{y^2\dot{y}}{x} = \frac{y^2\dot{y}}{\sqrt{a^2-y^2}}$ (becauſe $\sqrt{a^2-y^2}=x$.) Whence $Q = -\frac{1}{2}y\sqrt{a^2-y^2}+\frac{1}{2}az = -\frac{1}{2}yx+\frac{1}{2}az$*, and therefore $\dot{R}$ $\left(= \frac{Q\dot{y}}{x}\right) = -\frac{1}{2}y\dot{y}+\frac{\frac{1}{2}ax\dot{y}}{x} = -\frac{1}{2}y\dot{y}+\frac{1}{2}z\dot{z}$ (becauſe $\frac{a\dot{y}}{x} = \dot{z}$) and conſequently $R = -\frac{1}{4}y^2+\frac{1}{4}z^2$; and ſo, $\frac{az-yx}{2}\times z + a\times\frac{yy-zz}{4}$, or $\frac{az^2-2xyz+ay^2}{4}$, is the true Fluent of $\frac{zy^2\dot{y}}{x}$ $\left(= -zy\dot{x} = \frac{y^2z\dot{z}}{a}.\dagger\right)$

* Art. 279.

† Art. 344.

Again, let the Fluent of $-p\dot{x}\times\overline{z+y}|^2$ (expreſſing the Content of the Solid generated by the Revolution of the *Cycloid*) be required.

Here, the given Expreſſion, in ſimple Terms, will become $-pz^2\dot{x}-2pzy\dot{x}-py^2\dot{x}$: Whereof the Fluent of the firſt Term $-pz^2\dot{x}$, will be had, by making $n=2$, $m-1=0$, and $r+1=0$ (*Vid. Form.* 2. *in Corol.*) Where, we therefore, have $\dot{Q} = \frac{y\dot{y}}{x} = -\dot{x}$; whence $Q=-x$; alſo $\dot{R}$ $\left(\frac{Q\dot{y}}{x}\right) = -\dot{y}$, and $R=-y$; likewiſe $\dot{S}$ $\left(=\frac{R\dot{y}}{x}\right) = -\frac{y\dot{y}}{x} = \dot{x}$, $S=x$; and conſequently the Fluent of $-z^2\dot{x}$ $(Qz^n - naRz^{n-1} + n.\overline{n-1}.a^2Sz^{n-2}$ &c.$) = -xz^2+2ayz+2a^2x$: To which, adding the Fluent $\left(\frac{az^2-2xyz+ay^2}{2}\right)$ of the ſecond Term $-2zy\dot{x}$ (found in the preceding Example) and alſo that of $-y^2\dot{x}$ (or $-a^2\dot{x}+x^2\dot{x}$, found the common Way) we get, in the Whole, $\frac{1}{2}\overline{a-x}\times z^2+\overline{2ay-yx}\times z+\frac{1}{2}ay^2+a^2x+\frac{1}{3}x^3$; which, multiply'd by p, and corrected, gives, p *into* $\frac{1}{2}\overline{a-x}\times z^2+\overline{2ay-yx}\times z+\frac{1}{2}ay^2+a^2x+\frac{1}{3}x^3-\frac{4}{3}a^4$, for the true Fluent that was to be determined.

PROB. V.

346. *Suppoſing* H *to denote the Fluent of* $\overline{k+lz^n}|^r\times z^{vn-1}\dot{z}$; *to find the whole Fluent of* $H\times\overline{a-bz^n}|^m\times z^{qn-1}\dot{z}$, *(when* $a-bz^n$ *becomes equal to Nothing.)*

By reſolving $\overline{k+lz^n}|^r\times z^{vn-1}\dot{z}$ into ſimple Terms, and taking the Fluent, the ordinary Way, we get $H=\frac{k^rz^{vn}}{n}\times\frac{1}{v}+\frac{rlz^n}{\overline{v+1}\cdot k}+\frac{r\cdot\overline{r-1}\cdot l^2z^{2n}}{2\cdot\overline{v+2}\cdot k^2}$ &c. Which Value being ſubſtituted above, and p wrote inſtead of $q+v$, we ſhall have $H\times\overline{a-bz^n}|^m\times z^{qn-1}\dot{z}=\frac{k^r}{n}\times\overline{a-bz^n}|^m\times z^{pn-1}\dot{z}$ *into* $\frac{1}{v}+\frac{rlz^n}{\overline{v+1}\cdot k}+\frac{r\cdot\overline{r-1}\cdot l^2z^{2n}}{2\cdot\overline{v+2}\cdot k^2}+\frac{r\cdot\overline{r-1}\cdot\overline{r-2}\cdot l^3z^{3n}}{2\cdot 3\cdot\overline{v+3}\cdot k^3}$ &c.

Let, now, the Fluent of $\overline{a-bz^n}|^m\times z^{pn-1}\dot{z}$ (in the propoſed Circumſtance) be denoted by A, and put $t=p+m+1$; then it follows, from *Art.* 286. (by writing $\frac{1}{v}$ for e, $\frac{rl}{\overline{v+1}\cdot k}$ for f, &c.) that $\frac{k^r}{n}\times A$ *into* $\frac{1}{v}+$

$$\frac{p \cdot r}{t \cdot \overline{v+1}} \times \frac{al}{bk} + \frac{p \cdot \overline{p+1} \cdot r \cdot \overline{r-1}}{t \cdot \overline{t+1} \cdot 2 \cdot \overline{v+2}} \times \overline{\frac{al}{bk}}\Big|^{2} +$$

$$\frac{p \cdot \overline{p+1} \cdot \overline{p+2} \cdot r \cdot \overline{r-1} \cdot \overline{r-2}}{t \cdot \overline{t+1} \cdot \overline{t+2} \cdot 2.3\,\overline{v+3}} \times \overline{\frac{al}{bk}}\Big|^{3} + \text{\&c. will be}$$

the true Value of the Fluent. *Q. E. I.*

Note, p and $m+1$ muſt here be poſitive Quantities *; * Art. 286. and it is alſo requiſite that $\frac{l}{k}$ ſhould be greater than $-\frac{b}{a}$; otherwiſe the Fluent will fail.

Ex. 1. *Let* $\dot{H} = \overline{1-y^2}\big|^{-\frac{1}{2}} \times \dot{y}$; *and let the whole Fluent of* $H \times \overline{1-y^2}\big|^{-\frac{1}{2}}\dot{y}$, *be demanded.*

Then, k being $= 1$, $l = -1$, $z = y$, $n = 2$, $r = -\frac{1}{2}$, $v = \frac{1}{2}$; alſo $a = 1$, $b = 1$, $m = -\frac{1}{2}$, $q = \frac{1}{2}$; $p\ (= q+v) = 1$, $t\ (= p+m+1) = \frac{3}{2}$, and A (= the *whole* Fluent of $\overline{1-y^2}\big|^{-\frac{1}{2}} y\dot{y}$) $= 1$; we ſhall, by ſubſtituting theſe ſeveral Values above, get $1 + \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 5} + \frac{1}{7 \cdot 7} + \frac{1}{9 \cdot 9} + \frac{1}{11 \cdot 11}$ &c. = Fluent of $H \times \overline{1-y^2}\big|^{-\frac{1}{2}} \times \dot{y}$ (or $H\dot{H}$) when $y = 1$. Which Fluent being alſo expreſſed by $\frac{H^2}{2}$, it follows that $\frac{H^2}{2} = \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81}$ &c. Where H is $\frac{1}{4}$ of the Periphery of the Circle whoſe Radius is Unity.

Ex. 2. *Let* $\dot{H} = \overline{c^2 + z^2}\big|^{-\frac{3}{2}} \times \dot{z}$; *to find the Fluent of* $H \times \overline{b^2 - z^2}\big|^{-\frac{1}{2}} \times c^2\dot{z}$.

Here, $k = c^2$, $l = 1$, $n = 2$, $r = -\frac{3}{2}$, $v = \frac{1}{2}$; alſo $a = b^2$, $b = 1$, $m = -\frac{1}{2}$, $q = \frac{1}{2}$, $p\,(q+v) = 1$, $t\,(p+m+1) = \frac{3}{2}$, and A (= *whole* Fluent of $\overline{b^2 - z^2}\big|^{-\frac{1}{2}} \times z\dot{z}$) $= b$: Whence, by Subſtitution, we have $c^{-3} b \times \overline{1 - \frac{1}{3} \times \frac{b^2}{c^2} + \frac{1}{5} \times \frac{b^4}{c^4} - \frac{1}{7} \times \frac{b^6}{c^6}}$ &c. which, multiplied by c^2 (the Coefficient of $\dot{z}$) gives $\frac{1}{c} \times \overline{b - \frac{b^3}{3c^2} + \frac{b^5}{5c^4} - \frac{b^7}{7c^6}}$ &c. for the true Fluent in this Caſe: Where the Series is *that* expreſſing the Arch of the Circle whoſe Tangent is b and Radius c [*]; and is therefore equal to $c \times$ Arch, whoſe Radius is Unity and Tangent $= \frac{b}{c}$: Whence this laſt Arch (taken without the multiplicator c) is the true Value of the Fluent.

[*] Art. 142.

SECTION VII.

Shewing how Fluents, found by Means of Infinite Serieſes, are made to converge.

347. IT is found, in *Art.* 85. that the Fluent of $\overline{a + cz^n}\big|^{m} \times dz^{qn-1}\,\dot{z}$, in an infinite Series, (making $m+q=s$) is expreſſed by $\frac{\overline{a+cz^n}\big|^{m+1} \times dz^{qn}}{qna} \times$

$\overline{1 - \frac{\overline{s+1}\,.\,cz^n}{\overline{q+1}\,.\,a} + \frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2z^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^2}} - $ &c. Whence it follows (and is evident by bare Inſpection) that the Fluent of $\overline{a-cy^n}|^{r} \times y^{qn-1}\dot{y}$ (where the ſecond Term under the *Vinculum* is negative) will be truly defined by $\frac{\overline{a-cy^n}|^{r+1} \times y^{qn}}{qna}$ into $1 + \frac{\overline{s+1}\,.\,cy^n}{\overline{q+1}\,.\,a} + \frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^2}$ + &c. ſuppoſing $s = r+q$.

But, beſides the Series here given, and Thoſe, *in Art.* 83. 84. expreſſing the ſame Value, the Fluent of $\overline{a-cy^n}|^{r} \times y^{qn-1}\dot{y}$ will, yet, admit of another Form, different from all of them; by means whereof and *that* above, we ſhall be enabled to draw out ſome very uſeful Concluſions.

348. Put $z^n = \frac{ay^n}{a-cy^n}$; then $y^n = \frac{az^n}{a+cz^n}$, and therefore $ny^{n-1}\dot{y} = \frac{na^2z^{n-1}\dot{z}}{\overline{a+cz^n}|^{2}}$; alſo $a-cy^n = \frac{a^2}{a+cz^n}$, and $y^{qn-1}\dot{y}\ (= y^{qn-n} \times y^{n-1}\dot{y}) = \frac{a^{q+1}z^{qn-1}\dot{z}}{\overline{a+cz^n}|^{q+1}}$; and conſequently $\overline{a-cy^n}|^{r} \times y^{qn-1}\dot{y} = a^{2r+q+1} \times \overline{a+cz^n}|^{-r-q-1} \times z^{qn-1}\dot{z}$: Which Fluxion, ſo tranſformed, being compared with $\overline{a+cz^n}|^{m} \times dz^{qn-1}\dot{z}$; we have $m = -r-q-1$, $d = a^{2r+q+1}$, and $s\ (q+m) = -r-1$; whence, by ſubſtituting theſe Values in the firſt Series, above given, the Fluent ſought will be had $= \frac{\overline{a+cz}|^{-r-q} \times a^{2r+q} \times z^{qn}}{qn} \times \overline{1 + \frac{rcz^n}{\overline{q+1}\,.\,a} + \frac{r\,.\,\overline{r-1}\,.\,c^2z^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^2} + \frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3z^{3n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,\overline{q+3}\,.\,a^3}}$ &c.

Which, by reſtoring y (or writing $\frac{a^2}{a-cy^n}$ and $\frac{ay^n}{a-cy^n}$ for their Equals $a+cx^n$, and z^n) becomes

$$\frac{\overline{a-cy^n}\rceil^{r}\times y^{qn}}{qn}\times\overline{1+\frac{r}{q+1}\times\frac{cy^n}{a-cy^n}+\frac{r\cdot\overline{r-1}}{\overline{q+1}\cdot\overline{q+2}}\times\frac{c^2y^{2n}}{\overline{a-cy^n}\rceil^{2}}}$$

&c. the true Fluent, of $\overline{a-cy^n}\rceil^{r}\times y^{qn-1}\dot{y}$.

349. This Fluent may be otherwiſe found, independent of *that* above, in the following Manner:

It is evident, by taking the Fluxion of $\frac{\overline{a-cy^n}\rceil^{r}\times y^{qn}}{qn}$ (which Quantity would be the Fluent ſought, if $\overline{a-cy^n}\rceil^{r}$ was conſtant) that $\frac{\overline{a-cy^n}\rceil^{r}\times y^{qn}}{qn}$ is $=$ the Fluent of $\overline{a-cy^n}\rceil^{r}\times y^{qn-1}\dot{y}$ — Fluent of $\frac{rc}{q}\times\overline{a-cy^n}\rceil^{r-1}\times y^{qn+n-1}\dot{y}$: This Equation, by tranſpoſing the laſt Term, and writing x in the room of $a-cy^n$ (for the Sake of Brevity) will become $Flu.\ x^ry^{qn-1}\dot{y}=\frac{x^ry^{qn}}{qn}+\frac{rc}{q}\times Flu.\ x^{r-1}y^{qn+n-1}\dot{y}$. From the very ſame Argument (if, inſtead of r, we ſubſtitute $r-1$, $r-2$ &c. ſucceſſively; and, for q. write $q+1$, $q+2$, $q+3$, &c. reſpectively) we ſhall, alſo, have

$$Flu.\ x^{r-1}y^{qn+n-1}\dot{y}=\frac{x^{r-1}y^{qn+n}}{\overline{q+1}\cdot n}+\frac{\overline{r-1}\cdot c}{q+1}\times$$

$$Flu.\ x^{r-2}y^{qn+2n-1}\dot{y};$$

$$Flu.\ x^{r-2}y^{qn+2n-1}\dot{y}=\frac{x^{r-2}y^{qn+2n}}{\overline{q+2}\cdot n}+\frac{\overline{r-2}\cdot c}{q+2}\times$$

$$Flu.\ x^{r-3}y^{qn+3n-1}\dot{y};$$

&c. &c.

Whence, by ſubſtituting theſe Values, one by one, in that of, *Flu.* $x^r y^{qn-1}\dot{y}$, we get

$$Flu.\ x^r y^{qn-1}\dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rc}{q} \times \frac{x^{r-1} y^{qn+n}}{\overline{q+1}\,.\,n} + \frac{r\,.\,\overline{r-1}\,.\,c^2}{q\,.\,\overline{q+1}}$$

$$\times\ Flu.\ x^{r-2} y^{qn+2n-1}\dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rcx^{r-1} y^{qn+n}}{q\,.\,\overline{q+1}\,.\,n} +$$

$$\frac{r\,.\,\overline{r-1}\,.\,c^2}{q\,.\,\overline{q+1}} \times \frac{x^{r-2} y^{qn+2n}}{\overline{q+2}\,.\,n} + \frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3}{q\,.\,\overline{q+1}\,.\,\overline{q+2}} \times$$

$$Flu.\ x^{r-3} y^{qn+3n-1}\dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rcx^{r-1} y^{qn+n}}{q\,.\,\overline{q+1}\,.\,n} +$$

$$\frac{r\,.\,\overline{r-1}\,.\,c^2 x^{r-2} y^{qn+2n}}{q\,.\,\overline{q+1}\,.\,\overline{q+2}\,.\,n} + \frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3 x^{r-3} y^{qn+3n}}{q\,.\,\overline{q+1}\,.\,\overline{q+2}\,.\,\overline{q+3}\,.\,n}$$

&c. Where the Law of Continuation is manifeſt; and where, by making $\frac{x^r y^{qn}}{qn}$ a general Multiplicator, we ſhall have the very Series above exhibited.

350. From the Equality of the two foregoing Expreſſions, for the Fluent of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$, (or $x^r y^{qn-1}\dot{y}$) the Buſineſs of finding Fluents, by infinite Serieſes, will, in many Caſes, be very much facilitated.

For, in the firſt Place, it follows (by dividing both by $\frac{\overline{a-cy^n}|^{r+1} \times y^{qn}}{qna}$, or $\frac{x^{r+1} y^{qn}}{qna}$) that the Serieſes $1 + \frac{\overline{s+1}\,.\,cy^n}{\overline{q+1}\,.\,a} + \frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2 y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^2}$ &c. and $\frac{a}{x} \times$

$$\overline{1 + \frac{rcy^n}{\overline{q+1}\,.\,x} + \frac{r\,.\,\overline{r-1}\,.\,c^2 y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,x^2} + \frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3 y^{3n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,\overline{q+3}\,.\,x^3} +}$$

&c. muſt alſo be equal to each other, let the ſeveral

Quantities, therein concerned, be what they will (which may be otherwiſe proved, independent of Fluxions.) Therefore, if in the room of q and s we write any other Quantities p and t, the Equation will, *ſtill*, hold, and will then become $1 + \frac{\overline{t+1}\cdot cy^n}{\overline{p+1}\cdot a} + \frac{\overline{t+1}\cdot\overline{t+2}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot a^2} + \&c. = \frac{a}{x} \times \overline{1 + \frac{rcy^n}{\overline{p+1}\cdot x} + \frac{r\cdot\overline{r-1}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot x^2}\ \&c.}$ (t being $= p+r$.)

Moreover, if as many Terms of the firſt Series $1 + \frac{\overline{s+1}\cdot cy^n}{\overline{q+1}\cdot a} + \frac{\overline{s+1}\cdot\overline{s+2}\cdot c^2y^{2n}}{\overline{q+1}\cdot\overline{q+2}\cdot a^2} + \frac{\overline{s+1}\cdot\overline{s+2}\cdot\overline{s+3}\cdot c^3y^3}{\overline{q+1}\cdot\overline{q+2}\cdot\overline{q+3}\cdot a^3}$ &c. be taken as are denoted by any given Number v, and the laſt of them be repreſented by Q, it is evident, from the Law of the Series, that the firſt of the remaining Terms will be expreſſed by $Q \times \frac{s+v}{q+v} \times \frac{cy^n}{a}$; the ſecond, of them, by $Q \times \frac{\overline{s+v}}{q+v} \times \frac{\overline{s+v+1}}{q+v+1} \times \frac{c^2y^{2n}}{a^2}$ &c. and therefore the Sum of all of them (putting $q+v=p$ and $s+v$ $(=r+q+v) = t$) will be $= Q \times \frac{t}{p} \times \frac{cy^n}{a} + Q \times \frac{t}{p} \times \frac{t+1}{p+1} \times \frac{c^2y^{2n}}{a^2} + \&c. =$

$$\frac{tQcy^n}{pa} \times \overline{1 + \frac{\overline{t+1}\cdot cy^n}{\overline{p+1}\cdot a} + \frac{\overline{t+1}\cdot\overline{t+2}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot a^2} + \&c.}$$

$$= \frac{tQcy^n}{px} \times \overline{1 + \frac{rcy^n}{\overline{p+1}\cdot x} + \frac{r\cdot\overline{r-1}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot x^2}\ \&c.}$$

(by writing the Series found above in the room of its Equal) and conſequently the whole Series (including the v firſt Terms) $= 1 + \frac{\overline{s+1}\cdot cy^n}{\overline{q+1}\cdot a} +$

$$\frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^2}\ (v) + \frac{tQcy^n}{px} \times \overline{1 + \frac{rcy^n}{\overline{p+1}\,.\,x} + \frac{r\,.\,\overline{r-1}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,x^2} + \frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3y^{3n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,\overline{p+3}\,.\,x^3}} + \&c.$$

Which, drawn into the general Multiplicator $\frac{x^{r+1} \times y^{qn}}{qna}$ (*vid. Art.* 347.) will give the Fluent of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$ (or $x^r y^{qn-1}\dot{y}$) according to a new Form, compounded out of the two preceding ones; where the ſecond Series (the Value of p being large in reſpect of r) will always converge much faſter than the remaining Part of the firſt, for which it is ſubſtituted: But this will, more fully, appear from what follows hereafter. It will be proper to take notice here that the Fluent of $\overline{a+cz^n}|^m \times z^{qn-1}\dot{z}$ (the Fluxion firſt propoſed, where the ſecond Term under the Vinculum is poſitive) will alſo be had from hence (by writing z for y, m for r, and $-c$ for c) and is therefore equal to $\frac{x^{m+1}z^{qn}}{qna}$ drawn into the Sum of the two following Serieſes,

$$1 - \frac{\overline{s+1}.cz^n}{\overline{q+1}\,.\,a} + \frac{\overline{s+1}.\overline{s+2}.c^2z^{2n}}{\overline{q+1}.\overline{q+2}.a^2} - \frac{\overline{s+1}.\overline{s+2}.\overline{s+3}.c^3z^{3n}}{\overline{q+1}.\overline{q+2}.\overline{q+3}\,.\,a^3}\ (v)$$

$$-\frac{tQcz^n}{px} \times \overline{1 - \frac{mcz^n}{\overline{p+1}\,.\,x} + \frac{m\,.\,\overline{m-1}\,.\,c^2z^{2n}}{\overline{p+1}\,.\,\overline{p+2}.x^2} - \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,c^3z^{3n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,\overline{p+3}\,.\,x^3}} + \&c.$$

Where, $s = m + q$, $p = v + q$, $t = s + v$, $x = a + cz^n$, and Q = the laſt Term of the firſt Series continued to v Terms, v being any whole Number, at pleaſure. A few Examples will ſhew the Uſe of what is above delivered.

351. Ex. 1. *Let* $\frac{\dot{z}}{1+z}$, or $\overline{1+z}|^{-1}\dot{z}$, *be propounded.* Which being compared with $\overline{a+cx^n}|^m \times x^{qn-1}\dot{x}$, we have $a=1$, $c=1$, $n=1$, $x=1+z$, $m=-1$, $qn-1=0$, or $q=1$; whence alſo $s\ (m+q) = 0$, $p\ (v+q) = v+1$, $t\ (s+v) = v$, and conſequently the Fluent itſelf (by ſubſtituting theſe ſeveral Values in the laſt general Theorem) $= z$ into $1 - \frac{z}{2} + \frac{z^2}{3} - \frac{z^3}{4}\ (v) - \frac{v\,1\,\mathcal{Q}}{\overline{v+1}\,.\,x} \times \overline{1 + \frac{z}{\overline{v+2}\,.\,x} + \frac{2\,.\,z^2}{\overline{v+3}.\overline{v+3}\,x^2} + \frac{2\,.\,3\,.\,z^3}{\overline{v+2}.\overline{v+3}.\overline{v+4}.x^3}}$ &c. Where ($\mathcal{Q}$) the laſt Term of the firſt Series being $\pm\frac{z^{v-1}}{v}$, the Multiplicator $\left(\frac{vz\mathcal{Q}}{\overline{v+1}\,.\,x}\right)$ to the Second, will be $= \mp\frac{z^v}{\overline{v+1}\,.\,x}$; and ſo the Fluent itſelf will be reduced to $z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4}\ (v) \mp \frac{z^{v+1}}{\overline{v+1}.x} \times \overline{1 + \frac{z}{\overline{v+2}\,.\,x} + \frac{2\,.\,z^2}{\overline{v+2}\,.\,\overline{v+3}\,.\,x^2}}$ + &c. In which the Signs — and +, before z^{v+1}, obtain alternately, according as v is an odd or even Number. But, to ſhew the Advantage of expreſſing the Fluent in this Manner, by two different Serieſes, let $z=1$, and let v be taken $= 8$; then the Value of the firſt Series (continued to 8 Terms) being $= 0{,}6345238$ &c. and *That* of the ſecond Series $= \frac{1}{18} + \frac{A}{20} + \frac{2B}{22} + \frac{3C}{24} + \frac{4D}{26} + \frac{5E}{28}$ &c. (where *A*, *B*, *C*, *D* &c. denote the Terms preceding thoſe where they ſtand) $= 0{,}0555555 + 0{,}0027778 + 0{,}0002525 + 0{,}0000316 + 0{,}0000048 + 0{,}0000009 + 0{,}0000002 = 0{,}0586233$; it is evident that,

that the Fluent of $\frac{\dot{z}}{1+z}$, when z becomes $= 1$, will be $= 0{,}6345238 + 0{,}0586233 = 0{,}6931471$: Which is true to the very laſt Decimal Place; and would have required, at leaſt, 100000 Terms of the firſt, or common, Series.

352. Ex. 2. *Let the Fluent of* $\frac{\dot{z}}{1+z^2}$ *(expreſſing the Arch whoſe Radius is* 1 *and Tangent* z*) be required.*

In this Caſe we have $a=1$, $c=1$, $n=2$, $x=1+zz$, $m=-1$, $qn-1=0$, or $q=\frac{1}{2}$, $s=-\frac{1}{2}$, $p=v+\frac{1}{2}$, and the Fluent itſelf $= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7}\,(v) \pm \overline{\frac{z^{2v+1}}{\overline{2v+1}\cdot x} \times 1 + \frac{2\cdot z^2}{\overline{2v+3}\cdot x} + \frac{2\cdot 4\cdot z^4}{\overline{2v+3}\cdot\overline{2v+5}\cdot x^2} + \frac{2\cdot 4\cdot 6\cdot z^6}{\overline{2v+3}\cdot\overline{2v+5}\cdot\overline{2v+7}\cdot x^3}}$ &c. Where, if z be taken $=1$, and $v=6$, we ſhall have $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{26} \times \overline{1 + \frac{1}{15} + \frac{1}{15}\times\frac{2}{17} + \frac{1}{15}\times\frac{2}{17}\times\frac{3}{19}}$ &c. $= 0{,}785398 =$ the Fluent of $\frac{\dot{z}}{1+z^2}$ when $z = 1$ ($= \frac{1}{8}$ of the Periphery of the foreſaid Circle) Which Number, brought out by taking, only, 8 Terms of the ſecond Series, is more exact than if 100000 Terms of the common Series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ &c. had been uſed. And, if z be taken $= \sqrt{\frac{1}{3}}$ ($=$ Tangent of 30°) and $v=6$, as before, the ſame Number of Terms will be ſufficient to give the Anſwer, true to twice the Decimal Places above exhibited.

353. Ex.

353. Ex. 3. *Let the Fluxion propoſed be* $\overline{e^4+y^4}|^{\frac{1}{2}}\times\dot{y}$. Here we have, $a=e^4$, $c=1$, $z=y$, $n=4$, $x=e^4+y^4$, $m=\frac{1}{2}$, $q=\frac{1}{4}$, $s\ (m+q)=\frac{3}{4}$, $p\ (v+q)=v+\frac{1}{4}$; $t\ (s+v)=v+\frac{3}{4}$; and therefore the Fluent ſought (by Subſtitution) is $=\frac{x^{\frac{3}{2}}y}{e^4}$ into $1-\frac{7y^4}{5e^4}+\frac{7\cdot 11y^8}{5\cdot 9e^8}-\frac{7\cdot 11\cdot 15y^{12}}{5\cdot 9\cdot 13e^{12}}\ (v)-\frac{\overline{4v+3}\cdot Qy^4}{\overline{4v+1}\cdot x}\times 1-\frac{2y^4}{\overline{4v+5}\cdot x}-\frac{2\cdot 2y^8}{\overline{4v+5}\cdot\overline{4v+9}\cdot x^2}-\frac{2\cdot 6\cdot 2y^{12}}{\overline{4v+5}\cdot\overline{4v+9}\cdot\overline{4v+13}\cdot x^3}$ &c. in which (as in all other Caſes) Q denotes the laſt Term of the firſt Series. This Fluent approximates equally faſt with thoſe in the foregoing Examples: And it may be obſerved farther, that the Fluent will always converge, however great the Value of z is taken, if both a and c, in the general Fluxion $\overline{a+cz^n}|^m\times z^{pn-1}\dot{z}$, are poſitive Quantities. But, if the ſecond Term under the Vinculum be negative, the Caſe will be otherwiſe, when that Term becomes greater than half the Firſt; ſince the Powers of $\frac{cz^n}{x}$, in the latter Part of the Fluent, will then form an increaſing Geometrical Progreſſion. It may, therefore, be of uſe to ſhew how the Theorem may be varied ſo as to anſwer in this Caſe. In order thereto, if in the Equations $s=r+q$, and $1+\frac{\overline{s+1}\cdot cy^n}{\overline{q+1}\cdot a}+\frac{\overline{s+1}\cdot\overline{s+2}\cdot c^2y^{2n}}{\overline{q+1}\cdot\overline{q+2}\cdot a^2}$ &c. $=\frac{a}{x}\times 1+\frac{rcy^n}{\overline{q+1}\cdot x}+\frac{r\cdot\overline{r-1}\cdot c^2y^{2n}}{\overline{q+1}\cdot\overline{q+2}\cdot x^2}$ &c. (*given in Art.* 350.) you write k for r, and p for q, and multiply by $\frac{x}{a}$,

you will have $s = k + p$, and $\overline{1 + \frac{kcy^n}{\overline{p+1}\,.\,x} + \frac{k\,.\,\overline{k-1}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,x^2}}$ &c. $= \frac{x}{a} \times \overline{1 + \frac{\overline{s+1}\,.\,cy^n}{\overline{p+1}\,.\,a} + \frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,a^2}}$ &c.

Moreover, if the v first Terms of the above Series $1 + \frac{rcy^n}{\overline{q+1}\,.\,x} + \frac{r\,.\,\overline{r-1}\,.\,c^2y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,x^2}$ &c. be taken, and the last of them be denoted by Q, it is plain the first of the remaining Terms will be $= Q \times \frac{r-v+1}{q+v} \times \frac{cy^n}{x}$, the ſecond $= Q \times \frac{r-v+1}{q+v} \times \frac{r-v}{q+v+1} \times \frac{c^2y^{2n}}{x^2}$, &c. and the Sum of them all (putting $q + v = p$, and $r - v = k$) equal to $\frac{\overline{k+1}\,.\,Qcy^n}{px} \times \overline{1 + \frac{kcy^n}{\overline{p+1}\,.\,x} + \frac{k\,.\,\overline{k-1}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,x^2}}$ &c. $= \frac{\overline{k+1}\,.\,Qcy^n}{px} \times \frac{x}{a} \times \overline{1 + \frac{\overline{s+1}\,.\,cy^n}{\overline{p+1}\,.\,a} + \frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,a^2}}$ &c. (by the Equation above) and conſequently the Sum of the whole Series ($1 + \frac{rcy^n}{\overline{q+1}\,.\,x}$ &c.) $= 1 + \frac{rcy^n}{\overline{q+1}\,.\,x} + \frac{r\,.\,\overline{r-1}\,.\,c^2y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,x^2} + \frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3y^{3n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,\overline{q+3}\,.\,x^3}$ (v) $+ \frac{\overline{k+1} \times cy^n\,Q}{pa} \times \overline{1 + \frac{\overline{s+1}\,.\,cy^n}{\overline{p+1}\,.\,a} + \frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,a^2} + \text{\&c.}}$ Which, multiply'd by $\frac{x^r y^{qn}}{qn}$, gives the Fluent of $\overline{a-cy^n}\big|^{r}$ $\times y^{qn-1}\dot{y}$ (* or $x^r y^{qn-1}\dot{y}$) where $k = r - v$, $p = v + q$, s $(= k+p) = r+q$ and $x = a - cy^n$. I ſhall put down one Example of the Uſe of this laſt general Expreſſion; where we will take $\dot{y}\sqrt{2y-y^2}$ or $\overline{2-y}\big|^{\frac{1}{2}} \times y^{\frac{1}{2}}\dot{y}$ (being the Fluxion of the Area of the Circle whoſe Radius is Unity and verſed Sine y) In which Caſe, $a=2$, $c=1$, $n=1$, $r=\frac{1}{2}$, $qn-1=\frac{1}{2}$, or $q=\frac{3}{2}$, $k=-v+\frac{1}{2}$, $p=v+\frac{3}{2}$, $s=2$, $x=2-y$; and therefore the Fluent ſought $= \frac{2x^{\frac{1}{2}}y^{\frac{3}{2}}}{3}$ into

$$1 + \frac{y}{5x} - \frac{y^2}{5\cdot 7x^2} + \frac{3y^3}{5\cdot 7\cdot 9x^3} - \frac{3y^4}{7\cdot 9\cdot 11x^4} + \frac{3y^5}{9\cdot 11\cdot 13x^5}\ (v) \mp \frac{\overline{2v-3}\cdot y^{Q}}{\overline{2v+3}\cdot 2} \times \overline{1 + \frac{3y}{2v+5} + \frac{3\cdot 4y^2}{\overline{2v+5}\cdot\overline{2v+7}} + \frac{3\cdot 4\cdot 5y^3}{\overline{2v+5}\cdot\overline{2v+7}\cdot\overline{2v+9}}}\ \&c.$$

Which, if y be taken $=1$, and $v=5$, will become $= \frac{2}{3} + \frac{A}{5} - \frac{B}{7} + \frac{C}{3} - \frac{5D}{11} + \frac{7E}{2\times 13} + \frac{3F}{15} + \frac{4G}{17} + \frac{5H}{19}$ &c. $=0{,}785398$ (where A, B, C &c. denote the ſeveral Terms, reſpectively, without their Signs.) In bringing out which Concluſion, ſix Terms of the ſecond Series are required: But if y be taken $=\frac{1}{2}$ the Radius of the foreſaid Circle, then four Terms of each Series will be more than ſufficient to give the ſame Number of Decimal Places. And it may likewiſe be obſerved, that, although no general Rule can be laid down for aſſigning the Value of v, ſo as to anſwer the beſt in all Caſes, yet the Concluſion

* Art. 348, 349.

will, for the general Part, require the feweſt Terms, when the Number of thoſe, taken in each Series, is nearly the ſame.

354. But, after all, another Theorem or Series, ſtill, ſeems wanting, to expreſs the Value of the whole Fluent, when the Quantity under the *Vinculum* becomes equal to Nothing (which, in the Reſolution of Problems, is, commonly, what is required.) For, it is plain the laſt, above given, anſwers no better, *here*, than that preceding it; becauſe (the Diviſor (x) being Nothing) the former Part of it fails.

In order, therefore, to determine a proper *Form*, to obtain in this Circumſtance, it will be requiſite to obſerve, firſt of all, *from Article* 286. that the whole Fluent of $\overline{a-bz^n}|^m \times z^{pn+vn-1}\dot{z}$, ſuppoſing *that* of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$ to be denoted by A, will be truly expreſſed by $\frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}\ (v) \times \frac{a^v A}{b^v}$: In which $t=m+p+1$; and where it is requiſite that the Values of $m+1$ and p ſhould be poſitive, otherwiſe, A being infinite, the Fluent (or Compariſon) fails. Hence, becauſe the whole Fluent of $\overline{a-bz^n}|^m \times z^{n-1}\dot{z}$, (when $a-bz^n=0$) is found $=\frac{a^{m+1}}{\overline{m+1}\times nb}$, by the common Way*, it follows, by writing this Value in the Room of A, and expounding p by 1, that the *whole Fluent* of $\overline{a-bz^n}|^m \times z^{n+vn-1}\dot{z}$ is rightly expreſſed by $\frac{1}{m+2} \times \frac{2}{m+3} \times \frac{3}{m+4}\ (v) \times \frac{a^{m+v+1}}{\overline{m+1}\times nb^{v+1}}$, or by $\frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\ (v+1) \times \frac{a^{m+v+1}}{\overline{v+1}\times nb^{v+1}}$: Whence

That

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* Art. 77. and 78.

That of $\overline{a-bz^n}|^m \times z^{rn-1}\dot{z}$, by ſubſtituting r inſtead of $u+1$, will conſequently be equal to $\frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\,(r) \times \frac{a^{m+r}}{rnb^r}$. Let this Quantity be denoted by B; then, *by the ſame Article*, the Fluents of the ſeveral Terms of the Series $1, \frac{bz^n}{a}, \frac{b^2z^{2n}}{a^2}, \frac{b^3z^{3n}}{a^3}$ &c. drawn into the general Multiplicator $\overline{a-bz^n}|^m \times z^{rn-1}\dot{z}$, will be, reſpectively, expounded by thoſe of the Series $1, \frac{r}{t}, \frac{r\,.\,\overline{r+1}}{t\,.\,\overline{t+1}}, \frac{r\,.\,\overline{r+1}\,.\,\overline{r+2}}{t\,.\,\overline{t+1}\,.\,\overline{t+2}}$ &c. drawn into B; t being $=m+r+1$.

If now the Differences of the Quantities $1, \frac{r}{t}, \frac{r\,.\,\overline{r+1}}{t\,.\,\overline{t+1}}$ &c. be, continually, taken *; and for $r-t$ its Equal $-m-1$ be ſubſtituted, the Value of any Term of the Series, whoſe Diſtance from the firſt, excluſive, is denoted by s, or whoſe correſponding Term, in the preceding Series, is $\frac{b^sz^{sn}}{a^s}$, will be univerſally expreſſed by $1 - \frac{s\,.\,\overline{m+1}}{1\,.\,t} + \frac{s\,.\,\overline{s-1}\,.\,\overline{m+1}\,.\,\overline{m+2}}{1\,.\,2 \times t\,.\,\overline{t+1}} - \frac{s\,.\,\overline{s-1}\,.\,\overline{s-2}\,.\,\overline{m+1}\,.\,\overline{m+2}\,.\,\overline{m+3}}{1\,.\,2\,.\,3 \times t\,.\,\overline{t+1}\,.\,\overline{t+2}}$ + &c. Where, if s be interpreted by 0, 1, 2, 3 &c. ſucceſſively, you will have the Values $1, \frac{r}{t}, \frac{r\,.\,\overline{r+1}}{t\,.\,\overline{t+1}}$ &c. above exhibited: But, if s be taken as a Fraction, then the Value of ſuch an intermediate Term will be found as will give the Fluent of $\frac{b^s z^{sn}}{a^s} \times \overline{a - bz^n}|^m \times z^{rn-1}\dot{z}$, in any propoſed Circumſtance of s; which Fluent, it is evident, will therefore be expreſſed by $B \times \overline{1 - \frac{s . \overline{m+1}}{1 . t} + \frac{s . \overline{s-1} . \overline{m+1} . \overline{m+2}}{1 . 2 . t . \overline{t+1}}}$ &c. or its Equal $\frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\ (r) \times \frac{a^{m+r}}{rnb^r}$ into $\overline{1 - \frac{s . \overline{m+1}}{1 . t} - \frac{\overline{s-1} . \overline{m+2}}{2 . \overline{t+1}} \times E - \frac{\overline{s-2} . \overline{m+3}}{3 . \overline{t+2}} \times F - \frac{\overline{s-3} . \overline{m+4}}{4 . \overline{t+3}} \times G}$ &c. (where E, F, G &c. denote the Terms immediately preceding thoſe where they ſtand, under their proper Signs.) Whence, dividing by $\frac{b^s}{a^s}$, we have $\frac{1}{m+1} \times \frac{2}{m+2}\ (r) \times \frac{a^{m+r+s}}{rnb^{r+s}} \times \overline{1 - \frac{s . \overline{m+1}}{t} - \frac{\overline{s-1} . \overline{m+2}}{2 . \overline{t+1}} \times E}$, &c. for the true Fluent of $\overline{a - bz^n}|^m \times z^{rn+sn-1}\dot{z}$.

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* *See my Mathematical Eſſays*, p. 94.

From the laſt Fluent *that* of $\overline{a - bz^n}|^m \times z^{pn-1}\dot{z}$ (in which p denotes any poſitive Fraction, proper or improper) is very readily obtained: For, if the ſame (when $a - bz^n = 0$) be denoted *by* A; then the Fluent of $\overline{a - bz^n}|^m \times z^{pn+vn-1}\dot{z}$ will (according to the Article above quoted) be expreſſed by $\frac{p}{p+m+1} \times \frac{p+1}{p+m+2} \times \frac{p+2}{p+m+3}\ (v) \times \frac{a^v A}{b^v}$; ſuppoſing v any poſitive Integer. Therefore, by making $\overline{a-bz^n}\big|^m \times z^{rn+sn-1}\dot{z} = \overline{a-bz^n}\big|^m \times z^{pn+vn-1}\dot{z}$, or $r+s=p+v$, the correſponding Fluents muſt, alſo, be equal; that is, $\frac{p}{p+m+1} \times \frac{p+1}{p+m+2}\,(v) \times \frac{a^v A}{b^v} = \frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\,(r) \times \frac{a^{m+p+v}}{rnb^{p+v}} \times \overline{1 - \frac{s\,.\,\overline{m+1}}{t} - \&c.}$ And conſequently A (the whole Fluent of $\overline{a-bz^n}\big|^m \times z^{pn-1}\dot{z}) = \frac{p+m+1}{p} \times \frac{p+m+2}{p+1} \times \frac{p+m+3}{p+2}\,(v) \times \frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\,(r) \times \frac{a^{m+p}}{nrb^p} \times$ into the Series $1 - \frac{s\,.\,\overline{m+1}}{1\,.\,t} - \frac{\overline{s-1}\,.\,\overline{m+2}}{2\,.\,\overline{t+1}}\,E - \frac{\overline{s-2}\,.\,\overline{m+3}}{3\,.\,\overline{t+2}}\,F - \frac{\overline{s-3}\,.\,\overline{m+4}}{4\,.\,\overline{t+3}}\,G$ &c. where $t=r+m+1$ and $s=p+v-r$; v and r being any whole poſitive Numbers at pleaſure.

355. An Example, or two, of the Uſe of this Concluſion, may be proper.

1°. Let the *whole Fluent* of $\overline{1-x^2}\big|^{-\frac{1}{2}}\dot{x}$ (expreſſing the Length of $\frac{1}{4}$ of the Periphery of the Circle whoſe Radius is Unity) be demanded. In which Caſe, a being $+1$, $b=1$, $m=-\frac{1}{2}$, $n=2$, $p=\frac{1}{2}$, $t=r+\frac{1}{2}=\frac{2r+1}{2}$, and $s=v-r+\frac{1}{2}=\frac{2v-2r+1}{2}$, the Fluent ſought will, therefore, (by ſubſtituting theſe Values) be had $= \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5}\,(v) \times \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5}\,(r) \times$

$\frac{1}{2r}$ into $1 - \frac{1 \cdot \overline{2v - 2r + 1}}{2 \cdot \overline{2r + 1}} - \frac{3 \cdot \overline{2v - 2r - 1}}{4 \cdot \overline{2r + 3}} E - \frac{5 \cdot \overline{2v - 2r - 3}}{6 \cdot \overline{2r + 5}} F - \frac{7 \cdot \overline{2v - 2r - 5}}{8 \cdot \overline{2r + 7}} G$ &c. Which, by expounding v by 5 and r by 3, will become = 2,16719 &c. into $1 - \frac{1 \cdot 5}{2 \cdot 7} - \frac{3 \cdot 3}{4 \cdot 9} E - \frac{5 \cdot 1}{6 \cdot 11} F + \frac{7 \cdot 1}{8 \cdot 13} G + \frac{9 \cdot 3}{10 \cdot 15} H + \frac{11 \cdot 5}{12 \cdot 17} I +$ &c. = 1,5708. In the bringing out of which Value, all the Terms above exhibited are requiſite: But, of the common Series, $1 + \frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} +$ &c. more than 10 times that Number of Terms would be neceſſary to anſwer with the ſame Degree of Exactneſs.

Ex. 2°. *Let the Fluxion propoſed be* $\frac{d\dot{x}}{x^{\frac{1}{2}}\sqrt{d^2 - x^2}}$ (whoſe whole Fluent, when $x = d$, expreſſes the Time of Deſcent of a heavy Body in half the Arch of a Semicircle, whoſe Radius is d *.)

Here, by comparing $\overline{d^2 - x^2}|^{-\frac{1}{2}} \times x^{-\frac{1}{2}}\dot{x}$ with $\overline{a - bz^n}|^m \times z^{pn-1}\dot{z}$, we have $a = d^2$, $b = 1$, $n = 2$, $pn - 1 = -\frac{1}{2}$, or $p = \frac{1}{4}$; alſo $s\ (p + v - r) = v - r + \frac{1}{4}$, $t\ (r + m + 1) = r + \frac{1}{2}$: Whence, by taking r and v, each, equal to 4, the Fluent, itſelf, comes out = $\frac{3}{1} \times \frac{7}{5} \times \frac{11}{9} \times \frac{15}{13}$ into $\frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times \frac{d^{\frac{1}{2}}}{8}$ into $1 - \frac{1 \cdot 1}{4 \cdot 9} + \frac{3 \cdot 3}{8 \cdot 11} E + \frac{7 \cdot 5}{12 \cdot 13} F + \frac{11 \cdot 7}{16 \cdot 15} G +$ &c. $= \frac{11 \cdot 16 d^{\frac{1}{2}}}{13 \cdot 5}$ into $\overline{1 - \frac{1}{36} + \frac{9}{88} + \text{\&c.}}$

* Art. 207.

$= 2{,}6215d^{\frac{1}{2}}$: Which is to $2\sqrt{2d}$, the Time of Deſcent along the vertical Diameter of the foreſaid Circle, as 2,6215 to 2,8284, or as 100 to 108, nearly.

After the ſame Manner the Fluent will be found in other Caſes: But, with regard to the aſſigning of the Values of r and v, it may be obſerved, that the Anſwer will, commonly, be brought out with the leaſt Trouble when v is taken greater by an Unit or two than r; which laſt Quantity muſt be greater or leſs, according as a greater or leſs Degree of Exactneſs is neceſſary.——From the foregoing Expreſſions, by varying the Values of v and r, a great Number of Theorems, for the Summation of Serieſes, may be deduced. But this being foreign to my preſent Purpoſe, I am not at Leiſure to purſue it here.

356. Hitherto Regard has been had to Fluxions of the Binomial-Kind: But, from thence, the Fluents of Trinomials may alſo be found; when theſe laſt can be reduced to Binomials (*by Art.* 307.) without introducing new Radical Quantities.——Beſides which Method, I ſhall, here, give another, which will anſwer where that fails, and is alſo applicable to *Multinomials*.

In order thereto, let the Fluent of $\overline{a+cz^{n}}|^{m} \times z^{pn-1}\dot{z}$, be denoted by A; and let it be required to find, from thence, the Fluent of the Radical *Multinomial*, or Infinite Series, $\overline{a+cx^{n}+dx^{2n}+ex^{3n}+fx^{4n}\text{ \&c.}}|^{m} \times x^{pn-1}\dot{x}$.

Make $cz^{n} = cx^{n} + dx^{2n} + ex^{3n} + \text{\&c.}$ and $y = x^{pn}$; then, x^{n} being $= y^{\frac{1}{p}}$, if this Value be ſubſtituted for x^{n}, in the firſt Equation, it will become $cz^{n} = cy^{\frac{1}{p}} + dy^{\frac{2}{p}} + ey^{\frac{3}{p}}$ &c. Whence, by reverting the Series, (*by Art.*

Art. 275.) y (x^{pn}) is found $= z^{pn} + Rz^{pn+n} + Sz^{pn+2n} + Tz^{pn+3n} + \&c.$

Where $R = -\frac{pd}{c}$, $S = \frac{p\,.\,\overline{p+3}}{2} \times \frac{d^2}{c^2} - \frac{pe}{c}$, $T = -\frac{p\,.\,\overline{p+4}\,.\,\overline{p+5}}{6} \times \frac{d^3}{c^3} + p\,.\,\overline{p+4} \times \frac{de}{c^2} - \frac{pf}{c}$ &c.

Moreover, by taking the Fluxion of the Equation thus brought out, and dividing by pn, we have $x^{pn-1}\dot{x} = z^{pn-1}\dot{z} + \frac{p+1}{p} \times Rz^{pn+n-1}\dot{z} + \frac{p+2}{p} \times Sz^{pn+2n-1}\dot{z} + \frac{p+3}{p} \times Tz^{pn+3n-1}\dot{z} + \&c.$

Now let this Value, with that of $cx^n + dx^{2n} + ex^{3n} + \&c.$ (given above) be substituted in the proposed Fluxion, and it will become $\overline{a+cz^n}|^m \times \overline{z^{pn-1}\dot{z} + \frac{p+1}{p} \times Rz^{pn+n-1}\dot{z} + \frac{p+2}{p} \times Sz^{pn+2n-1}\dot{z} + \&c.}$

Alſo, let v denote the Place, or Diſtance, of any Term of this Series from the firſt, excluſive; then the Term itſelf, drawn into the general Multiplicator, will be expreſſed by $\overline{a+cz^n}|^m \times \frac{p+v}{p}\,\Delta\, z^{pn+vn-1}\dot{z}$ (Δ being the correſponding Coefficient R, S, T, &c.) and the Fluent thereof by $\frac{p+v}{p}\Delta \times \overline{a+cz^n}|^{m+1} \times z^{pn} \times$

$$\overline{\frac{z^{vn-n}}{\overline{s+1}\,.\,nc} - \frac{qaz^{vn-2n}}{\overline{s+1}\,.\,snc^2} + \frac{q\,.\,\overline{q-1}\,.\,a^2z^{vn-3n}}{\overline{s+1}\,.\,s\,.\,\overline{s-1}\,.\,nc^3}\,(v) \pm \frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}\,(v) \times \frac{p+v}{p} \times \Delta \times \frac{a^v A}{c^v}}$$ (*Art.* 283)

Where, $q=p+v-1$, $s=m+q$, $t=p+m+1$, and the Sign of the laſt Term is + or —, according as v is an even or odd Number. Now, if in the Fluent thus given, v be expounded by 1, 2, 3, 4, &c. ſucceſſively, it is evident the Fluent of the whole Expreſſion will, in all Circumſtances of z, be obtained. But, if the Coefficient c be negative, ſo that $a+cz^n$ may (by increaſing z) become equal to Nothing; then, in that Circumſtance, the Fluent of the foreſaid general Term $\overline{a+cz^n}|^m \times \frac{p+v}{p} \Delta z^{pn+vn-1}\dot{z}$ (or $\overline{a-bz^n}|^m \times \frac{p+v}{p} \Delta z^{pn+vn-1}\dot{z}$, making $-c=b$) being, *barely*, $=\frac{p}{t} \times$

* Art. 286. $\frac{p+1}{t+1} \times \frac{p+2}{t+2} (v) \times \frac{p+v}{p} \times \frac{\Delta a^v A}{b^v}$ *, it follows that the whole Fluent of the given Expreſſion, or its Equal, $\overline{a-bz^n}|^m \times \overline{z^{pn-1}\dot{z} + \frac{p+1}{p} Rz^{pn+n-1}\dot{z}}$ &c. will be truly repreſented by $A \times \overline{1 + \frac{\overline{p+1}\,.\,Ra}{tb} + \frac{\overline{p+1}\,.\,\overline{p+2}\,.\,Sa^2}{t\,.\,\overline{t+1}\,.\,b^2}}$ $+ \overline{\frac{\overline{p+1}\,.\,\overline{p+2}\,.\,\overline{p+3}\,.\,Ta^3}{t\,.\,\overline{t+1}\,.\,\overline{t+2}\,.\,b^3}}$ &c. In which, $R=\frac{pd}{b}$, $S=\frac{p\,.\,\overline{p+3}}{2} \times \frac{d^2}{b^2}+\frac{pe}{b}$, $T=\frac{p\,.\,\overline{p+4}\,.\,\overline{p+5}}{6} \times \frac{d^3}{b^3}+\frac{p\,.\,\overline{p+4}}{1} \times \frac{de}{bb}+\frac{pf}{b}$, &c. and $A=$ the Fluent $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$, when $a-bz^n=0$.

357. Hence, if the Fluxion given be of the Trinomial Kind (then, e, f, &c. vaniſhing the *whole Fluent*

of $\overline{a-bx^n+dx^{2n}}\big|^m \times x^{pn-1}\dot{x}$ (when $a-bx^n+dx^{2n}=0$) will, by ſubſtituting for *R*, *S*, *T*, &c. be $= A \times$

$$\overline{1+\frac{p\,.\,\overline{p+1}}{1\,.\,t}\times\frac{ad}{bb}+\frac{p\,.\,\overline{p+1}\,.\,\overline{p+2}\,.\,\overline{p+3}}{1\,.\,2\,.\,t\,.\,\overline{t+1}}\times\overline{\frac{ad}{bb}}\Big|^2+\frac{p\,.\,\overline{p+1}\,.\,\overline{p+2}\,.\,\overline{p+3}\,.\,\overline{p+4}\,.\,\overline{p+5}}{1\,.\,2\,.\,3\,.\,t\,.\,\overline{t+1}\,.\,\overline{t+2}}\times\overline{\frac{ad}{bb}}\Big|^3+\&c.}$$

358. If $m+1$ and p are the Halves of any odd Affirmative-Numbers, the *Fluent* of $\overline{a-bz^n}\big|^m \times z^{pn-1}\dot{z}$, when $a-bz^n=0$, will be equal to

$$\frac{1\,.\,3\,.\,5\,.\,7\,(m+\frac{1}{2})\times 1\,.\,3\,.\,5\,.\,7\,(p-\frac{1}{2})}{2\,.\,4\,.\,6\,.\,8\,.\,10\,.\,12\,(m+p)}\times\frac{a^{m+p}G}{nb^p}.^{*}$$

* Art. 295. and 298.

G being the Periphery of the Circle whoſe Diameter is Unity. Therefore the Fluent of $\overline{a-bx^n+dx^{2n}+ex^{3n}\,\&c.}\big|^m \times x^{pn-1}\dot{x}$, or its Equal, $\overline{a-bx^n}\big|^m \times x^{pn-1}\dot{x}+\frac{\overline{p+1}}{p}\times\overline{Rz^{pn+n-1}}\,\dot{z}$ &c. is found, in this Caſe, by multiplying the Expreſſion here given, into the foregoing Series, $1+\frac{\overline{p+1}\,.\,Ra}{tb}+$ &c.

359. An Example or two will help to ſhew the Uſe of what is above delivered.

Firſt, let the *Fluent* of $\frac{\dot{x}}{\sqrt{a^2-x^2-\frac{x^4}{raa}}}$ (when the Diviſor becomes equal to Nothing) be required.

Then, by comparing $\overline{a^2-x^2-\frac{x^4}{raa}}\Big|^{-\frac{1}{2}}$ with the general *Trinomial* $\overline{a - bx^n + dx^{2n}}\big|^m \times x^{pn-1}\dot{x}$, it appears that a^2 muſt be, here, wrote in the room of a, and that n, m, p, b and d, will be interpreted by 2; $-\frac{1}{2}$, $\frac{1}{2}$, 1, and $-\frac{1}{raa}$ reſpectively: Whence we have $t\,(p+m+1) = 1$, $\frac{1.3.5\,(m+\frac{1}{2}) \times 1.3.5\,(p-\frac{1}{2})}{2.4.6\,(m+p)}$ $\times \frac{a^{m+p}G}{nb^p} = \frac{G}{2}$, and the Fluent ſought $= \frac{G}{2} \times$

$$\overline{1 - \frac{1.3}{2.2r} + \frac{1.3.5.7}{2.2.4.4r^2} - \frac{1.3.5.7.9.11}{2.2.4.4.6.6r^3} + \&c.}$$

360. The ſecond Example ſhall be, to find the Fluent expreſſing the *Apſide Angle* in an Orbit deſcribed by means of a centripetal Force varying according to any Power of the Diſtance.

In which Caſe the given Fluxion being

$$\frac{\pm p\dot{x}}{x\sqrt{p^2 + \frac{2}{n+1} \times x^2 - p^2 + \frac{2x^{n+3}}{n+1}}}$$ (*Vid. Art.* 242.

where A is ſuppoſed the higher Apſe, and CA (and conſequently Cb) equal to Unity) we ſhall, by putting $1 - p^2 = \beta$, $\frac{n+3}{2} = v$, and $1 - x^2 = y$, reduce it to

$$\frac{\frac{1}{2}\sqrt{1-\beta} \times \dot{y}}{\overline{1-y} \times \sqrt{\beta y + \frac{1 - vy - \overline{1-y}\big|^v}{1-v}}} = \frac{1}{2}\sqrt{1-\beta} \times$$

$$\overline{\beta - \frac{vy}{2} + \frac{v.\overline{v-2}}{2.3}.y^2 - \frac{v.\overline{v-2}.\overline{v-3}}{2.3.4}.y^3 + \&c.}\Big|^{-\frac{1}{2}}.$$

$\times\, y^{-\frac{1}{2}}\dot{y} + y^{\frac{1}{2}}\dot{y} + y^{\frac{3}{2}}\dot{y} + y^{\frac{5}{2}}\dot{y} + \&c.$ Where the Quantity under the Radical Sign (now anſwering to the Form above preſcribed) being compared with $\overline{a - bx^{n} + dx^{2n} + ex^{3n}\ \&c.}\,\big|^{m}$, we have $m = -\frac{1}{2}$, $n = 1$, $b = \frac{v}{2}$, $\frac{d}{b}$, $= \frac{v-2}{3}$ $\frac{e}{b} = -\frac{\overline{v-2}\cdot\overline{v-3}}{3\cdot 4}$ &c. Alſo the Value of p with regard to the firſt Term $(y^{-\frac{1}{2}}\dot{y})$ will be $= \frac{1}{2}$ (becauſe $pn - 1 = -\frac{1}{2}$) likewiſe its Value in the ſecond Term $(y^{\frac{1}{2}}\dot{y})$ is $= \frac{3}{2}$; in the third $= \frac{5}{2}$ &c. In the firſt of theſe Caſes we, therefore, have $t\ (m+p+1) = 1$, $R\ \left(p \times \frac{d}{b}\right) = \frac{v-2}{6}$, $S = \frac{\overline{v-2}\cdot\overline{4v-5}}{72}$, $T = \frac{\overline{v-2}\cdot\overline{16v^{2}-37v+22}}{16\times 45}$. Whence it follows, that the Fluent of the firſt Term $\left(\overline{\beta - \frac{vy}{2} + \frac{v\cdot\overline{v-2}}{2\cdot 3}\cdot y^{2}\ \&c.}\,\Big|^{-\frac{1}{2}} \times y^{-\frac{1}{2}}\dot{y}\right)$ when the Quantity under the Radical Sign becomes equal to Nothing (or the Body arrives at its lower Apſe) will be truly expreſſed by $\frac{G}{\sqrt{\frac{1}{2}v}}$ into $1 + \frac{v-2}{2v}\cdot\beta + \frac{5\cdot\overline{v-2}\cdot\overline{4v-5}}{48v^{2}}\cdot\beta^{2} + \frac{7\cdot\overline{v-2}\cdot\overline{16v^{2}-37v+22}}{6\times 48v^{3}}\cdot\beta^{3}$ + &c.

In the ſame Manner it will appear, that the Fluent of the ſecond Term, in that Circumſtance, is $= \frac{G}{\sqrt{\frac{1}{2}v}} \times \overline{\frac{1}{v}\cdot\beta + \frac{5\cdot\overline{v-2}}{4v^{2}}\cdot\beta^{2} + \frac{35\cdot\overline{v-2}\cdot\overline{2v-3}}{48v^{3}}\cdot\beta^{3}}$

&c. that of the Third $= \frac{G}{\sqrt{\frac{1}{2}v}} \times \overline{\frac{3}{2v^2} \cdot \beta^2 + \frac{35 \cdot \overline{v-2}}{12v^3} \cdot \beta^3}$ &c. that of the Fourth $= \frac{5}{2v^3} \cdot \beta^3$ &c. &c.

Whence, the Fluent of the whole Series, by collecting theſe ſeveral Values together, will come out $= \frac{G}{\sqrt{\frac{1}{2}v}} \times \overline{1 + \frac{1}{2}\beta + \frac{20v^2 - 5v + 2}{48v^2} \cdot \beta^2 + \frac{112v^3 - 63v^2 - 42v - 8}{6 \times 48v^3} \cdot \beta^3 + \&c.}$ Which, drawn into $\frac{1}{2} \times \overline{1 - \frac{1}{2}\beta - \frac{1}{8}\beta^2 - \frac{1}{16}\beta^3 - \&c.}$ (the Value of the general Multiplicator $\frac{1}{2}\sqrt{1-\beta}$) gives $\frac{G}{\sqrt{2v}} \times \overline{1 + \frac{\overline{v-2} \cdot \overline{2v-1}}{48} \cdot \frac{\beta^2}{v^2} + \frac{\overline{v-2} \cdot \overline{2v-1} \cdot \overline{2v-1}}{72} \times \frac{\beta^3}{v^3}}$ &c. for the true Meaſure of the Angle required, in Parts of the Radius, or Unity: From whence, by writing 180 inſtead of G, we ſhall have the ſame in Degrees: Which, last of all, by reſtoring n, becomes

$$\frac{180^\circ}{\sqrt{n+3}} \times \overline{1 + \frac{\overline{n-1} \cdot \overline{n+2}}{24} \times \overline{\frac{\beta}{n+3}}\Big|^2 + \frac{\overline{n-1} \cdot \overline{n+2} \cdot \overline{n+2}}{18} \times \overline{\frac{\beta}{n+3}}\Big|^3 \&c.}$$

Where n is the Exponent of the Law of the Force, whereby the Orbit is deſcribed; and β, the Defect of the Square of the Meaſure of the Celerity, at the higher *Apſis*, below That which the Body ought to have to revolve in a Circle, this laſt being denoted by Unity.

The ſame Concluſion may be otherwiſe derived, by bringing $1-y$, in the transformed Fluxion, under the *Vinculum*; but this Way of going to work, though we have but one Series to manage, will prove rather more troubleſome than the foregoing.

It will appear from the two preceding Examples, eſpecially the firſt of them, that this laſt Method of finding Fluents is, chiefly, uſeful when all the Terms of the given Expreſſion, after the two firſt, in reſpect of theſe, are but ſmall. Which is a Circumſtance that frequently occurs in the Reſolution of phyſical Problems; ſuch as determining the Effect of the Atmoſphere's Reſiſtance upon the Vibration of Pendulums; and the *Inequalities* of the Planets ariſing from their Action on each other.——In ſhort, wherever the Fluent, or the Quantity it expreſſes, would belong to the Circle, or ſome other of the Conic-Sections, were it not for the Interpoſition of ſome ſmall perturbating Force (whereby new Terms, ſmall in Compariſon of the two firſt, are introduced) the ſaid Method will be found of very great Service.

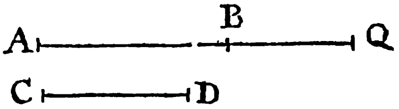
SECTION VIII.

The Uſe of Fluxions in determining the Motion of Bodies in reſiſting Mediums.

PROB. I.

361. *Suppoſing that a Body, let go from a given Point* A, *with a given Celerity, in a Right-line* AQ, *is reſiſted by a Medium (or any Force) acting according to a given Power of the Velocity: To determine the Velocity, and alſo the Space run over, at the End of a given Time.*

LET the given Celerity at A (meaſur'd by the Space which would be uniformly deſcribed in any propoſed Time r) be put $= c$, and that at any other Point B, $= v$; moreover put AB $= x$, and the Time of its Deſcription $= z$; and let the Reſiſtance, or Force, acting upon the Body at A, be ſuch, that, if the ſame was to be uniformly continued, the Body would have all its Motion deſtroyed thereby, in the Time wherein it might move, uniformly, over a given Diſtance d (CD) with its firſt Velocity c: Which Time, let be denoted, by t.



Then, ſince the whole Celerity c would be deſtroy'd in the Time t, that Part of it which would be uniformly taken away in the Time r, above propoſed, will be truly repreſented by $\frac{r}{t} \times c$; or by $\frac{cc}{d}$; which is equal to it, becauſe the Spaces (c and d) deſcribed with the ſame Ce-

Celerity are always as the Times (r and t) of their Deſcription; and therefore $\frac{r}{t} = \frac{c}{d}$.

Hence, the Reſiſtance at B being to that at A (*by Hypotheſis*) as v^n to c^n, it follows that the Velocity which might be deſtroyed in the given Time r, by a Force equal to the Reſiſtance at B, will be expreſſed by $\frac{cc}{d} \times \frac{v^n}{c^n}$, or its Equal $\frac{v^n}{dc^{n-2}}$: Which Expreſſion is, therefore, the true Meaſure of the Force of the ſaid Reſiſtance.

Now, it appears, from *Art.* 218. that, if the Force with which the Body is acted on (or the Velocity it would generate in the given Time r) be repreſented by F, the Relation of the Meaſures of the Velocity and Space gone over, will be expreſſed by the Equation $\pm v\dot{v} = F\dot{x}$: From whence, by writing $\frac{v^n}{dc^{n-2}}$ inſtead of F, we have $-v\dot{v} = \frac{v^n\dot{x}}{dc^{n-2}}$ (the Sign of $v\dot{v}$ being negative, becauſe v decreaſes while x increaſes *.) From this Equation, we get $\dot{x} = -dc^{n-2}v^{1-n}\dot{v}$; whoſe Fluent is $x = -\frac{dc^{n-2} \times v^{2-n}}{2-n} +$; which, corrected (by taking $x = 0$, and $v = c$) becomes $x =$

$$\frac{-dc^{n-2} \times v^{2-n} + d}{2-n} = \frac{d}{n-2} \times \overline{\frac{c}{v}\Big|^{n-2} - 1}.$$

Moreover, ſince the Time ($\dot{z}$) is to the Time r, as the Diſtance $\dot{x}$ to the Diſtance v, we alſo have $\dot{z}$ ($= \frac{r\dot{x}}{v}$) $= -rdc^{n-2}v^{-n}\dot{v}$; and conſequently $z =$

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* Art. 5.

$\frac{rd}{\overline{n-1}\times c}\times\overline{\frac{c}{v}}\big|^{n-1}-1=\frac{t}{n-1}\times\overline{\frac{c}{v}}\big|^{n-1}-1$ (by writing t for its Equal $\frac{rd}{c^n}$): From which Equation we get $\frac{c}{v}=\overline{1+\overline{n-1}\times\frac{z}{t}}\big|^{\frac{1}{n-1}}$: Likewiſe, from the preceding Equation, we get $\frac{c}{v}=$ $\overline{1+\overline{n-2}\times\frac{x}{d}}\big|^{\frac{1}{n-2}}$: Which two equal Values being compared together, there, at length, reſults $x=$ $\frac{d}{n-2}$ into $\overline{1+\overline{n-1}\times\frac{z}{t}}\big|^{\frac{n-2}{n-1}}-1$, for the required Relation of x and z. *Q. E. I.*

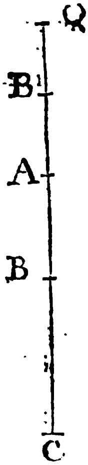
Corollary.

362. If $n=2$, or, the Reſiſtance be in the Duplicate Ratio of the Velocity, the Equation exhibiting the Relation of z and v, will be $\frac{c}{v}=1+\frac{z}{t}$, or $v=$ $\frac{c}{1+\frac{z}{t}}$: But the other Equation (the Fluent failing) becomes impracticable. Here x, the Fluent of $-\frac{d\dot{v}}{v}$, will be explicable by $d\times$ *hyp. Log.* $\frac{c}{v}$ *, or by $d\times$ *hyp. Log.* $1+\frac{z}{t}$; becauſe $v=\frac{c}{1+\frac{z}{t}}$.

* Art. 126.

In the like Manner, when $n = 1$, or the Reſiſtance is as the Velocity, the Relation of v, x and z, will be exhibited by the Equations $v = c \times \frac{\overline{d-x}}{d}$, and $z = t \times$ *hyp. Log.* $\frac{c}{v} = t \times$ *hyp. Log.* $\frac{d}{d-x}$. Which Caſe, and that above, are the only two wherein the general Solution fails.

PROB. II.



363. *If a Body, let go from a given Point* A *with a given Celerity, in a vertical Line* CAQ, *is acted on by an uniform Gravity, and alſo by a Medium, reſiſting according to any given Power of the Velocity; 'tis propoſed to determine the Relation of the Times, the Velocities, and the Spaces gone over.*

Let the Notation in the preceding Problem be retained; and let the Force of Gravity, in the given Medium (meaſured by the Velocity it might generate in the propoſed Time r *) be repreſented by b. Then, this Value being added to, or ſubtracted from $\left(\frac{v^n}{dc^{n-2}}\right)$ the Meaſure of the Reſiſtance †, according as the Body is in its Aſcent, or Deſcent, we thence get $\frac{v^n}{dc^{n-2}} \pm b$ for the whole Force (F) whereby the Motion, at B, is affected: Whence (*by Art.* 218) $\dot{x}\ \left(= \frac{-v\dot{v}}{F}\right) = \frac{-dc^{n-2}v\dot{v}}{v^n \pm bdc^{n-2}}$, and $\dot{z}\ \left(= \frac{r\dot{x}}{v}\ ‡\right) = \frac{-rdc^{n-2}\dot{v}}{v^2 \pm bdc^{n-2}}$: Whoſe Fluents may

* Art. 361.

† Art. 361.

‡ Art. 362.

may be had, by the Means of circular Arcs, and Logarithms, *from Art.* 331. *Q. E. I.*

COROLLARY I.

364. It appears that the Force $\left(\frac{v^n}{dc^{n-2}}\right)$ of the Reſiſtance is to (b) *that* of Gravity, in the given Medium, as v^n to bdc^{n-2}: Therefore, if this Ratio be expounded by that of v^n to a^n, or a^n be put $= bdc^{n-2}$, it follows that a will expreſs the Celerity with which the Reſiſtance would be equal to the Gravity (ſince, when $v = a$, the ſaid Ratio becomes that of Equality.) Hence, alſo, by ſubſtituting $\frac{a^n}{b}$ for its Equal dc^{n-2}, we get $\dot{x} = \frac{-a^n v\dot{v}}{b \times \overline{v^n \pm a^n}}$, and $\dot{z} = \frac{-ra^n \dot{v}}{b \times \overline{v^n \pm a^n}}$.

COROLLARY II.

365. If the Reſiſtance be in the Duplicate Ratio of the Celerity, our two laſt Equations will become $\dot{x} = \frac{-a^2 v\dot{v}}{b \times \overline{vv \pm aa}}$, and $\dot{z} = \frac{-ra^2\dot{v}}{b \times \overline{vv \pm aa}}$: From the former whereof we get $x = -\frac{a^2}{2b} \times hyp.\ Log.\ \frac{vv \pm aa}{cc \pm aa}$* $= \frac{a^2}{2b} \times hyp.\ Log.\ \frac{cc \pm aa}{vv \pm aa} = \frac{d}{2} \times hyp.\ Log.\ \frac{cc \pm bd}{vv \pm bd}$ (becauſe, here, $a^2 = bd$.) From whence, when $v = 0$, (ſuppoſing the Body to aſcend) there comes out $x = \frac{d}{2} \times hyp.\ Log.\ \overline{1 + \frac{cc}{aa}}$, for the Height (AQ) of the whole Aſcent. But, if c be taken $= 0$, or the Body

* Art. 126.

be ſuppoſed to deſcend from Reſt, we ſhall then have $-\frac{d}{2}\times$ *hyp. Log.* $\overline{1-\frac{vv}{aa}}$ = the Diſtance *AB* deſcended. Whence, if *N* be put for the Number whoſe Hyperbolical Logarithm is $\frac{2x}{d}$, it follows, (becauſe, Log. $1-\frac{vv}{aa}=-\frac{2x}{d}=-$ Log. N) that $1-\frac{vv}{aa}=\frac{1}{N}$, and conſequently $v=a\sqrt{\frac{N-1}{N}}$. From which, the Diſtance *AB* being given, the Velocity acquired in the Fall will be determined. But, if the Body, firſt, aſcends from a given Point *A*, with a given Celerity *c*, and the Celerity, acquired in falling, when it arrives, again, at that Point, be required; the ſame may be exhibited in a more commodious Form, independent of Logarithms, and will be equal to $\frac{c}{\sqrt{1+\frac{cc}{aa}}}$; becauſe *N*, in this Caſe, is found above to be $=1+\frac{cc}{aa}$. Furthermore, with regard to the Time (x), we have already found that $\dot{x}$ is $=\frac{-ra^2\dot{v}}{b\times\overline{vv+aa}}$, or $=\frac{-ra^2\dot{v}}{b\times\overline{vv-aa}}$ $\left(=\frac{ra^2\dot{v}}{b\times\overline{aa-vv}}\right)$ according as the Motion of the Body is from, or towards the Center of Force. Therefore the Time itſelf, in the former Caſe, will be $=\frac{ra}{b}$ drawn into the Difference of the two circular Arcs whoſe Tangents are $\frac{c}{a}$ and $\frac{v}{a}$, and whereof the common Radius is Unity *: Whence it follows that the

Time

* Art. 142.

Time of the whole Aſcent will be denoted by $\frac{ra}{b}$ multiplied into the former of the ſaid Arcs.

But, in the other Caſe, the Fluent, exhibiting the Time of Deſcent, is not explicable by the Arcs of a Circle, but by the Difference of the hyperbolical Logarithms of $\frac{a+v}{a-v}$ and $\frac{a+c}{a-c}$ drawn into $\frac{ra}{2b}$ *. Therefore, when $c = 0$, or the Body falls from Reſt, the Time z will be barely $= \frac{ra}{2b} \times$ *hyp. Log.* $\frac{a+v}{a-v} = \frac{ra}{b} \times$ *hyp. Log.* $N^{\frac{1}{2}} + \overline{N-1}|^{\frac{1}{2}}$ (by ſubſtituting the Value of v found above, and ordering the Logarithm as in *Art.* 303.) This Equation, in the forementioned Circumſtance, where $N = 1 + \frac{cc}{aa}$, and $v = \frac{c}{\sqrt{1+\frac{cc}{aa}}}$, becomes $z = \frac{ra}{b} \times$ *hyp. Log.* $\sqrt{1+\frac{cc}{aa}} + \frac{c}{a}$.

* Art. 126.

SCHOLIUM.

366. If, according to Sir *Iſaac Newton*, we ſuppoſe the Reſiſtance of the Air, to Bodies moving in it, to be in the Duplicate Ratio of the Celerities *; and that a Ball,

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* *That the Reſiſtance is as the Square of the Celerity, the Learner may, in ſome meaſure, conceive, by conſidering that the ſame Body, with a double Velocity, not only puts twice the Number of reſiſting Particles in Motion, in the ſame time, but alſo acts upon each with a double Force; and therefore muſt ſuffer a four-fold Reſiſtance, or a Reſiſtance proportional to the Square of the Velocity. This would be ſtrictly true, were it not that the Particles ſo put in Motion impel others lying before them, and thereby prevent, as it were, the Action of the Body. What Deviation from the foregoing Law may hence ariſe, is not eaſy to determine. This, however, ſeems plain, that the Reſiſtance at the Beginning of any very ſwift Motion (till the Air in the Way of the Body comes duly to participate of that Motion) will be greater than That ſuſtained by another equal Body, moving with the ſame Celerity, that has been in Motion ſome time.*

a Ball, in the Time it might move, uniformly, over a Space (d) which is to $\frac{8}{3}$ of its Diameter as the Denſity of the Ball to that of the Medium, would have all its Motion taken away by a Force equal to that of the Reſiſtance, uniformly continued: Then, from theſe *Data*, applied to the Theorems in the preceding Article, we ſhall be able to determine the Velocities, and the Times of the perpendicular Aſcent and Deſcent of Bodies near the Earth's Surface; allowing for the Reſiſtance of the Atmoſphere.

Thus, for Inſtance, let a Cannon Ball, of four Inches Diameter (whereof the Denſity, or ſpecific Gravity, is to that of Air as 6000 to 1, nearly) be ſuppoſed to be projected, perpendicular to the Horizon, with a Velocity ſufficient to cauſe it to aſcend to the Height of half a Mile, or 2640 Feet, *in vacuo*; which Velocity (*by Art.* 203.) will be found to anſwer to the Rate of about 412 Feet *per Second:* Then, according to the Proportion juſt now mentioned, it will be as $1 : 6000 :: \frac{8}{3} \times 4 : 64000$ Inches, or 5333 Feet; which is the Value of d in this Caſe. Therefore, if the Time r, in the preceding Article (which may be aſſumed at pleaſure) be here interpreted by *one Second*, the correſponding Values of d, c and b will be expounded by 5333 F. 412 F. and $32\frac{1}{12}$ F. * reſpectively. Which Values being ſubſtituted in the ſeveral Equations in the laſt Article, we ſhall get

* Art. 202.

1°. $a\ (= \sqrt{bd}) = 414$ F. the Velocity, *per Second*, wherewith the Reſiſtance would be equal to the Gravity, or Weight, of the Ball.

2°. $\frac{d}{2} \times$ *hyp. Log.* $1 + \frac{cc}{aa} = 1835$ Feet, the whole Height of the Aſcent.

3°. $\frac{ra}{b} \times$ *Arch.* whoſe Tang. is $\frac{c}{a} = 10{,}08$ *Seconds*, the whole Time of the Aſcent (which is leſs than the Time, *in vacuo*, by 2,73.)

4°. $v \left(= \frac{c}{\sqrt{1+\frac{cc}{aa}}}\right) = 292$, the Velocity, *per Second*, acquired in the Deſcent.

5°. Laſtly, $\frac{ra}{b} \times$ *hyp. Log.* $\sqrt{1+\frac{cc}{aa}} + \frac{c}{a} =$ 11,30 Seconds, the Time of the Deſcent.

Note, In this Example the Meaſure of the abſolute Gravity of the Body, *in vacuo*, is taken, inſtead of its Gravity in Air (the Difference, *there*, being too inconſiderable to be regarded.) But, in Caſes where the ſpecific Gravity of the Medium bears a ſenſible Proportion to that of the Body, the Force of Gravity (b) muſt be expounded by $32\frac{1}{12} \times \frac{B-M}{B}$ (inſtead of $32\frac{1}{12}$) where B is to M as the ſpecific Gravity of the Body to that of the Medium.

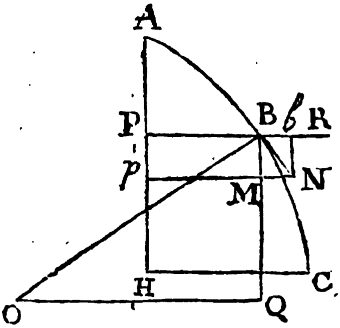
PROB. III.

367. *To determine the Reſiſtance, by means whereof a Body, gravitating uniformly in the Direction of parallel Lines, may deſcribe a given Curve.*

Let ABC be the given Curve, and BQ, parallel to the Axis (or any given Line) AH, be the Direction of Gravitation at any Point B: Make PBR perpendicular to AH and BQ; and let AP $= x$, PB $= y$, AB $= z$, BM $(Nb) = \dot{x}$, MN $(Bq) = \dot{y}$, BN $= \dot{z}$, and the Velocity of the Body at B in the Direction PBR $= v$. Then, the Decreaſe of Velocity in the ſaid Direction, which is wholly owing to the Reſiſtance *, being repreſented by $-\dot{v}$, it follows that the correſponding Decreaſe of Motion in the Direction BN, ariſing from the ſame Cauſe, will be expreſſed by $\frac{\dot{z}}{\dot{y}} \times -\dot{v} = -\frac{\dot{v}\dot{z}}{\dot{y}}$; and, that in the Direction BM, by $-\frac{\dot{v}\dot{x}}{\dot{y}}$. But, the

* Art. 209.

Celerity in this laſt Direction being, every where, repreſented by $v \times \frac{\dot{x}}{\dot{y}}$, its Fluxion $\frac{v\ddot{x} + \dot{v}\dot{x}}{\dot{y}}$ will be the



whole Alteration of Motion in the ſaid Direction, ariſing from the Reſiſtance and the Force of Gravity, conjunctly: From which deducting the Part owing to the Reſiſtance, found above to be $\frac{\dot{v}\dot{x}}{\dot{y}}$, the Remainder $\frac{v\ddot{x}}{\dot{y}}$ will be the Effect of the Gravity. Which being to $\left(-\frac{\dot{v}\dot{x}}{\dot{y}}\right)$ the Effect of the abſolute Reſiſtance in the Direction BN, as 1 to $-\frac{\dot{v}\dot{x}}{v\ddot{x}}$, the Force of Gravity, muſt therefore be to that of the required Reſiſtance, in the ſame Ratio of 1 to $-\frac{\dot{v}\dot{x}}{v\ddot{x}}$.

Moreover, the Force of Gravity, meaſured by the Velocity it would generate in a given Part of Time (1), being denoted by Unity, the Velocity generated thereby, in the Time $\left(\frac{\dot{y}}{v}\right)$ of deſcribing B*b*, with the Celerity v, will likewiſe be truly expreſed by, $\frac{\dot{y}}{v}$, the Meaſure of the ſaid Time: Which being put = to $\left(\frac{v\ddot{x}}{\dot{y}}\right)$ the Value of the ſame Quantity, given above, we thence have $v^2 = \frac{\dot{y}\dot{y}}{\ddot{x}}$: From whence, not only the Velocity, but the Reſiſtance will be found. But, if you would have the Reſiſtance expreſſed independent of v; then let the Fluxion $\left(2v\dot{v} = -\frac{\dot{y}^2\dddot{x}}{\ddot{x}\ddot{x}}\right)$ of the laſt Equation be divided by the Fluent, which will give $\frac{\dot{v}}{v} = -\frac{\frac{1}{2}\dddot{x}}{\ddot{x}}$: And then, by ſubſtituting this Value in $-\frac{\dot{v}\dot{z}}{v\ddot{x}}$, you will get $\frac{\dot{z}\dddot{x}}{2\ddot{x}\ddot{x}}$, *for* the true Force of the Reſiſtance, *that* of Gravity (or the Weight of the Body) being expounded by Unity.

The ſame otherwiſe.

Let BO be the Radius of Curvature at B, and let OQ be parallel to PB, meeting BM, produced, in Q; Then, if the abſolute Gravity, acting in the Direction BQ, be denoted by Unity, its Force in the Direction BO, whereby the Body is retained in the Curve, will be repreſented by $\frac{\text{BQ}}{\text{BO}}$. Therefore, ſince the Velocities in Circles are known to be in the Subduplicate Ratio
* Art. 212. of the Radii and of the Forces conjunctly *, the Velocity at B will be rightly expreſſed by $\sqrt{\text{BO} \times \frac{\text{BQ}}{\text{BO}}}$, or its Equal $\sqrt{\text{BQ}}$. (For the Curve at, and indefinitely near, B may be taken as an Arch of a Circle whoſe Radius is BO: And it is evident that the Reſiſtance has nothing to do in forcing the Body from the Tangent,

Tangent, but only ſerves to retard its Motion ſo, that it may, every where, bear a due Proportion to the given Force of Gravity acting in the Direction BO.) Hence, putting BQ $= s$, the Increaſe of the Celerity in the Time $\left(\frac{\dot{z}}{\sqrt{s}}\right)$ of deſcribing BN, will be expreſſed by the Fluxion of $\sqrt{s}$, or $\frac{\dot{s}}{2\sqrt{s}}$. Moreover, the Celerity that might be generated by Gravity in the ſaid Time $\frac{\dot{z}}{\sqrt{s}}$ being meaſured thereby, the Increaſe, in BN, ariſing from the ſame Cauſe, will therefore be $= \frac{\dot{z}}{\sqrt{s}} \times \frac{\dot{x}}{\dot{z}} = \frac{\dot{x}}{\sqrt{s}}$; Which, being taken from $\left(\frac{\dot{s}}{2\sqrt{s}}\right)$ the whole Increaſe, found above, the Remainder, $\frac{\dot{s} - 2\dot{x}}{2\sqrt{s}}$, will be the Effect of the Reſiſtance: Which is to the Effect, $\frac{\dot{x}}{\sqrt{s}}$, of the abſolute Gravity as $\frac{\dot{s} - 2\dot{x}}{2\dot{x}}$ to 1. Therefore the Reſiſtance is to the Gravity (or Weight of the Body) as $\frac{2\dot{x} - \dot{s}}{2\dot{x}}$ to Unity: Where the Signs are changed, becauſe the two Forces act in contrary Directions.

Becauſe BO $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$ *, therefore s $\left(\text{BO} \times \frac{\dot{y}}{\dot{z}}\right) = \frac{\dot{z}^2}{\ddot{x}} = \frac{\dot{y}^2 + \dot{x}^2}{\ddot{x}}$ (= the Square of the Celerity) whence $\dot{s} = \frac{2\dot{z}\ddot{z}\ddot{x} - \overline{\dot{y}^2 + \dot{x}^2} \times \dddot{x}}{\ddot{x}\ddot{x}}$, and conſequently the Reſiſtance

* Art. 63.

ſiſtance $\frac{2\ddot{x}-\ddot{s}}{2\dot{x}} = \frac{\overline{\dot{y}^2+\dot{x}^2}\times\dddot{x}}{2\dot{x}\ddot{x}\ddot{x}} = \frac{\dot{x}\dddot{x}}{2\ddot{x}\ddot{x}}$, *the very ſame as before.*

COROLLARY.

368. If the Reſiſtance be ſuppoſed as any given Power of the Velocity drawn into (D) the Denſity of the Medium; then, from hence, the Denſity of the Medium, at every Point of the Curve, may be determined: For, the abſolute Celerity at B being repreſented by $\frac{v\dot{z}}{\dot{y}}$, the Reſiſtance at that Point will, according to the ſaid Hypotheſis, be as $\overline{\frac{v\dot{z}}{\dot{y}}}\Big|^{n} \times D$; and therefore the Velocity that would be deſtroyed thereby, in the Time $\left(\frac{\dot{y}}{v}\right)$ of deſcribing BN, as $\overline{\frac{v\dot{z}}{\dot{y}}}\Big|^{n} \times \frac{D\dot{y}}{v}$: Which being put $= \left(-\frac{\dot{v}\dot{z}}{\dot{y}}\right)$ the Effect of the ſame Reſiſtance, found above, we thence get $D = \frac{-\dot{v}\dot{y}^{n-2}}{\overline{v\dot{z}}|^{n-1}}$: Which, by ſubſtituting for v and $\dot{v}$, becomes $D = \frac{\dddot{x}}{2\dot{x}^{n-1}\times\ddot{x}^{2-\frac{1}{2}n}}$.

In this Corollary, and what, elſewhere, relates to unequal Denſities, the Gravity of the Body in the Medium is ſuppoſed to continue, every where. the ſame, or, that the Attraction increaſes with the Denſity, ſo that the Difference between the ſpecific Gravities of the Body and Medium may, at every Point, be a conſtant Quantity.

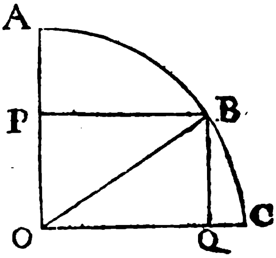
EXAMPLE I.

369. *Let the propoſed Curve* ABC *be the common Parabola:*

Then; x being here $=\frac{y^2}{a}$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, $\ddot{x} = \frac{2\dot{y}\dot{y}}{a}$ and $\dddot{x} = 0$; and therefore $\frac{\dot{x}\dddot{x}}{2\ddot{x}\ddot{x}}$ * is alſo $= 0$: * Art. 367.
Whence it appears that a Body, to deſcribe this Curve, muſt move in Spaces intirely void of Reſiſtance.

EXAMPLE II.

370. *Let the Curve* ABC *be taken as a Quadrant of a Circle, whoſe Radius* BO *is* $= a$.

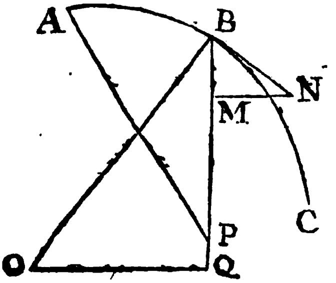


In this Caſe we have s (BQ) † $= a - x$ ($=$ AO $-$ AP) whence $\dot{s} = -\dot{x}$, † Art. 367.
and therefore $\frac{2\dot{x} - \dot{s}}{2\dot{z}} = \frac{3\dot{x}}{2\dot{z}} = \frac{3PB}{2AO}$ ‡. From which it is evident, that the ‡ Art. 142.
Velocity is, every where, as $\sqrt{BQ}$, and the Reſiſtance to the Gravity (or Weight of the Body) as 3PB to 2OB.

PROB. IV.

371. *The Centripetal Force (F) being given; to find the Reſiſtance and Velocity whereby a Body may deſcribe a given Spiral (or any other, poſſible, Curve) about the Center of Force.*

Let P be the Center of Force, and BO the Radius of Curvature at any Point B in the propoſed Curve,



and let OQ be perpendicular to BPQ; alſo let BP $= y$, BQ $= s$, AB $= z$, BM $= -\dot{y}$ *, and BN $= \dot{z}$. Then, it is evident from *Art.* 367. that the Velocity at B will be expreſſed by $\sqrt{\text{BO} \times \frac{\text{BQ}}{\text{BO}} \times F}$; or, its Equal, $\sqrt{sF}$: And therefore its increaſe in the Time $\left(\frac{\dot{z}}{\sqrt{sF}}\right)$ of deſcribing BN will be $\frac{s\dot{F} + F\dot{s}}{2\sqrt{sF}}$: From which, deducting $\left(F \times \frac{\dot{z}}{\sqrt{sF}} \times \frac{-\dot{y}}{\dot{z}}\right)$ the Effect of the centripetal Force, in the ſame Time and Direction, the Remainder, $\frac{s\dot{F} + F\dot{s} + 2F\dot{y}}{2\sqrt{sF}}$, is the Effect of the Reſiſtance. Therefore the Reſiſtance is to the centripetal Force as $\frac{s\dot{F} + F\dot{s} + 2F\dot{y}}{2\sqrt{sF}}$ to $\frac{F\dot{z}}{\sqrt{sF}}$, or as $\frac{s\dot{F} + F\dot{s} + 2F\dot{y}}{2F\dot{z}}$ to Unity. *Q. E. I.*

EXAMPLE.

372. Let the Meaſure *(F)* of the centripetal Force be expounded by any Power y^n of the Diſtance; and let the Curve be the logarithmic Spiral; putting the Co-ſine of the given Angle PBN † (to the Radius r) $= c$. Then, s being here $= y$ ‡, and $\dot{F} = ny^{n-1}\dot{y}$, we

* Art. 5.

† Art. 61.

‡ Art. 74.

we have $\frac{s\dot{F}+F\dot{s}+2F\dot{y}}{2F\dot{z}} = \frac{ny^n\dot{y}+y^n\dot{y}+2y^n\dot{y}}{2y^n\dot{z}} = \frac{n+3}{2} \times \frac{\dot{y}}{\dot{z}} = \frac{n+3}{2} \times \frac{c}{r}$.

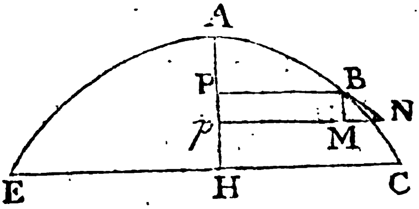
Hence it appears that the Velocity muſt be, every where, as $y^{\frac{n+1}{2}}$; and the Reſiſtance, to the centripetal Force, as $\frac{n+3}{2} \times \frac{c}{r}$ to Unity. But, when $n = -3$, $\frac{n+3}{2} \times \frac{c}{r}$ becomes $= 0$; therefore the Body, in this Caſe, muſt move in Spaces intirely void of Reſiſtance; agreeable to *Art.* 233. And, if $n+3$ be negative, an accelerating, inſtead of a reſiſting Force, will be required.

SCHOLIUM.

373. If the Denſity of a Medium, wherein a Body moves, be either uniform, or varies according to a given Law, the Nature of the Curve, or Trajectory may be determined from what is delivered in the preceding Pages.

Thus, for Example, let the Denſity be ſuppoſed every where the ſame, and the Reſiſtance as the Square of the Celerity; then, from *Art.* 368. we have $\frac{\ddot{x}}{\dot{z}\dot{x}} = D$; which, in order to exterminate $\dot{z}$, may be transformed to $\ddot{x}\ddot{x} = \overline{\dot{y}\dot{y}+\dot{x}\dot{x}} \times D^2\dot{x}\dot{x}$: Where, D being a conſtant Quantity (depending upon the given Denſity of the Medium) the Value of x will be found, as is taught in *Sect.* 2. *Art.* 268, 271. and comes out $= \frac{y^2}{p} + \frac{Dy^3}{3p} + \frac{D^2y^4}{12p}$ &c. In which p is put to denote the Parameter of the Curve at the Vertex, or higheſt Point A, (to be determin'd from the Force of Gravity and the given Velocity of the Body at that Point.) This Solution anſwers near enough when the Reſiſtance is but ſmall in Proportion to the Gravity; in other Circumſtances, the Series not converging, it becomes uſeleſs: For which Reaſon, and becauſe the Caſe above ſpecified is That ſuppoſed to obtain, in reſpect to the Air near the Earth's Surface, and its Reſiſtance to Bodies moving therein, I ſhall ſhew, by a different Method, how the Nature of the Curve may be inveſtigated.

In order thereto, let the Celerity at the higheſt Point, A, above the Plane of the Horizon EC, be denoted by c; and let a be the Celerity with which the Reſiſtance is equal to the Gravity (*vid. Art.* 365. *and* 366.)



Moreover, let d be put for the Diſtance over which the Ball might uniformly move in the Time that the Medium would deſtroy all its Motion, was the Reſiſtance to continue the ſame, all along, as at the firſt Inſtant (Which Diſtance, according to Sir *Iſaac Newton*, is, always, in Proportion to $\frac{8}{3}$ of the Ball's Diameter, as the Denſity of the Ball to that of the Medium.)

Then it will be, as $d : \dot{x}$ (BN) :: $\frac{v\dot{x}}{\dot{y}}$, the abſolute Celerity at B, to $\left(\frac{v\dot{x}^2}{d\dot{y}}\right)$ the Part thereof that would be uniformly deſtroyed by the Reſiſtance in the Time of deſcribing BN, with the Velocity at B: Which Value being alſo expreſſed by $\frac{-\dot{v}\dot{x}}{\dot{y}}$ (*vid. Art.* 367.) we therefore have $\frac{v\dot z^2}{d\dot y} = -\frac{\dot v\dot z}{\dot y}$; whence $\frac{\dot z}{d} = \frac{-\dot v}{v}$, and conſequently, by taking the Fluent, $\frac{z}{d} = -$ hyp. Log. v; which corrected (by putting $z=0$, and $v=c$) gives $\frac{z}{d}$ ($=$ hyp. Log. $c -$ hyp. Log. v) $=$ hyp. Log. $\frac{c}{v}$.

Furthermore, ſince (by Hypotheſis) the Reſiſtance with the Celerity $\frac{v\dot z}{\dot y}$ (at B) is to the Force of Gravity, or the Reſiſtance with the Celerity a, as $\frac{vv\dot z\dot z}{\dot y\dot y}$ to a^2; and it appears, from the aforeſaid Article, that the ſame Ratio is alſo univerſally expreſſed by *that* of $\frac{-\dot v\dot z}{v\ddot x}$ to 1, it follows, from the Equality of theſe Ratios, that $\frac{\dot z\ddot x}{\dot y\dot y}$ is $= -\frac{a^2\dot v}{v^3}$. But, in order to the Reſolution of the Equation thus given, let the Tangent of the Angle PBA (or N) which the Ordinate, PB, makes with the Curve (ſuppoſing Radius Unity) be, every where, repreſented by w: Then, becauſe $\dot x = w\dot y$, $\dot z$ $(\sqrt{\dot y^2+\dot x^2})$ $= \dot y\sqrt{1+w^2}$, and $\ddot x = \dot w\dot y$ ($\dot y$ being conſtant) we ſhall, by ſubſtituting theſe Values in the foreſaid Equation, get $-\frac{a^2\dot v}{v^3} = \dot w\sqrt{1+w^2}$; whereof the Fluent will be given, $\frac{\frac{1}{2}a^2}{v^2} = \frac{1}{2}w\sqrt{1+w^2} + \frac{1}{2}$ hyp. Log. $w+\sqrt{1+w^2}$ *: Which corrected (by taking $v=c$ and $w=0$) becomes $\frac{\frac{1}{2}a^2}{v^2} - \frac{\frac{1}{2}a^2}{c^2} = \frac{1}{2}w\sqrt{1+w^2}$ $+ \frac{1}{2}$ hyp. Log. $w+\sqrt{1+w^2}$. But, to ſhorten the

* Art. 126, and 281.

remaining Part of the Proceſs, let the latter Part of the Equation, or the Fluent of $\dot{w}\sqrt{1+w^2}$ be denoted by Q; then $\frac{aa}{2vv}$ being $= \frac{aa}{2cc} + Q$, we have $v = \frac{ac}{\sqrt{aa+2ccQ}}$; and conſequently $\frac{z}{d}$ $\left(= \text{hyp. Log. } \frac{c}{v}\right)$ $= \text{hyp. Log. } \frac{\sqrt{aa+2ccQ}}{a} = \frac{1}{2} \text{ hyp. Log. } \overline{1+\frac{2ccQ}{aa}}$. From which two Equations, the Velocity of the Ball, and the Diſtance it has moved, when its Direction makes any given Angle with the Horizon, may be computed, let the Medium be as denſe as it will: Alſo, from hence, if the Celerity anſwering to any one given Angle of Direction be known, the Celerity correſponding to any other given Direction may be found, together with the Diſtance deſcribed between the two Poſitions. For v (in the Deſcent of the Body) being, *univerſally*, equal to $\frac{ac}{\sqrt{aa+2c^2Q}}$, the Value of c, expreſſing the Celerity at the Vertex *A*, will be had from that Equation, and comes out $= \frac{av}{\sqrt{aa-2v^2Q}}$; whence alſo z $\left(= d \times \text{hyp. Log. } \frac{c}{v}\right) = d \times \text{hyp. Log. } \frac{a}{\sqrt{aa-2v^2Q}} = -\frac{1}{2} d \times \text{hyp. Log. } \overline{1-\frac{2vvQ}{aa}}$. From which, the Celerity at A being known, the reſt is obvious.

But, in the aſcending Part of the Curve EA, both z and Q muſt be conſidered as negative, or wrote with contrary Signs: And then, from the foregoing Equations, we ſhall alſo get $v = \frac{ac}{\sqrt{aa-2ccQ}}$, $c = \frac{av}{\sqrt{aa+2vvQ}}$, and $-x = \frac{1}{2}d \times$ hyp. Log. $\overline{1 - \frac{2ccQ}{aa}} = -\frac{1}{2}d \times$ hyp. Log. $\overline{1 + \frac{2vvQ}{aa}}$; and, conſequently, $x = -\frac{1}{2}d \times$ hyp. Log. $\overline{1 - \frac{2ccQ}{aa}} = \frac{1}{2}d \times$ hyp. Log. $\overline{1 + \frac{2vvQ}{aa}} = d \times$ hyp. Log. $\frac{v}{c}$: Anſwering in this Caſe.

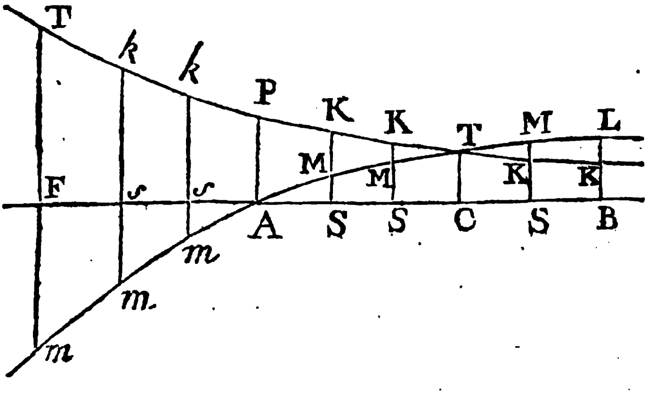
It ſtill remains to take ſome notice of the Values of x and y (in order to have the Form, as well as the Length of the Curve.) Theſe, indeed, are not ſo eaſy to bring out as That of x, given above; nor can they be exhibited in a general Manner, either by circular Arcs, or Logarithms (that I have been able to diſcover) but may, however, be approximated to any required Degree of Exactneſs, as will appear from what follows.

Since x (= AB) is found $= \frac{1}{2}d \times$ hyp. Log. $\frac{aa + 2c^2Q}{aa}$, by taking the Fluxion thereof, we get $\dot{x} = \frac{ccd\dot{Q}}{aa + 2ccQ} = \frac{c^2d\dot{w}\sqrt{1 + w^2}}{aa + 2c^2Q}$ (becauſe $\dot{Q} = \dot{w}\sqrt{1 + w^2}$) Therefore $\dot{y}$ $\left(= \frac{\dot{x}}{\sqrt{1 + w^2}}\right) = \frac{c^2d\dot{w}}{aa + 2c^2Q}$; and $\dot{x}$ $(= w\dot{y}) = \frac{c^2dw\dot{w}}{aa + 2c^2Q}$: Which Equations, by taking r to 1, as a^2 to c^2 (or as the Square of the Force of Gravity to the Square of the Reſiſtance at A) are reduced to $\dot{y} = \frac{d\dot{w}}{r + 2Q}$, and $\dot{x} = \frac{dw\dot{w}}{r + 2Q}$: Whence we get $y = d$ into $\frac{w}{r + 2Q} + \frac{\frac{2}{3} \times \overline{1 + ww}|^{\frac{3}{2}} - \frac{2}{3}}{\overline{r + 2Q}|^{2}} +$

$\frac{\frac{3}{5}w\times\overline{1+\frac{2}{3}w^2+\frac{1}{5}w^4}-\frac{3}{5}Q}{\overline{r+2Q}|^3}$ &c. And $x = d$ into

$\frac{\frac{1}{2}w^2}{r+2Q}+\frac{\frac{1}{4}w\times\overline{1+ww}|^{\frac{3}{2}}-\frac{1}{4}Q}{\overline{r+2Q}|^2}+$

$\frac{\frac{1}{6}\times\overline{1+ww}|^3-\frac{1}{6}-\frac{1}{2}Q^2}{\overline{r+2Q}|^3}$ &c. Theſe Expreſſions (brought out by aſſuming $\frac{A}{r+2Q}+\frac{B}{\overline{r+2Q}|^2}+$ &c. for the Fluent ſought, and proceeding as in *Art.* 340.) converge very faſt when r is large in compariſon to Q; but in other Caſes the required Values will be had, with leſs Trouble, from the following Method.



Let PKTK and AMTM be two Curves, whereof the Ordinates SK and SM, to the common *Abſciſſa* w (= AS) are expreſſed by $\frac{1}{r+2Q}$ and $\frac{w}{r+2Q}$ reſpectively: Then it is plain, from the foregoing Equations, that the Meaſures of the Areas of the ſaid Curves, multiplied by d, will truly exhibit the Values of y and x; an-

anſwering to any given Value of w (or AS) the Tangent of the Angle of Direction; or, or ſpeak more geometrically, a Square upon AC (ſuppoſing AC = Radius = Unity) will be to either of the ſaid Areas ASKP, or ASM as the given Diſtance d, to the Value of y or x required——But now as to a Way for computing theſe Areas (without which what has been ſaid about them would be to very little Purpoſe) the Method of *Equi-diſtant Ordinates* may here be applied to very good Advantage (when the foregoing Serieſes do not converge) By means whereof the required Quantities may, with a little Trouble, be brought out to a ſufficient Degree of Exactneſs, let the Reſiſtance be as great as it will.

According to the ſame Way of proceeding, the Values of x and y, in the Aſcent of the Ball, will alſo be found, if the Ordinates sk and sm, generating the required Areas, be taken, every where, equal to $\frac{1}{r - 2Q}$ and $\frac{w}{r - 2Q}$ (inſtead of $\frac{1}{r + 2Q}$ and $\frac{w}{r + 2Q}$).

From what has been thus far delivered, it will not be very difficult to calculate (according to the foregoing Hypotheſis) all the principal Requiſites concerning the Motion and Track of a Ball in the Air, projected with a given Velocity, at a given Elevation; as will be more clearly ſeen by the Example ſubjoined.

Suppoſe a Cannon Ball of 4 Inches Diameter (whereof the Weight is nearly 9 Pounds) to be diſcharged at an Elevation of 45 Degrees, with a Velocity ſufficient to carry it to the Diſtance of one Mile, on the Plane of the Horizon, were it not for the Reſiſtance of the Air.—— Then that Velocity, being the ſame as might be freely acquired in a perpendicular Deſcent of half a Mile *, * Art. 366. will be found to anſwer to the Rate of 412 Feet, *per Second*, according to *Art.* 202. and 366. From whence it is alſo plain, that the Diſtance d (ſo often mentioned above) will here be expounded by 5333 Feet; and that the Celerity (a) with which the Reſiſtance would be equal to the Gravity (or Weight of the Ball) anſwers to the Rate of about 414 Feet *per Second.*

Moreover, ſince the Tangent of the Angle of Elevation, or the firſt Value of w, is given *equal to* Unity (or Radius) we have $\mathcal{Q}\left(\frac{1}{2}w\sqrt{w^2+1}+\frac{1}{2}\text{ hyp. Log. } \overline{w+\sqrt{w^2+1}}\right)=1.1478$: From which, and $v\left(=412\sqrt{\tfrac{1}{2}}\right)$, we get $z\left(=\frac{1}{2}d\times\text{hyp. Log. } \overline{1+\frac{2vv\mathcal{Q}}{aa}}\right)$ $=2025$ Feet $=$ the Arch deſcribed in the whole Aſcent. Alſo $\left(c=\frac{v}{\sqrt{1+\frac{2vv\mathcal{Q}}{aa}}}\right)=199\frac{1}{3}$ Feet, for the Rate of the Velocity, *per Second*, at the higheſt Point: Whence $r\left(=\frac{aa}{cc}\right)=4{,}314$; by Means whereof the greateſt Altitude of the Ball, and the horizontal Diſtance correſponding thereto will likewiſe be found: For let AF, *in the preceding Figure*, be taken $=1$ (the given Value of w) and let the ſame be divided into three Parts by equi-diſtant Ordinates (which Number will anſwer ſufficiently exact) then the ſucceſſive Values of w, for the Ordinates AP, *ks*, *ks* and TF, being 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1, *thoſe* of $\mathcal{Q}$ will be 0, 0.3394, 0.713, and 1.1478, and the Ordinates themſelves (or the correſponding Values of $\left.\frac{1}{r-2\mathcal{Q}}\right)=$ to 0.2318, 0.2751, 0.3463 and 0.4953, reſpectively. From whence, by adding the two Extremes to three times the Sum of the two middle Terms, and dividing the whole by 8, we get 0.3239 for the Value of a mean Ordinate *: Which, as AF is here equal to Unity, is alſo the Meaſure of the required Area AFTP: Which, therefore, being multiplied by 5333 (d) gives 1727 Feet, for the horizontal Diſtance made good in the whole Aſcent. In the ſame Way the Area AF*m* is found = 0.1828. Whence the greateſt Height of the Ball appears to be (= 0.1828 × 5333) = 975 Feet.

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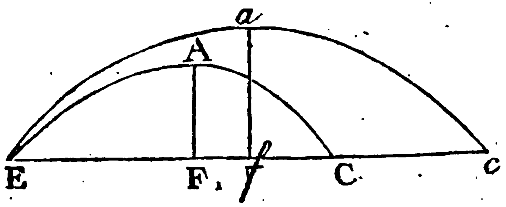
* *See* p. 117. *of my Mathematical Diſſertations.*

By taking AC = 1, and repeating the Operation (only changing $r - 2\mathcal{Q}$ to $r + 2\mathcal{Q}$) the Area ACTP will come out = 0.1883, and ATC = 0.0875; which multiplied by 5333 (as above) give 1004 F. and 467 F. for the Amplitude, and the Diſtance deſcended, from the higheſt Point, when the Direction of the Ball makes an Angle with the Horizon equal to that in which it was projected.

But, to have the Direction when the Ball ſtrikes the Ground, and the whole Amplitude of the Projection, we muſt find the Value of the Tangent AB, when the Area ABL is equal to (0.1828) the Area AF*m* (ſo that the Deſcent, from the higheſt Point, may become equal to the whole Aſcent.) In order thereto, let 0.0875 (ATC) be deducted from 0.1828 (AF*m*) and the Remainder 0.0953 will be = CTBL; this, divided by TC (0.1513) quotes 0.63; which would be the Value of CB, if all the Ordinates CT, SM, &c. were equal: But, as it is obvious from the Nature of the Problem, and from the Law of the Ordinates already computed, that BL will be ſomething greater than CT, and conſequently CB leſs than 0.63——I therefore ſuppoſe the Value of CB may be about 0.56; and, accordingly, proceed to compute the Area of CBLT anſwering to this Number; by means of CT (0.1513) and BL (0.1852) and one intermediate Ordinate SM (0.1715) and find it (from the Approximation $\frac{CT + BL + 4SM}{6} \times CB$) to come out = 0.0955: Which is ſo near the required Value 0.0953, that it will be altogether needleſs to repeat the Operation. It is evident from hence, that the Tangent (AB) of the Angle of Direction, when the Ball ſtrikes the Ground, is 1.56; anſwering to 57° : 20′: From whence, CBKT being found = 0.0752, the whole Area ABKP will be had = 0.2635, and conſequently 0.2635 × 5333 = 1405 F. = the Amplitude in the whole Deſcent.

Furthermore, from the ſaid Value of w and that of c ($= 199\frac{1}{3}$) given above, we get z ($= \frac{1}{2}d \times$ hyp. Log. $\overline{1 + \frac{2cc\mathcal{Q}}{aa}}$) $= 1788$ Feet, for the Arch deſcribed in the Deſcent; and alſo $v = 142\frac{1}{2}$ F. which multiplied by 1.8527, the Secant of 57° : 20′, gives 264 F. for the Celerity of the Ball, *per Second*, at the End of its Flight.

Now, by collecting the principal of the foregoing Concluſions, it appears,



1°. That the Velocity at the higheſt Point A of the Trajectory will be at the Rate of $199\frac{1}{3}$ Feet, *per Second*: Which is to the Velocity at the higheſt Point a of the Parabola (Eac) that would be deſcribed, were it not for the Reſiſtance, as 2 to 3, nearly.

$$\left.\begin{array}{lll} 2^\circ. & EA = 2025 & \text{and } Ea = 3030 \\ 3^\circ. & EF = 1727 & \text{and } Ef = 2640 \\ 4^\circ. & AF = 975 & \text{and } af = 1320 \\ 5^\circ. & AC = 1788 & \text{and } ac = 3030 \\ 6^\circ. & FC = 1405 & \text{and } fc = 2640 \end{array}\right\} \text{Feet}$$

7°. Angle C = 57° : 20′ and c (= E) = 45°.

8°. Velocity at C to that at E, as 264 to 412, or as 2 to 3, nearly.

Theſe Proportions, between the Diſtances, in Air and *in vacuo*, hold at an Elevation of 45°, when the Reſiſtance, at going off, is nearly equal to the Gravity, or Weight, of the Ball. If the Velocity be greater than that above ſpecified, or the Body, projected, be, either, leſs, or leſs denſe, the Curve will differ, *ſtill*, more from a Parabola.

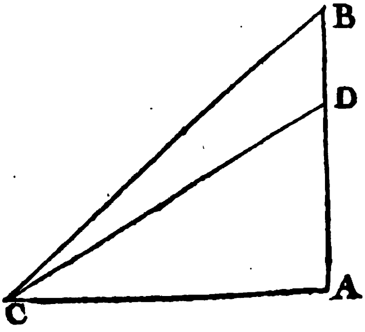
Hence it evidently appears, that the Effect of the Air's Reſiſtance upon very ſwift Motions, is too conſiderable to be intirely diſregarded in the Art of Gunnery.——'Tis true the Method given above is, by much, too intricate for common Practice; but when the Law of the Reſiſtance to very ſwift Motions is once ſufficiently eſtabliſhed (which, according to ſome late Experiments, ſeems to be in a Ratio greater than that of the Square of the Celerity) it will be no very difficult Matter to find out proper Approximations to correct the Proportions in common Uſe.

SECTION IX.

The Uſe of Fluxions in determining the Attraction of Bodies under different Forms.

PROB I.

374. *Suppoſing* AC *perpendicular to* AB, *and that a Corpuſcle at* C *is attracted towards every Point or Particle of the Line* AB, *by Forces in the reciprocal duplicate Ratio of the Diſtances; to determine the Ratio of the whole Force whereby the Corpuſcle is urged in the Direction* CA.

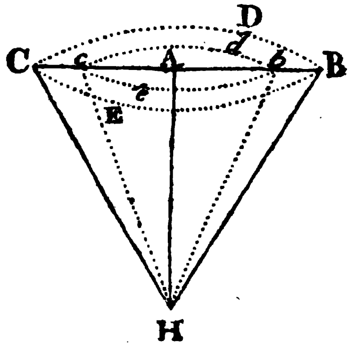


Put AC$=a$, and let AD (conſidered as variable by the Motion of D towards B) be denoted by x: Then, the Force of a Particle at D being as $\frac{1}{CD^2}$ (by Hypotheſis) its Efficacy in the propoſed Direction AC will (by the Reſolution of Forces) be as $\frac{AC}{CD}\times\frac{1}{CD^2}=\frac{AC}{CD^3}=\frac{a}{\overline{a^2+x^2}|^{\frac{3}{2}}}$: Therefore $\frac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$ is the Fluxion of the whole Force; whoſe Fluent, which (*by Art.* 85.) is $=\frac{x}{a\times\overline{a^2+x^2}|^{\frac{1}{2}}}$ $=\frac{AD}{CA\times CD}$, will, when AD=AB, be as the Force itſelf.

Q. E. I.

PROB. II.

375. *Suppoſing* BCDE *to repreſent a circular Plane, and that a Corpuſcle* H, *in the Axis thereof* AH, *is attracted by every Point or Particle of the Plane by Forces in the reciprocal duplicate Ratio of the Diſtances; to find the whole Force by which the Corpuſcle is urged towards the Plane.*



Let AH $=a$, and H$b=x$; then A$b^2=x^2-a^2$; which multiply'd by ($p=$ 3,14159 &c.) the Area of the Circle whoſe Radius is Unity, gives $p\times\overline{x^2-a^2}$ for the Area of the Circle A$cdbe$: whoſe Fluxion is $=2px\dot{x}$. But the Force of a ſingle Particle at b, in the Direction HA, is as $\frac{AH}{Hb^3}$, or $\frac{a}{x^3}$ (ſee the laſt Problem) therefore the Fluxion of the whole Force is truly defined by $2px\dot{x} \times \frac{a}{x^3}$ or its Equal $\frac{2p\dot{x}}{x^2}$ and the Force itſelf by the Fluent of $\frac{2pa\dot{x}}{x^2}$; which (properly corrected) is $-\frac{2pa}{x} + \frac{2pa}{a} = 2p \times \overline{1 - \frac{a}{x}} = 2p \times \overline{1 - \frac{AH}{BH}}$, when $x = HB$. *Q. E. I.*

376. In the preceding Problems, we have ſuppoſed the Attraction of each Particle, to be as the Square of the Diſtance inverſely; that being the Law which is found to obtain in Nature: But if the Force, according to any other Law of Attraction, be required, the Proceſs will be very little different.

Thus, let the Attraction be as any Power (n) of the Diſtance: Then (in the laſt *Prob.*) the Force of a Particle at b (upon H) being as x^n, its Force in the Direction HA will be as $\frac{a}{x} \times x^n$ or ax^{n-1}; which multiply'd by $2px\dot{x}$ (as before) gives $2pax^n\dot{x}$: whereof the Fluent $\frac{2pax^{n+1} - 2pa^{n+2}}{n+1}$ $\left(= \frac{2p}{n+1} \times \overline{AH \times BH^{n+1} - AH^{n+2}}\right)$ will be as the Force required.

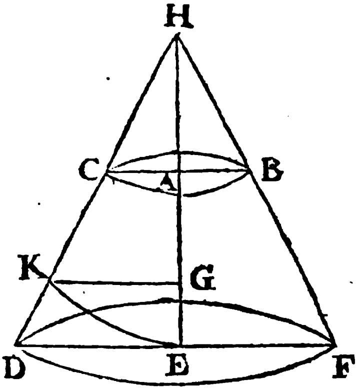
PROB. III.

377. *To determine the Attraction of a Cone* DHF *at its Vertex; the Attraction of each Particle being as the Square of the Diſtance inverſely.*

Put the Axis $EH = q$, the Length of the Slant-Side HD (or HF) $= b$, and AH (conſidered as variable) $= x$: Then (*by ſim. Triangles*) q (HE) : b (HF)

:: x

:: x (HA) : HB $= \frac{bx}{a}$. But, by the laſt Problem,



the Attraction of all the Particles in the Circle BC will be meaſured by $2p \times \overline{1 - \frac{AH}{BH}} = 2p \times \overline{1 - \frac{a}{b}}$ (becauſe HB $= \frac{bx}{a}$): Which therefore being multiply'd by $\dot{x}$, and the Fluent taken, we thence have $x - \frac{ax}{b}$ for the Attraction of ACHB: And this, when $x = a$, will be, $2p \times \overline{EH - \frac{EH^2}{DH}}$, the Force of the whole Cone DEHF: Which, if HK be made = HE, and KG perpendicular to HE, will likewiſe be truly defined by $2p \times EG$ (becauſe HG $= \frac{EH^2}{DH}$). *Q. E. I.*

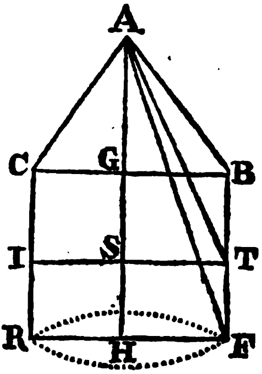
COROLLARY.

378. Seeing the Attraction of ACHB is, every where, as $x - \frac{ax}{b}$, or $\frac{b-a}{b} \times x$, it follows that the Forces of ſimilar Cones, at their Vertexes, are directly as their Altitudes.

PROB.

PROB. IV.

379. *To find the Force of a Cylinder* CBRF, *at any Point* A *in the produced Axis; the Law of Attraction being ſtill as in the preceding Problems.*

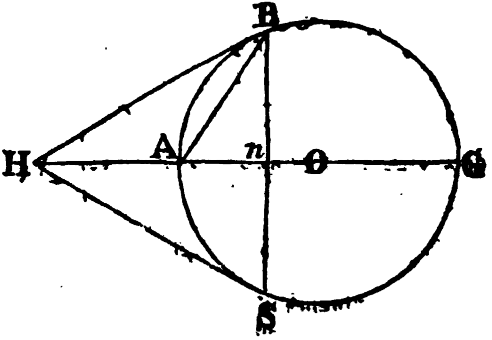


Put BG ($=$ CG $=$ RH) $= b$; and let AS (taken as variable) $= x$: Therefore $AT = \sqrt{b^2+x^2}$, and $2p \times \overline{1 - \frac{AS}{AT}} = 2p \times \overline{1 - \frac{x}{\sqrt{b^2+x^2}}}$: Which (*by Prob.* 2.) expreſſes the Force of all the Particles in the circular Surface IST.

Therefore $2p \times \overline{\dot{x} - \frac{x\dot{x}}{\sqrt{b^2+x^2}}}$ is the Fluxion of the required Force: Whoſe Fluent $(2p \times \overline{x - \sqrt{b^2+x^2}})$ when $x =$ AG, will be $= 2p \times \overline{AG - AB}$; but when $x =$ AH, it will be $= 2p \times \overline{AH - AF}$: Hence, by taking the former of theſe Values from the latter, we have $2p \times \overline{AB + BF - AF}$ for the Meaſure of the true Force by which a Corpuſcle at A is urged towards the Cylinder.

PROB. V.

380. *The Law of the Force being* ſtill *ſuppoſed the ſame; to determine the Attraction of a Sphere* OABGS, *at any given Point* H *above its Surface.*



Let BS be perpendicular to HG, and let HB be drawn; alſo put the Radius AO $= a$, OH $= b$, AH $(b-a)$ $= c$, Hn $= y$, and HB $= c+x$; then A$n = y-c$, G$n = 2a-y+c$, and conſequently $\overline{y-c} \times \overline{2a-y+c}$ (= An × Gn = Bn^2 = BH^2 − Hn^2) $= \overline{c+x}^2 - y^2$: From which Equation we get $y = \frac{2ac+2c^2+2cx+x^2}{2a+2c} = \frac{2bc+2cx+x^2}{2b}$ (becauſe $a+c=b$) Whence alſo $2p \times$

$$\overline{1-\frac{Hn}{HB}}\,{}^{*} = 2p \times \overline{1-\frac{2bc+2cx+x^2}{2b\times\overline{c+x}}} = \frac{2p\times\overline{2ax-x^2}}{2b\times\overline{c+x}}:$$

* Art. 375.

Which multiply'd by $\frac{c\dot{x}+x\dot{x}}{b} = \dot{y}$ gives $\frac{p\times\overline{2ax\dot{x}-x^2\dot{x}}}{b^2}$ for the Fluxion of the required Force; whereof the Fluent

$\frac{p \times \overline{ax^2 - \frac{1}{3}x^3}}{b^2}$ will be the Attraction of the Segment ABS: Which therefore, when B coincides with G and x is $= 2a$, becomes $\frac{4pa^3}{3b^2}$, for the Meaſure of the Attraction of the whole Sphere. *Q. E. I.*

Corollary I.

381. Hence the Attraction $\left(\frac{4pa^3}{3b^2}\right)$ at the Surface of the Sphere, where b is $= a$, will be $\frac{4pa}{3}$; and therefore is directly as the Radius of the Sphere.

Corollary II.

382. Since $\frac{4pa^3}{3}$ is known to expreſs the Content of Sphere whoſe Radius is a *, it is evident that the Attraction $\left(\frac{4pa^3}{3b^2}\right)$ of any Sphere is, univerſally, as its Quantity of Matter directly, and the Square of the Diſtance from its Center inverſely; and is, moreover, the very ſame as it would be, was all the Matter in the Sphere to be united in a Point at the Center.

* Art. 148.

Corollary III.

383. If inſtead of a Corpuſcle, or a ſingle Particle of Matter, at H, we ſuppoſe another Sphere, having its Center at H: Then, ſince the two Spheres, at O and H, act upon each other with the very ſame Forces, as if each Maſs was contracted into its Center, it follows that the abſolute Force, with which two ſpherical Bodies tend towards each other, is as the Product of their Maſſes directly, and the Square of the Diſtance of their

Centers inverſely: And therefore, if the Maſſes are given, will be barely as the Square of the Diſtance.

PROB. VI.

384. *To determine the ſame as in the laſt Problem, the Force of each Particle being as any Power* (n) *of the Diſtance.*

Let HB $= x$, and let every thing elſe remain as above; then we ſhall have $y = \frac{c^2+2ac+x^2}{2b} = d + \frac{x^2}{2b}$ (by putting $d = \frac{c^2+2ac}{2b}$) and conſequently $\dot{y} = \frac{x\dot{x}}{b}$.

Now the Attraction of all the Particles in the circular Surface BS, is as $\frac{2p}{n+1} \times \overline{Hn \times HB^{n+1} - Hn^{n+2}}$ (*by Art.* 376.) $= \frac{2p}{n+1} \times \overline{yx^{n+1} - y^{n+2}}$: Which, multiply'd by $\dot{y}$, gives $\frac{2p}{n+1} \times \overline{x^{n+1}y\dot{y} - y^{n+2}\dot{y}}$ for the Fluxion of the required Force: Which, becauſe $y\dot{y}$ is $= \overline{d + \frac{x^2}{2b}} \times \frac{x\dot{x}}{b} = \frac{dx\dot{x}}{b} + \frac{x^3\dot{x}}{2b^2}$, will likewiſe be expreſſed by $\frac{2p}{n+1} \times \overline{\frac{dx^{n+2}\dot{x}}{b} + \frac{x^{n+4}\dot{x}}{2b^2} - y^{n+2}\dot{y}}$: Whereof the Fluent is $\frac{2p}{n+1} \times \overline{\frac{dx^{n+3}}{n+3 \times b} + \frac{x^{n+5}}{n+5 \times 2b^2} - \frac{y^{n+3}}{n+3}}$: Which, when B coincides with A, or $x = y = c$, will be $= \frac{2p}{n+1} \times \overline{\frac{dc^{n+3}}{n+3 \times b} + \frac{c^{n+5}}{n+5 \times 2b^2} - \frac{c^{n+3}}{n+3}}$: But, when

B coincides with G, or $x = y = 2a + c$ $(=f)$ it will become $= \frac{2p}{n+1} \times \overline{\frac{df^{n+3}}{\overline{n+3} \times b} + \frac{f^{n+5}}{\overline{n+5} \times 2b^2} - \frac{f^{n+3}}{n+3}}$: Therefore the Difference of theſe two, which is $=$

$$\frac{2pf^{n+3}}{n+1} \times \frac{\overline{n+5} \times \overline{2bd - 2b^2} + \overline{n+3} \times f^2}{\overline{n+3} \times \overline{n+5} \times 2b^2} - \frac{2pc^{n+3}}{n+1} \times \frac{\overline{n+5} \times \overline{2bd - 2b^2} - \overline{n+3} \times c^2}{\overline{n+3} \times \overline{n+5} \times 2b^2} = \frac{\overline{\overline{1+n} \times ab - cc} \times 2pf^{n+3} + \overline{\overline{5+n} \times ab + cc} \times 2pc^{n+3}}{\overline{n+1} \times \overline{n+3} \times \overline{n+5} \times b^2}$$

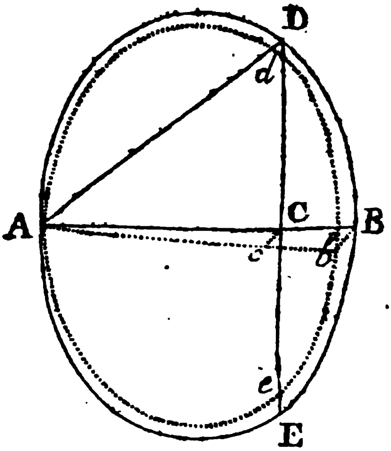
(becauſe $f = a+b$, and $2db = c^2 + 2ac$) will be the Attraction of the whole Sphere. *Q. E. I.*

COROLLARY.

385. Hence, the Attraction at the Surface of the Sphere (where $c = 0$) will be $\frac{2p}{n+1} \times \frac{\overline{1+n} \times \overline{2a}|^{n+3} + \overline{n+5} \times 0^{+3}}{\overline{n+1} . \overline{n+3} . \overline{n+5}}$: Which, if $n+3$ be poſitive, will be $= \frac{2p \times \overline{2a}|^{n+3}}{\overline{n+3} \times \overline{n+5}}$; but, otherwiſe, infinite.

386. *Suppoſing* ADB*b*A *to be a* Cuneus *of uniformly denſe Matter, compriz'd by two equal and ſimilar elliptic Planes* ADBEA *and* A*dbe*A, *inclin'd to each other, at the common Vertex* A, *of either their firſt or ſecond Axes, in an indefinitely ſmall Angle* BA*b*; *To determine the Attraction thereof at the Point* A, *ſuppoſing the Force of each Particle of Matter to be as the Square of the Diſtance inverſely.*

Let DE be any Ordinate to the Axis AB, and let AD be drawn; alſo put AB$=a$, BC$=x$, CD$=y$, and the Sine of the Angle BA*b*, formed by the two Planes



(to the Radius 1) $= d$; and let the Equation of either Curve be $y^2 = fx - x^2 - gx^2$: Which will anſwer, to the Conjugate, or Tranſverſe Axis thereof, according as the Value of g is poſitive or negative.

Now it will be, 1 (Radius) $: d :: a - x$ (AC) $:$ C*c* $= d \times \overline{a-x}$, the Thickneſs of the *Cuneus* at the Ordinate (or Section) DE: Moreover, becauſe $AD^2 = AC^2 + CD^2$, we have $AD = \sqrt{\overline{a-x}|^2 + fx - x^2 - gx^2}$: Whence, $\frac{DE \times Cc}{AC \times AD}$, expreſſing (*by Art.* 374.) the Attraction of the Particles in the indefinitely narrow Rectangle DE$\times$C*c*, will be defined by $\frac{2d\sqrt{fx - x^2 - gx^2}}{\sqrt{\overline{a-x}|^2 + fx - x^2 - gx^2}}$: Which therefore, multiply'd by $\dot{x}$, will give the Fluxion of the Force to be found. But when $fx - x^2 - gx^2$ be-

becomes $= 0$, x will be $= \frac{f}{1+g}$ ($=$AB) $= a$; therefore, by ſubſtituting for f, our Fluxion will be tranſformed to

$$\frac{2d\dot{x}\sqrt{\overline{1+g}\times ax-\overline{1+g}\times x^2}}{\sqrt{\overline{a-x}|^2+\overline{1+g}\times ax-\overline{1+g}\times x^2}} = \frac{2d\dot{x}\sqrt{\overline{1+g}\times \overline{ax-x^2}}}{\sqrt{\overline{a-x}|^2+\overline{1+g}\times\overline{ax-x^2}}} = \frac{2d\dot{x}\sqrt{\overline{1+g}\times x}}{\sqrt{a-x+\overline{1+g}\times x}} =$$

$$\frac{2d\times\overline{1+g}|^{\frac{1}{2}}\times x^{\frac{1}{2}}\dot{x}}{\overline{a+gx}|^{\frac{1}{2}}} = \frac{\overline{1+g}|^{\frac{1}{2}}\times 2dx^{\frac{1}{2}}\dot{x}}{a^{\frac{1}{2}}} \times \overline{1-\frac{gx}{2a}+\frac{3g^2x^2}{2.4a^2}-\frac{3.5g^3x^3}{2.4.6a^3}}$$ &c.

Whereof the Fluent, when $x = a$, will be $\overline{1+g}|^{\frac{1}{2}}\times 2ad\times\overline{\frac{2}{3}-\frac{2}{5}\times\frac{g}{2}+\frac{2}{7}\times\frac{3g^2}{2.4}-\frac{2}{9}\times\frac{3.5g^3}{2.4.6}}$ &c. Which, becauſe $\overline{1+g}|^{\frac{1}{2}}\times a$ is $= f\times\overline{1+g}|^{-\frac{1}{2}} = f\times\overline{1-\frac{g}{2}+\frac{3g}{2.4}-\frac{3.5g^2}{2.4.6}}$ &c. will (by multiplying the two Serieſes together &c.) be reduced to $2df\times\overline{\frac{2}{3}-\frac{2.4g}{3.5}+\frac{2.4.6g^3}{3.5.7}-\frac{2.4.6.8g^3}{3.5.7.9}}$ &c.

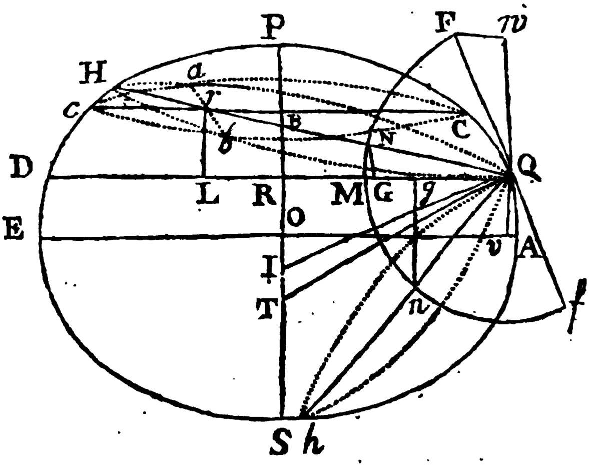
Q. E. I.

It may be obſerved, that the Fluent given above may be brought out without an Infinite Series (*by Art.* 126. *and* 278.) But the Solution here exhibited is beſt adapted to what follows hereafter; to which the Propoſition itſelf is premiſed as a *Lemma*.

PROB. VIII.

387. *To determine the Attraction at any Point* Q *in the Surface of a given Spheroid* OAPES.

Let QRL be perpendicular to the Axis PS of the Spheroid, and QT perpendicular to the Tangent F*f* of the generating Ellipſis at Q, meeting PS in T: Moreover, let Q*a*H*b* be a Section of the Spheroid by a Plane perpendicular to that of the Ellipſis APES, and thro' any Point *r*, in the Axis thereof, draw CB*c* and *r*L parallel to AE and PS: And make the Abſciſſa Q*r* $= x$, its correſponding Semi-Ordinate *ra* (or *rb*) $= y$, QR $= a$,



and RT $= b$; alſo let the Sine (NG) of the Angle HQD (to the Radius NQ $= 1$) $= p$, its Co-ſine QG $= q$, and the Ratio of OA^2 to OP^2, as any given Quantity b to Unity. Now, by reaſon of the ſimilar Triangles Q*r*L and QNG, we have *r*L (BR) $= px$, and QL $= qx$, and therefore B*r* (RL) $= qx - a$: Alſo, from the Nature of the Ellipſis, AO^2 : PO^2 $(b : 1) :: \text{RT}\ (b) : \text{OR} = \frac{b}{b}$: Likewiſe AO^2 : PO^2 $(b : 1)$

$(b:1)::\mathrm{QR}^2:\mathrm{OP}^2-\mathrm{OR}^2$; and $\mathrm{PO}^2:\mathrm{AO}^2\ (1:b)$ $::\mathrm{OP}^2-\mathrm{OB}^2:\mathrm{BC}^2=b\times\overline{\mathrm{OP}^2-\mathrm{OB}^2}=b\times$ $\overline{\mathrm{OP}^2-\overline{\mathrm{OR}+\mathrm{RB}}^2}=b\times\overline{\mathrm{OP}^2-\mathrm{OR}^2-2\mathrm{OR}\times\mathrm{RB}-\mathrm{RB}^2}$ $=\mathrm{QR}^2+b\times\overline{-2\mathrm{OR}\times\mathrm{RB}-\mathrm{RB}^2}$; becauſe (by the former Proportion) $\mathrm{QR}^2=b\times\overline{\mathrm{OP}^2-\mathrm{OR}^2}$: Whence, by the Property of the Circle, C*acb*, we get $y^2\ (\mathrm{BC}^2-\mathrm{B}r^2)=\mathrm{QR}^2-$ $\mathrm{B}r^2-b\times\overline{2\mathrm{OR}\times\mathrm{RB}+\mathrm{RB}^2}=a^2-\overline{qx-a}^2-b\times$ $\overline{\frac{2b}{b}\times px+p^2x^2}=\overline{aq-bp}\times 2x-\overline{q^2+bp^2}\times x^2$: Which Equation, by making $1+B=b$, becomes $y^2=$ $\overline{aq-bp}\times 2x-\overline{q^2+p^2+Bp^2}\times x^2=\overline{aq-bp}\times 2x-x^2-$ Bp^2x^2 (becauſe $q^2+p^2=1=\mathrm{QN}^2$: Which being only of two Dimenſions, the Curve Q*a*H*b*, whereto it belongs, is an Ellipſis.

The Equation of the Curve Q*a*H*b* being now obtained, let its Axis QH be ſuppoſed to revolve about Q, as a Center (the Plane of the Curve being always perpendicular to that of the Ellipſis APES) and let the Fluxion of the Arch MN (expreſſing the Angle deſcribed from the time the ſaid Axis begins its Motion at the Poſition ALD) be denoted by $\dot{A}$: Then, it is evident from the preceding Problem, that, $\overline{2aq-2bp}\times 2\dot{A}\times$ $\overline{\frac{2}{3}-\frac{2\cdot 4Bp^2}{3\cdot 5}+\frac{2\cdot 4\cdot 6B^2p^4}{3\cdot 5\cdot 7}}$ &c. will be the Fluxion of the Attraction of the correſponding Part DQH of the Solid, upon a Corpuſcle at Q, conſidered as acting in the Direction HQ (which Expreſſion is found, by, barely, writing $2aq-2bq$, $\dot{A}$, and Bp^2, in the ſaid Problem, for f, d, and g reſpectively.)

Hence, by the Reſolution of Forces, the Fluxion of the Attraction, in the Directions QR and Q*w* (perpendicular to QR) will be truly exhibited by $\overline{2aq-2bp} \times 2\dot{A}q \times \overline{\frac{2}{3} - \frac{2\cdot 4Bp^2}{3\cdot 5} + \frac{2\cdot 4\cdot 6B^2p^4}{3\cdot 5\cdot 7}\ \&c.}$ and $\overline{2aq-2bp} \times 2\dot{A}p \times \overline{\frac{2}{3} - \frac{2\cdot 4Bp^2}{3\cdot 5} + \frac{2\cdot 4\cdot 6B^2p^4}{3\cdot 5\cdot 7}\ \&c.}$

Let now another Plane Q*b* be ſuppoſed to revolve about the Point Q, the contrary Way to the former, from QD towards Q*f*; and let (*ng*) the Sine of the Angle RQ*b* be denoted by P, and its Co-ſine (Q*g*) by Q: Then the Fluxion of the Attraction of the Part DQ*b*, in the foreſaid Directions QR and Q*w* (by writing $-P$ inſtead of p and Q inſtead of q) will appear to be $\overline{2aQ+2bP} \times 2\dot{A}Q \times \overline{\frac{2}{3} - \frac{2\cdot 4BP^2}{3\cdot 5} + \frac{2\cdot 4\cdot 6B^2P^4\ \&c.}{3\cdot 5\cdot 7}}$ and $\overline{2aQ+2bP} \times -2\dot{A}P \times \overline{\frac{2}{3} - \frac{2\cdot 4BP^2}{3\cdot 5} + \frac{2\cdot 4\cdot 6B^2P^4}{3\cdot 5\cdot 7}}\ \&c.$ Which being added to thoſe of the former Part, in the ſame Directions, and $\frac{\dot{p}}{q}$ and $\frac{\dot{P}}{Q}$ reſpectively ſubſtituted inſtead of $\dot{A}$*, we have

* Art. 142.

$$4a \text{ into } \frac{2}{3} \times \overline{q\dot{p}+Q\dot{P}} - \frac{2\cdot 4B}{3\cdot 5} \times \overline{qp^2\dot{p}+QP^2\dot{P}}\ \&c.$$
$$+\ 4b \text{ into } \frac{2}{3} \times \overline{P\dot{P}-p\dot{p}} - \frac{2\cdot 4B}{3\cdot 5} \times \overline{P^3\dot{P}-p^3\dot{p}}\ \&c.$$

And

$$4a \text{ into } \frac{2}{3} \times \overline{p\dot{p}-P\dot{P}} - \frac{2\cdot 4B}{3\cdot 5} \times \overline{p^3\dot{p}-P^3\dot{P}}\ \&c.$$
$$-4b \text{ into } \frac{2}{3} \times \overline{\frac{p^2\dot{p}}{q}+\frac{P^2\dot{P}}{Q}} - \frac{2\cdot 4B}{3\cdot 5} \times \overline{\frac{p^4\dot{p}}{q}+\frac{P^4\dot{P}}{Q}}\ \&c.$$

for the Fluxion of the Attraction of both Parts together in the foreſaid Directions: Whereof the Fluents, when N coincides with F, and n with f, will be the Attraction of the whole Spheroid in thoſe Directions. But now, in order to determine theſe Fluents with as little Trouble as poſſible, let m be aſſumed to denote any whole poſitive Number; then the Fluent of $\frac{p^{2m}\dot{p}}{q}$, or $\frac{p^{2m}\dot{p}}{\sqrt{1-p^2}}$, will be univerſally $= \frac{-q}{2m} \times \overline{p^{2m-1}}$

$$\overline{+ \frac{2m-1}{2m-2} \times p^{2m-3} + \frac{2m-1 \,.\, 2m-3}{2m-2 \,.\, 2m-4} \times p^{2m-5}} \quad (m)$$

$$+ \frac{1 \cdot 3 \cdot 5 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \ldots 2m} \times \text{the Arch (MN) whoſe Sine}$$

is p *: And that of $\frac{P^{2m}\dot{P}}{Q}$, or $\frac{P^{2m}\dot{P}}{\sqrt{1-P^2}}$ (in the * Art. 296. ſame Manner) $= \frac{-Q}{2m} \times \overline{P^{2m-1} + \frac{2m-1}{2m-2} \times P^{2m-3}}$

&c. $+ \frac{1 \cdot 2 \cdot 3 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \ldots 2m} \times$ Arch (Mn) whoſe Sine is P. But when N coincides with F, and n with f, the Sines p and P, of the Arches MF and Mf, becoming equal, and (the Co-ſine) $Q = -$ (Co-ſine) q, it is evident that the Sum of the Fluents of $\frac{p^{2m}\dot{p}}{q}$ and $\frac{P^{2m}\dot{P}}{Q}$, will, in that Caſe, be truly exhibited by $\frac{1 \cdot 3 \cdot 5 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \ldots 2m} \times \text{MF} + \frac{1 \cdot 3 \cdot 5 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \ldots 2m} \times$ Mf, or its Equal $\frac{1 \cdot 3 \cdot 5 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \ldots 2m} \times$ FMf; becauſe,

cauſe, then all the reſt of the Terms (by reaſon of the equal Quantities P, p and Q, $-q$) deſtroy one another. After the ſame Manner the Sum of the Fluents of $qp^{2m}\dot{p}$ and $QP^{2m}\dot{P}$, in the foreſaid Circumſtance, will

* Art. 297. appear to be $= \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \cdot 8 \ldots \overline{2m+3}} \times \mathrm{FM}f$*.

Now, to apply this to the Matter in hand, let the Exponent of B, in any Term of either of the above found Fluxions be, univerſally, expreſſed by n; then the numeral Coefficient (annexed to B) will be defined by $\frac{2 \cdot 4 \cdot 6 \ldots \overline{2n+2}}{1 \cdot 3 \cdot 5 \ldots \overline{2n+3}}$, and the variable Quantities multiplied thereby, in the firſt Line of the former Fluxion, will be $qp^{2n}\dot{p} + QP^{2n}\dot{P}$: Therefore $\frac{2 \cdot 4 \cdot 6 \ldots \overline{2n+2}}{1 \cdot 3 \cdot 5 \ldots \overline{2n+3}} \times B^n \times \overline{qp^{2n}\dot{p} + QP^{2n}\dot{P}}$ is a General Term, (from whence, if n be expounded by 1, 2, 3 &c. ſucceſſively, that whole Line will be produced.) But, the Fluent of $qp^{2n}\dot{p} + QP^{2n}\dot{P}$, in the Circumſtance above ſpecified, (putting $m = n$ and $\mathrm{FM}f = k$) appears to be $= \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots \overline{2n-1}}{2 \cdot 4 \cdot 6 \cdot 8 \ldots \overline{2n+2}} \times k$: Which, therefore, multiplied by $\frac{2 \cdot 4 \cdot 6 \ldots \overline{2n+2}}{3 \cdot 5 \ldots \overline{2n+3}} \times B^n$, gives $\frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots \overline{2n-1}}{2 \cdot 4 \cdot 6 \cdot 8 \ldots \overline{2n+2}} \times \frac{2 \cdot 4 \cdot 6 \ldots \overline{2n+2}}{3 \cdot 5 \ldots \overline{2n+3}} \times B^n k = \frac{B^n k}{\overline{2n+1} \times \overline{2n+3}}$, for the true Fluent of the ſaid General Term: Which, if n be expounded by

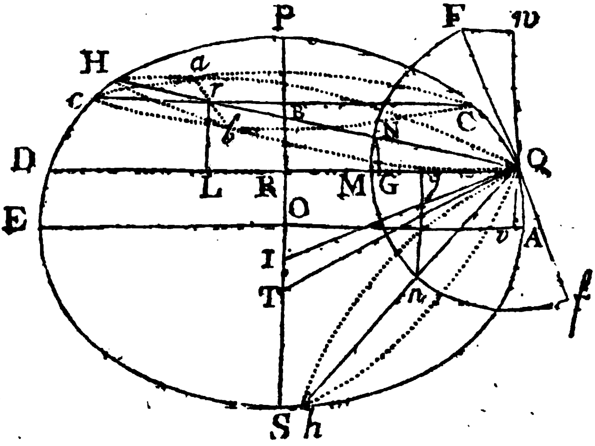
O,

0, 1, 2, 3 &c. ſucceſſively, will become equal to $\frac{k}{1\cdot 3}$, $\frac{Bk}{3\cdot 5}$, $\frac{B^2k}{5\cdot 7}$, $\frac{B^3k}{7\cdot 9}$ &c. reſpectively; and therefore the Fluent of the whole Line (drawn into the general Multiplicator $4a$) is $= 4ak \times \overline{\frac{1}{1\cdot 3} - \frac{B}{3\cdot 5} + \frac{B^2}{5\cdot 7} - \frac{B^3}{7\cdot 9}}$ &c. But now, for the Fluent of the ſecond Line: This, it is plain, will be $= 4b$ into $\frac{2}{3} \times \overline{\frac{P^2}{2} - \frac{p^2}{2}} - \frac{2\cdot 4B}{3\cdot 5} \times \overline{\frac{P^4}{4} - \frac{p^4}{4}}$ &c. Which, in the foreſaid Circumſtance, when $P = p$, intirely vaniſhes. Therefore it appears, that the Attraction of the whole Spheroid, in the Direction QR, is truly expreſſed by $4ak \times \overline{\frac{1}{1\cdot 3} - \frac{B}{3\cdot 5} + \frac{B^2}{5\cdot 7} - \frac{B^3}{7\cdot 9}}$, or its Equal $4k \times \overline{\frac{1}{1\cdot 3} - \frac{B}{3\cdot 5} + \frac{B^2}{5\cdot 7}}$ &c. $\times$ *QR*.

After the ſame Manner the Fluent of the firſt Line, in the latter of our two Fluxions, will be found to vaniſh: And $\frac{2\cdot 4\cdot 6\ldots\overline{2n+2}}{1\cdot 3\cdot 5\ldots\overline{2n+3}} \times B^n \times \overline{\frac{p^{2n+2}\dot{p}}{q} + \frac{P^{2n+2}\dot{P}}{Q}}$ will be a General Term to the ſecond Line. Whereof the Fluent (by expounding $2m$ by $\overline{2n+2}$) appears, from above, to be $= \frac{2\cdot 4\cdot 6\ldots\overline{2n+2}}{3\cdot 5\cdot 7\ldots\overline{2n+3}} \times B^n k \times \frac{1\cdot 3\cdot 5\ldots\overline{2n+1}}{2\cdot 4\cdot 6\ldots\overline{2n+2}} = \frac{B^n k}{2n+3}$: Which, when n is

n is interpreted by 0, 1, 2, 3 &c. ſucceſſively, comes out equal to $\frac{k}{3}$; $\frac{Bk}{5}$, $\frac{B^2k}{7}$ &c. reſpectively: Therefore the Attraction of the Spheroid, in the Direction Qw, is exhibited by $-4bk \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}}$ &c. and conſequently, That in the oppoſite Direction Qv, by $4bk \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}} = 4k \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}}$ &c. $\times RT = 4k \times \overline{1 + B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}}$ &c. $\times OR$ (becauſe $\overline{1 + B} \times OR = RT$.)

From which and the Force in the Direction QR (found above) not only the Direction of the abſolute



Attraction, but that Attraction itſelf will be known: For, let RI be taken to QR, as the Force in the Direction Qv to that in the Direction QR; and then, by the Compoſition of Forces, QI will be the Direction of the Attraction, or the Line in which a Corpuſcle at Q tends to deſcend: And the Attraction itſelf, in that Direction, (being to that in QR, as QI to QR) will be defined by $4k \times \overline{\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}\ \&c.} \times QI$; which, ſince $4k$ is conſtant, will alſo be as $\overline{\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}\ \&c.} \times QI$. *Q.E.I.*

Corollary.

388. Since, by Conſtruction, RI : QR :: $\overline{1+B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}\ \&c.} \times QR : \overline{\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}\ \&c.} \times QR$, it follows that $\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}\ \&c. : \overline{1+B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}\ \&c.}$ :: RO : RI; whence (by Diviſion) $\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}\ \&c. : \frac{3B}{3\cdot5} - \frac{3B^2}{5\cdot7} + \frac{3B^3}{7\cdot9}\ \&c.$ :: OR $\left(:\frac{OT}{B}\right)$: OI; and conſequently, $\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}\ \&c. : 3 \times \overline{\frac{1}{3\cdot5} - \frac{B}{5\cdot7} + \frac{B^2}{7\cdot9}\ \&c.}$ :: OT : OI.

Hence it appears that the Direction QI, of the abſolute Attraction, divides the Part of the Axis OT, intercepted by the Center and Normal, in a given Ratio: And that the Attraction itſelf (being defined by $\overline{\frac{1}{1\cdot 3} - \frac{B}{3\cdot 5} + \frac{B^2}{5\cdot 7}\ \&c. \times QI}$) is every where as the ſaid Line of Direction QI.

SCHOLIUM.

389. Although the foregoing Concluſions are exhibited by infinite Serieſes, yet the Sums of thoſe Serieſes are explicable by means of the Arch of a Circle.

Thus, let the Series $\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}\ \&c.$ (which is one of the two original ones above found) be put $= S$, and let $B = t^2$; then by Subſtitution, and multiplying the whole Equation by t^3, we ſhall have $\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7}\ \&c. = t^3 S$, and conſequently $t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7}\ \&c. = t - t^3 S$: Where, the former Part of the Equation is known to expreſs the Arch of a Circle, whoſe Tangent is $t\ (B^{\frac{1}{2}})$ and Radius Unity *: Wherefore, putting that Arch $= A$, we have $A = t - t^3 S$, and conſequently $S = \frac{t - A}{t^3} = \frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}\ \&c.$

Moreover, ſince it appears that

$$\left.\begin{array}{l} \frac{B}{3} - \frac{B^2}{5} + \frac{B^3}{7}\ \&c. \\ -\frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}\ \&c. \end{array}\right\}\ \text{is} = \frac{2B}{3\cdot 5} - \frac{2B^2}{5\cdot 7} + \frac{2B^3}{7\cdot 9}\ \&c.$$

(where the Sum of $\frac{B}{3} - \frac{B^2}{5} + \frac{B^3}{7}\ \&c.$ is already found $= \frac{t - A}{t^3} \times B = \frac{t - A}{t}$, and where That of

[*] Art. 142.

of $-\frac{B}{5}+\frac{B^2}{7}$ &c. by the ſame Method will come out $=\frac{t-A-\frac{1}{3}t^3}{t^3}$) it is evident that $\frac{2B}{3\cdot 5}-\frac{2B^2}{5\cdot 7}+\frac{2B^3}{7\cdot 9}$ &c. $=\frac{t-A}{t}+\frac{t-A-\frac{1}{3}t^3}{t^3}=\frac{\frac{2}{3}t^3+t-A\times\overline{1+t^2}}{t^3}$; and conſequently $\frac{1}{1\cdot 3}-\frac{B}{3\cdot 5}+\frac{B^2}{5\cdot 7}$ &c. $\left(=\frac{1}{3}-\frac{\frac{2}{3}t^3+t^2-A\times\overline{1+t^2}}{2t^3}\right)=\frac{A\times\overline{1+t^2}-t}{2t^3}$: Which is the Value of the other original Series found above: From whence that of $\frac{3}{3\cdot 5}-\frac{3B}{5\cdot 7}+\frac{3B^2}{7\cdot 9}$ will alſo be had $=\frac{3t+2t^3-3A\times\overline{1+t^2}}{2t^5}$.

Hence, if

$$\frac{t-A}{t^3}\left(=\frac{1}{3}-\frac{B}{5}+\frac{B^2}{7}-\frac{B^3}{9}\right) \text{ be put } = f;$$

$$\frac{A\times\overline{1+t^2}-t}{2t^3}\left(=\frac{1}{1\cdot 3}-\frac{B}{3\cdot 5}+\frac{B^2}{5\cdot 7}\ \text{\&c.}\right)=g;$$

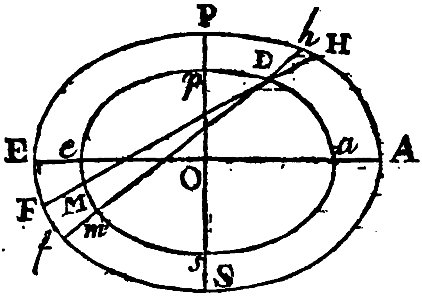
And

$$\frac{3t+2t^3-3A\times\overline{1+t^2}}{2t^5}\left(=\frac{3}{3\cdot 5}-\frac{3B}{5\cdot 7}+\frac{3B^2}{7\cdot 9}\ \text{\&c.}\right)=h$$

it is evident that OT will be to OI, in the conſtant Ratio of g to h; and that the Forces in the Directions QI, QR, and Qv, will be as $g\times$ QI, $g\times$ QR, and $f\times\overline{1+B}\times$ OR reſpectively: Where $1+B$ is $=\frac{\text{AO}^2}{\text{PO}^2}$.

PROB. IX.

390. *To determine the Attraction at any Point* D *within a given Spheroid* OAPES.



Let O*apes* be another Spheroid, concentric with, and ſimilar to, the given one; whoſe Surface D*m*M &c. paſſes through the given Point D; alſo let FD*f* and HD*h* be taken as two oppoſite, indefinitely ſlender, Cones (or Pyramids) conceived to be formed by drawing innumerable Lines HDF, *h*D*f* &c. through the common Vertex D) which Cones (or Pyramids) having the ſame Angle, may be conſidered as ſimilar; and ſo their Forces, at D, will be as the Altitudes DF and DH*: And, therefore, the Exceſs of the former, above the latter, or the Force whereby a Corpuſcle at D, tends towards F, through the contrary Action of the two oppoſite Cones, will be as DF—DH, or as DM; becauſe (*by the Property of the Ellipſis*) MF is, in all Poſitions, equal to DH.

* Art. 378.

Hence it appears that the Parts of Matter FM*mf* and HD*h*, without the Spheroid *apes* (acting equally, in contrary Directions) can have no Effect at D: And this, being every where the Caſe, the whole, efficacious, Force at D muſt therefore be that of the Spheroid O*apes*.

Hence, if the Ratio of Oa^2 to Op^2 (or of OA^2 to OP^2) be denoted by that of $1+B$ to 1, as in the laſt Problem, it follows, from thence, that the Attraction at D, in the Directions DM and DN (perpendicular to PS and AE; *ſee the next Fig.*) will be expounded by $\overline{\frac{1}{1\cdot 3} - \frac{B}{3\cdot 5} + \frac{B^2}{5\cdot 7}\ \&c.} \times DM$, and $\overline{1+B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}\ \&c.} \times DN$ reſpectively, or by their Equals $g \times DM$ and $f \times \overline{1+B} \times DN$: Where the Values of f and g are the ſame as given in the preceding Article.

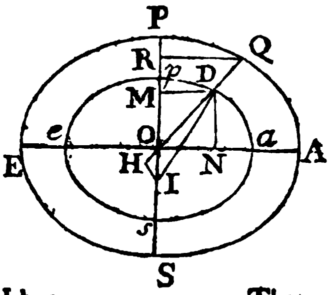
COROLLARY.

391. Hence the Force wherewith a Corpuſcle, any where within a given Spheroid, is attracted, either, towards the Axis, or the Plane of its Equator, is directly as the Diſtance therefrom.

PROB. X.

392. *Suppoſing every Particle of Matter in a Spheroid to have a Tendency to recede, both, from the Axis* PS, *and from the Plane of the greateſt Circle, by Means of Forces that are as the Diſtances from the ſaid Axis, and Plane, reſpectively; to find the Direction* DI *wherein a Corpuſcle, at any Point* D, *tends to move through the Action of the ſaid Forces and the Attraction conjunctly; and likewiſe the whole compound Force in that Direction.*

Let DM and DN be perpendicular to PS and AE, and let the given Forces, in the Direction of thoſe Lines (independent of the Attraction) be expreſſed by $m \times DM$ and $n \times DN$ reſpectively.



Therefore, ſince (by the laſt Problem) the Force of Attraction in the ſaid Directions is defined by $g \times$ DM and $f \times \overline{1+B} \times$ DN, the whole reſulting Forces will be truly denoted by $\overline{g-m} \times$ DM, and $\overline{f \times \overline{1+B}-n}$ $\times$ DN: Whence (by the Compoſition of Forces) it will be, $g-m : f \times \overline{1+B}-n$ :: DN (OM) : MI; whence the Point I is given:

Alſo DM : DI :: $\overline{g-m} \times$ DM (the Force in the Direction DM) : $\overline{g-m} \times$ DI, the Force in DI. *Q.E.I.*

PROB. XI.

393. *Every thing being ſuppoſed as in the preceding Problems, it is required to determine the Force of all the Particles in the Line (or Column)* QDO *tending to the Center* O *of the Spheroid.*

Let IH be perpendicular to QO produced (*ſee the laſt Fig.*) then the abſolute Force, in the Direction DI, being $\overline{g-m} \times$ DI, that in the Direction DH, whereby a Corpuſcle at D is urged towards the Center, will be $\overline{g-m} \times$ DH. Let now OD (conſidered as variable) be denoted by x; then becauſe the Ratio of OM to MI is given (being every where as $g-m$ to $f \times \overline{1+B}-n$, *by the Precedent*) and the Triangles ODM and IOH are ſimilar, it follows that the Ratio of OD to OH will be given, or conſtant; and conſequently that of DH to OH, likewiſe: Let therefore this Ratio of DH to OH be expreſſed by that of r to s, and we ſhall have DH $= \frac{rx}{s}$, and conſequently ($\overline{g-m} \times$ DH) the Force at D, equal to $\overline{g-m} \times \frac{rx}{s}$: Which therefore being multiplied by $\dot{x}$, and the Fluent taken, there comes out $\frac{\overline{g-m}\times rx^2}{2s}=\frac{g-m}{2}\times$ DO × DH, for the whole Force of the Line or Column OD at the Center.

Q. E. I.

Corollary.

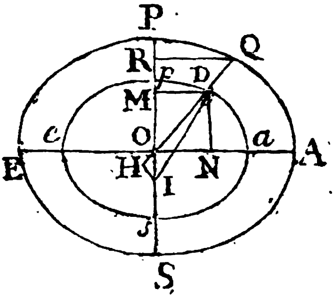
394. If the given Forces m and n be ſuch that the Ratio of OM to MI, (which is found to be univerſally as $g-m$ to $f\times\overline{1+B}-n$) may become as $1:1+B$ (or as $pO^2:aO^2$) it is evident (from the Property of the Ellipſis) that the Line of Direction DI will be always perpendicular to the Surface of the Spheroid *Oapes*. In which Caſe OD × DH is alſo (by the Nature of the Ellipſis) $=Oa^2$; And therefore the Force $\left(\frac{g-m}{2}\times OD\times DH\right)$ of OD is $=\frac{g-m}{2}\times Oa^2$: Which, when D coincides with Q, will become $\frac{g-m}{2}\times AO^2$; and is, therefore, a conſtant Quantity.

Moreover, ſince in this Caſe, $g-m:f\times\overline{1+B}-n::1:1+B$ (by Hypotheſis) we have $m-\frac{n}{1+B}=g-f$: Which Equation, if n be taken $=0$, gives $m=g-f=\frac{2B}{3\cdot5}-\frac{4B^2}{5\cdot7}+\frac{6B^3}{7\cdot9}$ &c. $=\frac{\overline{3+t^3}\times A-3t}{2t^3}$ *; But, if m be taken $=0$, it will then give $n=-\overline{1+B}\times\overline{g-f}=-\overline{1+B}\times\overline{\frac{2B}{3\cdot5}-\frac{4B^2}{5\cdot7}+\frac{6B^3}{7\cdot9}}$ &c. Where, $t=B^{\frac{1}{2}}$, and $A=$ the Arch whoſe Tangent is t, and Radius Unity.

* Art. 389.

PROP. XII.

395. *If an oblate Spheroid* OAPES, *whereof the Square of the Equatoreal Diameter* AE, *is to that of the Axis* PS, *in any given Ratio of* $1 + B$ *to* 1, *revolves about its Axis, in ſuch a Time, that the centrifugal Force, at the Equator* A, *is to the Attraction at the Surface of a Sphere whoſe Radius is* OA, *in the Ratio of* $\frac{2B}{3\cdot5} - \frac{4B^2}{5\cdot7} + \frac{6B^3}{7\cdot9}$ *&c.* to $\frac{1}{3}$: *I ſay, in that Caſe, every Particle of the Spheroid will be* in Equilibrio; *ſo that, though the Coheſion of the Parts was to ceaſe, the Figure itſelf would remain unchanged.*



For, the Attraction of the Spheroid, at A, being defined by $\overline{\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}\ \&c.} \times AO$ (*Art.* 387.) it is evident (by conceiving $B = 0$) that $\frac{AO}{3}$ will represent the Attraction at the Surface of the Sphere whoſe Radius is AO: Whence (by Hypotheſis) the centrifugal Force at A (putting $m = \frac{2B}{3\cdot5} - \frac{4B^2}{5\cdot7} + \frac{6B^3}{7\cdot9}$ *&c.*) will be truly defined by $m \times AO$; and conſequently That, at any other Point D, by $m \times DM$ (becauſe the centrifugal Forces of Bodies deſcribing unequal Circles, in equal Times, are known to be directly as the Radii *.) Hence, and from the Corollary to the laſt Problem, it appears that the Direction of Gravitation DI is always perpendicular to the Surface *apes*; and that the Force of all the Particles in the Line (or Canal) OD or OQ, towards the Center O, will continue invariable, take the Point Q in what Part of the Arch APE you will: From which laſt Conſideration, it follows that the Force, or Preſſure of every Canal QO, at the Center O, (conſidering the Body in a fluid State) will be the ſame: Whence (by the Principles of Hydroſtatics) a Corpuſcle at D has no Tendency to move, either Way, in the Line OQ: And therefore, as it hath no Tendency to move in the Direction of the Surface D*pe* (the Gravitation being perpendicular thereto) it is evident, *from Mechanics*, that no Motion at all can enſue, in any Direction. *Q. E. D.*

COROLLARY I.

396. Since m is $= \frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9}$ &c. the Gravitation $(\overline{g - m} \times DI)$ at any Point D in the Spheroid will therefore be as $\overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} \text{ \&c.}} \times DI = \frac{t - A}{t^3} \times DI$ (*ſee Art.* 389.

COROLLARY II.

397. If the Time of Revolution be given $= p$, and q be put to denote the Time wherein a (ſolid) Sphere, of the ſame Denſity with the Spheroid, muſt revolve; ſo that the centrifugal Force, at the Equator thereof, may be equal to the Gravity: Then, as this laſt Time is known to continue the ſame, whatever the Magnitude of that Sphere is †; and the centrifugal Forces, in equal

* Art. 213.

† Art. 213, and 381.

Circles, are alſo known to be inverſely as the Squares of the periodic Times——it follows, that $p^2 : q^2 :: \frac{1}{3}$AO (the Attraction, or centrifugal Force, reſpecting the Sphere OA, revolving in the Time q) : $\overline{\frac{2B}{3\cdot 5} - \frac{4B^2}{5\cdot 7} + \frac{6B^3}{7\cdot 9}\ \&c.} \times$ AO, the centrifugal Force of the Spheroid at A, revolving in the Time p. From which Proportion we get $\frac{q^2}{3p^2} = \frac{2B}{3\cdot 5} - \frac{4B^2}{5\cdot 7} + \frac{6B^3}{7\cdot 9}$ &c. $= \frac{\overline{3+t^2}\times A - 3t}{2t^3}$ (*Art.* 394.) Whence, by Help of the Trigonometrical-Canon, the Value of t ($= B^{\frac{1}{2}}$) and, conſequently, the Ratio of the two principal Diameters, will be found; ſo that all the Parts of the Spheroid may remain *in Equilibrio*. But, when $\frac{q^2}{3p^2}$ is ſmall, the Solution by an Infinite Series is preferable: For, then the Series $\frac{2B}{3\cdot 5} - \frac{4B^2}{5\cdot 7}$ &c. ($= \frac{q^2}{3p^2}$) converging ſufficiently ſwift, we ſhall, by the Reverſion thereof, find $B = \frac{5q^2}{2p^2} + \frac{25\times 6q^4}{4\times 7p^4} + \frac{125\times 37q^6}{8\times 49p^6}$ &c. In which Caſe the Ratio of the Equatoreal Diameter to the Axis, if we take only the firſt Term of the Series, will be, as $\sqrt{1+\frac{5q^2}{2p^2}} : 1$, or as $1 + \frac{5q^2}{4p^2}$, nearly.

Which, if $\frac{p^2}{q^2} = 289$, or the centrifugal Force at the Equator be to the Gravity as 1 to 289 (that being the Proportion at the Equator of the Earth *) will come out as 231 to 230.

* Art. 237.

Cor

COROLLARY III.

398. Becauſe, $\frac{\overline{3+t^2}\times A-3t}{2t^3}$, the latter Part of our foregoing Equation will be equal to Nothing, both when t is Nothing and Infinite, it is evident that the Value thereof cannot, in any intermediate Circumſtance of t, exceed a certain aſſignable Quantity.

Wherefore, to determine this Limit of the Value of $\frac{q^2}{3p^2}$ (beyond which the Problem becomes impoſſible) let the Fluxion of $\frac{\overline{3+t^2}\times A-3t}{2t^3}$, or its Double $\frac{\overline{3+t^2}\times A}{t^3}-\frac{3}{t^2}$ be taken and put $=0$, and you will have $-\overline{9+t^2}\times A\dot{t}+\overline{3t+t^3}\times\dot{A}+6t\dot{t}=0$: Which, becauſe $\dot{A}=\frac{\dot{t}}{1+t^2}$ * will be reduced to $9t+7t^3-\overline{1+t^2}\times\overline{9+t^2}\times A=0$; where t is found $=2{,}5293$, from whence the correſponding Values of $\sqrt{1+t^2}$, and $\frac{q}{p}$ come out $=2{,}7198$, and 0.5805 &c. reſpectively. Hence it appears that it is impoſſible for the Parts of the Spheroid, in a fluid State, to continue at Reſt among themſelves, when the Time of Revolution is ſo great that $\frac{q}{p}$ exceeds $0{,}5805$ &c. And that, of all the Spheroids which can be aſſumed by a Fluid revolving about an Axis, That whoſe Equatoreal Diameter is to its Axis as 2,7198 to Unity, will perform its Revolutions in the ſhorteſt Time.

Thus, for Example, if a (ſolid) Sphere of the ſame common Denſity with the Earth was to revolve about its Axis in the Time of 84½ Minutes, the centrifugal Force

* Art. 142.

Force at the Equator thereof would, it is known, be equal to the Gravity *: Therefore, by taking $\frac{84\frac{3}{4}}{p}\left(=\frac{q}{p}\right) = 0{,}5805$ &c. the Time p will come out $= \overset{\text{M}}{146}$ or $\overset{\text{H}}{2}\ \overset{\text{M}}{26}$. Which Time is the leaſt, poſſible, wherein a Fluid, of the ſame common Denſity with the Earth, can revolve, ſo as to preſerve its ſpheroidal Figure. And this holds univerſally, let the Magnitude of the Body, or Fluid, be what it will.

* Art. 217.

COROLLARY IV.

399. Hence alſo may be determined the Spheroid, which a ſpherical Body (of Ice or any other Matter) revolving in a given Time s, will converge to, when reduced to a fluid State *.

For, ſince the Momenta of Rotation, in equal Spheres and Spheroids, are to one another, in a Ratio compounded of the direct Ratio of their Equatoreal Diameters, and the inverſe Ratio of the Times of their Rotation, it follows, if d be put = the Diameter of the given Sphere, and E = the Equatoreal Diameter of the required Spheroid, that $\frac{d}{s} = \frac{E}{p}$ (becauſe the Quantity of Motion about the Axis is not affected by the Action of the Particles one upon another, while the Figure of the Fluid is changing.) Moreover, ſince the Maſſes of the Sphere and Spheroid are alſo equal to each other (by Hypotheſis) we have $d^3\ (= AE^2 \times PS) = \frac{E^3}{\overline{1+t^2}|^{\frac{1}{2}}}$: From which two Equations, exterminating d, there ariſes $p = \overline{1+t^2}|^{\frac{1}{6}} \times s$, for the Time of Revolution of the required Spheroid: Whence, by ſubſtituting this Value of p in the general Equation $\frac{q^2}{3p^2}$

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* The Author in a Note, page 135 of his *Miſcellaneous Tracts in 4to*, has corrected an Overſight in this Corollary, by taking here $\frac{e}{d} \times s$ (inſtead of $\frac{e^2}{d^2} \times s$) whereby the remaining Part of this Article is rendered erroneous.

$= \frac{\overline{3+t^2}\times A-3t}{2t^3}$, we get $\frac{q^2}{3s^2} = \overline{1+t^2}|^{\frac{1}{3}} \times$ $\frac{\overline{3+t^2}\times A-3t}{2t^3}$; from the Solution of which the Value of t, and the Spheroid itſelf, will be given.

But, ſince the Value of the latter Part of the Equation can never exceed a certain aſſignable Quantity, the Matter propoſed can therefore be only poſſible under certain Limitations: In order to determine theſe Limitations, let the Fluxion of $\overline{1+t^2}|^{\frac{1}{3}} \times \frac{\overline{3+t^2}\times A-3t}{2t^3}$ be taken and put $= 0$, and it will be found that $\overline{t^4+24t^2+27}\times A - 15t^3 - 27t = 0$: Whence t comes out $= 7.5$, and the correſponding Value of $\frac{q}{s} = 0{,}927$, nearly.

Hence the Parts of the Fluid cannot poſſibly come to an Equilibrium among themſelves, when the Time s is leſs than $\frac{q}{0{,}927}$, but will continue to recede from the Axis, *in Infinitum.*

If q be taken $= \overset{M}{84\frac{1}{4}}$ (as in the Example to the preceding Corollary) s will be equal $\overset{M}{91} = \overset{H}{1} : \overset{M}{31}$. From which it appears, that, if the Earth (or a ſpherical Body of the ſame Denſity) was to revolve about its Axis in leſs than $\overset{H}{1} : \overset{M}{31}$; and, in the mean time, be reduced to a State of Fluidity, the Parts thereof towards the Equator would aſcend, and continue to recede from the Axis, *in Infinitum.*

COROLLARY V.

400. Seeing the Values of t and A are given when the Spheroid is given, it follows that the Gravi-

tation $\left(\frac{t-A}{t^3}\times QI\right)$ at any Point in the Surface of a Spheroid, whereof the Parts are kept *in Equilibrio*, by their Rotation about the Axis, will be accurately as a Perpendicular to the Surface at that Point, continued to the Axis of the Figure. Therefore the Gravitation at the Equator is to that at either of the Poles, as the Equatoreal Diameter to the Axis inverſly.

COROLLARY VI.

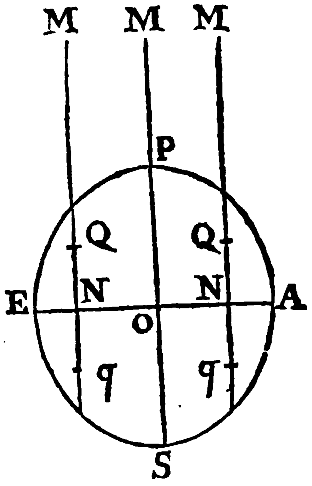
401. But, if the Spheroid differs but little from a Sphere, the Exceſs of QI above AO will (by the Property of the Ellipſis) be nearly as OR^2. Whence it appears that the Increaſe of Gravitation, in going from the Equator to the Pole, is as the Square of the Sine of Latitude, nearly.

COROLLARY VII.

402. Moreover, ſince the Ratio of the Equatoreal Diameter to the Axis is found, in this Caſe, to be that
† Art. 397. of $1+\frac{5q^2}{4p^2}$ to 1 †, the Exceſs of that Diameter above the Axis will be to the Axis as $\frac{5q^2}{4p^2}$ to Unity; that is, as $\frac{5}{4}$ of the centrifugal Force at the Equator to the mean Force of Gravity. Whence, as the centrifugal Forces, in unequal Circles, are univerſally as the Radii directly, and the Squares of the periodic Times inverſly, it follows that the foreſaid Exceſs (in Figures nearly ſpherical) will be as the Radii directly, and as the Denſity and the Square of the Time of Rotation inverſly: From which Proportions, the Ratios of the greateſt and leaſt Diameters of the Planets may be inferred from each other; ſuppoſing the Times of their Rotation, about their Axes, to be known.

PROB. XIII.

403. *To determine the Figure which a Fluid will acquire when, beſides the mutual Gravitation of the Parts thereof, it is attracted by another Body, ſo remote, that all Lines drawn from it to the Surface of the Fluid, may be taken as Parallels.*



Let OAPES be the propoſed Fluid, and let MPS and MQq be Right-lines, drawn from the remote Body *M*; whereof the former MPS paſſes thro' the Center of Gravity O: Moreover, let the Plane AE be perpendicular to the Axis MOS; and put NQ$=a$ and OM (the Diſtance of the remote Body) $=d$; alſo put the Semi-diameter of the Body (at M) $=r$, and let its Denſity be to that of the Fluid APES, as any Quantity v to Unity. Then, ſince, according to the foregoing Calculations, the Attraction at the Surface of a Sphere (of a given Denſity) is expreſſed by $\frac{1}{3}$ of the Radius, it follows that the Attraction of the Body *M*, at its Surface, will be explicable by $\frac{vr}{3}$: And therefore, the Force varying according to the Square of the Diſtance inverſly *, it will be, d^2 (MN^2) : r^2 :: $\frac{vr}{3} : \frac{vr^3}{3d^2}$, the Attraction of *M*, at the Diſtance MN: Alſo $\overline{d-a}|^2$ (MQ^2) : r^2 :: $\frac{vr}{3} : \frac{vr^3}{3\times\overline{d-a}|^2}$, its Attraction at the Diſtance

* Art. 382.

Diſtance MQ. Whence the Difference of theſe two, or $\frac{vr^3}{3 \times \overline{d-a}|^2} - \frac{vr^3}{3d^3}$ ($= \frac{vr^3}{3d^3} \times \overline{2a + \frac{3a^2}{d} + \frac{4a^3}{d^2}}$ *&c.*) will be as the Force whereby a Corpuſcle at Q endeavours to recede from the Plane AE: Which becauſe (by Hypotheſis) d is very great in reſpect of a, will (by rejecting all the Terms after the firſt) be expreſſed by $\frac{2vr^3}{3d^3} \times a$, or its Equal $\frac{2vr^3}{3d^3} \times$ NQ.

In the very ſame Manner, the Force whereby a Corpuſcle at q, below the Plane AE, tends to recede therefrom, will be defined by $\frac{2vr^3}{3d^3} \times$ Nq.

Now, therefore, ſeeing theſe Forces are, every where, as the Diſtances NQ, Nq, from the Plane AE, it appears (*by Art.* 393. *and* 394.) that the Figure OAPES will be a Spheroid; whereof the Equation, for the Relation of its two principal Diameters (putting $n = \frac{2vr^3}{3d^3}$) is $n = -\overline{1+B} \times \overline{\frac{2B}{3\cdot5} - \frac{4B^2}{5\cdot7} + \frac{6B^3}{7\cdot9}}$ *&c.* (In which, the Ratio of PS^2 to AE^2 is denoted by that of 1 to $1+B$.) Hence, by reverting the Series, we have $B = -\frac{15n}{2} - \frac{225n^2}{28}$ *&c.* and conſequently PS : AE :: 1 : $\sqrt{1 - \frac{15n}{2} - \frac{225n^2}{28}}$ *&c.* :: 1 : $1 - \frac{15n}{4}$, nearly: Which, by reſtoring the Value of n, becomes PS : AE :: 1 : $1 - \frac{5vr^3}{2d^3}$.

Q. E. I.

Co-

COROLLARY.

404. Becauſe $\frac{r}{d}$ expreſſes the Sine of the apparent Semi-diameter of the Body *M*, to the Radius 1) ſeen at the Diſtance OM, it follows, if the ſaid Sine be denoted by *c*, that $PS : AE :: 1 : 1 - \frac{5v}{2} \times c^2$; and conſequently, by Diviſion, $PS : PS - AE :: 1 : \frac{5v}{2} \times c^2$.

Hence it appears, that the Forces of the Planets, to produce Tides at the Earth's Surface, are to one another as their Denſities, and the Cubes of their apparent Diameters conjunctly. (For the Sines of ſmall Arcs are nearly as the Arcs themſelves.)

EXAMPLE.

405. If *c* be taken = the Sine of 16′ (expreſſing the mean Apparent Semi-diameter of the Moon) and $v = \frac{5}{4}$ (the Ratio of her Denſity with reſpect to that of the Earth) our laſt Proportion will become $PS : PS - AE :: 1 : 0{,}000000315$: Whence, if PS be taken = 42000000 Feet (the Meaſure of the Earth's Diameter) $PS - AE$ will come out $= \overset{F}{13{,}23}$.

SECT.

SECTION X.

Of the Application of FLUXIONS *to the Reſolution of ſuch Kinds of Problems* DE MAXIMIS ET MINIMIS, *as depend upon a particular Curve, whoſe Nature is to be determined.*

I SHALL begin this Section with premiſing the following uſeful

THEOREM.

406. *If the Relation of two flowing Quantities* y *and* u *be required; ſo that, when the Fluent of* $y^m\dot{u}$ *becomes equal to a given Value, that of* $\frac{y^r \times \overline{\dot{u}\dot{u} \pm \dot{y}\dot{y}}|^{n}}{\dot{y}^{2n-1}}$ *may be a* Maximum *or a* Minimum; *I ſay, their Relation muſt be ſuch that* $\frac{y^{r-m}\dot{u} \times \overline{\dot{u}\dot{u} \pm \dot{y}\dot{y}}|^{n-1}}{\dot{y}^{2n-1}}$ *may be, every where, the ſame, or equal to a conſtant Quantity.*

The Demonſtration hereof depends upon the ſubſequent

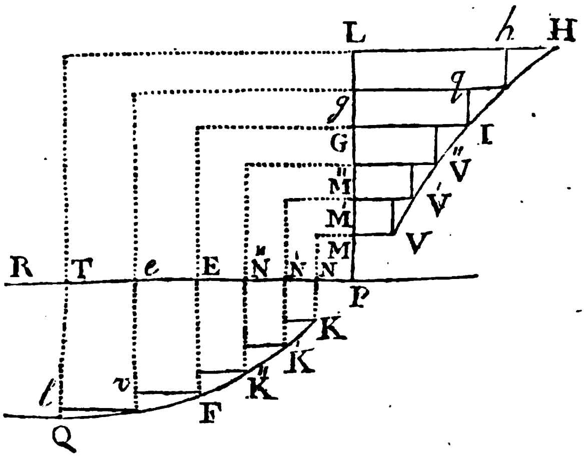
LEMMA.

407. If $a\alpha + b\beta = Q$, wherein α and β are indeterminate, the Value of $A \times \overline{\alpha\alpha \pm pp}|^{n} + B \times \overline{\beta\beta + pp}|^{n}$ will be a *Maximum* or *Minimum*, when $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$ and $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$ are equal to each

other. For, by taking the Fluxions of both Expreſſions we have $a\dot{\alpha} + b\dot{\beta} = 0$, and $2nA\alpha\dot{\alpha} \times \overline{\alpha\alpha \pm pp}|^{n-1} + 2nB\beta\dot{\beta} \times \overline{\beta\beta \pm pp}|^{n-1} = 0$: From whence, $\dot{\alpha}$ and $\dot{\beta}$ being exterminated, there reſults $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1} = \frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$. *Q. E. D.*

Hence, if $a\alpha + b\beta + c\gamma + d\delta$ &c. $= Q$ (where $\alpha, \beta, \gamma, \delta$ &c. are indeterminate) it follows that $A \times \overline{\alpha\alpha \pm pp}|^{n} + B \times \overline{\beta\beta \pm pp}|^{n} + C \times \overline{\gamma\gamma \pm pp}|^{n} + D \times \overline{\delta\delta \pm pp}|^{n}$ &c. will be a *Maximum* or *Minimum*, when all the Quantities $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$, $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$, $\frac{C\gamma}{c} \times \overline{\gamma\gamma \pm pp}|^{n-1}$ &c. are equal to each other. For that Expreſſion is a *Maximum* (or *Minimum*) when it cannot be increaſed (or decreaſed) by altering the Values of the indeterminate Quantities involved therein; but it may be increaſed (or decreaſed) by altering only two of them (as α and β) whilſt the reſt remain unchanged; unleſs $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$ and $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$ are equal to each other. (This is proved above.) Therefore, when $A \times \overline{\alpha\alpha \pm pp}|^{n} + B \times \overline{\beta\beta \pm pp}|^{n} + C \times \overline{\gamma\gamma \pm pp}|^{n} +$ &c. is a *Maximum* or *Minimum*, the Quantities $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$ and $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$ cannot be unequal: And, by the very ſame Argument, no other two of the Quantities above ſpecified can be unequal.

If, in the Right-line PR, there be now aſſumed $N\acute{N} = \alpha$, $\acute{N}\acute{\acute{N}} = \beta$, &c. and upon theſe, as Baſes,



Rectangles $\acute{N}K$, $\acute{\acute{N}}\acute{K}$ be ſuppoſed, whoſe Altitudes NK, $\acute{N}\acute{K}$ &c. are denoted by a, b, c, d &c. it is evident that $a\alpha + b\beta + c\gamma + d\delta$ &c. ($= \mathcal{Q}$) will be expreſſed by the Sum of all the ſaid Rectangles, or the whole Polygon Nl.

Moreover, if, in the Right-line PL (perpendicular to PR) there be taken $M\acute{M}$, $\acute{M}\acute{\acute{M}}$ &c. each equal to p, and, upon theſe equal Baſes, Rectangles $\acute{M}V$, $\acute{\acute{M}}\acute{V}$ &c. be conſtituted, whoſe Altitudes are denoted by $A \times \frac{\overline{\alpha\alpha \pm pp}|^{n}}{p^{2n}}$, $B \times \frac{\overline{\beta\beta \pm pp}|^{n}}{p^{2n}}$, &c. it is likewiſe plain that the Value of $\frac{A \times \overline{\alpha\alpha \pm pp}|^{n}}{p^{2n-1}} + \frac{B \times \overline{\beta\beta \pm pp}|^{n}}{p^{2n-1}} + \frac{C \times \overline{\gamma\gamma \pm pp}|^{n}}{p^{2n-1}}$ will be truly repreſented by the whole whoſe Polygon Mb. Which Polygon (as p is conſtant) will be a *Maximum* or *Minimum*, when $A \times \overline{\alpha\alpha \pm pp}|^{n} + B \times \overline{\beta\beta \pm pp}|^{n} +$ &c. is a *Maximum* or *Minimum*; that is, when all the Quantities $\frac{A\alpha}{a} \times \frac{\overline{\alpha\alpha \pm pp}|^{n-1}}{p^{2n-1}}$, $\frac{B\beta}{b} \times \frac{\overline{\beta\beta \pm pp}|^{n-1}}{p^{2n-1}}$, &c. are equal to each other (as has been proved above.)

Let now, A, B, C, D &c. be expounded by any Powers, (MP^{r}, $\overset{\prime}{\mathrm{M}}\mathrm{P}^{r}$, $\overset{\prime\prime}{\mathrm{M}}\mathrm{P}^{r}$, &c.) of the reſpective Diſtances from a given Point P; and let, at the ſame time, the correſponding Values of a, b, c, d &c. be interpreted by any other propoſed Powers MP^{m}, $\overset{\prime}{\mathrm{M}}\mathrm{P}^{m}$, $\overset{\prime\prime}{\mathrm{M}}\mathrm{P}^{m}$ &c. of the ſame given Diſtances: Then the Area of the Polygon Nl will be expreſſed by $\mathrm{MP}^{m} \times \alpha + \overset{\prime}{\mathrm{M}}\mathrm{P}^{m} \times \beta + \overset{\prime\prime}{\mathrm{M}}\mathrm{P}^{m} \times \gamma$ &c. $(= Q)$; and that of the Polygon Mb, by $\mathrm{MP}^{r} \times \frac{\overline{\alpha\alpha \pm pp}|^{n}}{p^{2n-1}} + \overset{\prime}{\mathrm{M}}\mathrm{P}^{r} \times \frac{\overline{\beta\beta \pm pp}|^{n}}{p^{2n-1}} + \overset{\prime\prime}{\mathrm{M}}\mathrm{P}^{r} \times \frac{\overline{\gamma\gamma \pm pp}|^{n}}{p^{2n-1}} +$ &c. And the foreſaid equal Quantities $\frac{A\alpha}{a} \times \frac{\overline{\alpha\alpha \pm pp}|^{n-1}}{p^{2n-1}}$, $\frac{B\beta}{b} \times \frac{\overline{\beta\beta \pm pp}|^{n-1}}{p^{2n-1}}$ &c. will become $\mathrm{MP}^{r-m} \times \frac{\alpha \times \overline{\alpha\alpha \pm pp}|^{n-1}}{p^{2n-1}}$, $\overset{\prime}{\mathrm{M}}\mathrm{P}^{r-m} \times \frac{\beta \times \overline{\beta\beta \pm pp}|^{n-1}}{p^{2n-1}}$, &c. reſpectively.

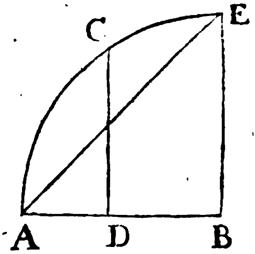
Now let the Number of the Rectangles be ſuppoſed indefinitely great, and their Breadths indefinitely ſmall, ſo that the Area of each of the two Polygons N*l* and M*b* may be taken for that of its circumſcribing Curve: Moreover, let *u* and *y* be put to repreſent the Diſtances of any two correſponding Ordinates EF and GI from the given Point P; and let $\dot{y}$ be every where expreſſed by p ($=\overset{\prime}{M}M=\overset{\prime}{M}\overset{\prime\prime}{M}=$ &c.) Then, $\dot{u}$ being a general Value for any of the Quantities α, β, γ, δ &c. (or NN, NN &c.) it follows; Firſt, that the Fluxion of the Area of the Curve NEFK (the Ordinate being, every where, $=y^{m}$) will be truly defined by $y^{m}\dot{u}$; Secondly, that the Fluxion of the Area MGIV (by ſubſtituting y, $\dot{u}$ and $\dot{y}$ inſtead of their Equals) will be $\frac{y^{r}\times\overline{\dot{u}\dot{u}\pm\dot{y}\dot{y}}|^{n}}{\dot{y}^{2n-1}}$; and, laſtly, that the Value of each of the equal Quantities, $\mathrm{MP}^{r-m}\times\frac{\alpha\times\overline{\alpha\alpha\pm pp}|^{n-1}}{p^{2n-1}}$, $\overset{\prime}{\mathrm{M}}\mathrm{P}^{r-m}\times\frac{\beta\times\overline{\beta\beta\pm pp}|^{n-1}}{p^{2n-1}}$, &c. above ſpecified, will be expreſſed by $\frac{y^{r-m}\times\dot{u}\times\overline{\dot{u}\dot{u}\pm\dot{y}\dot{y}}|^{n-1}}{\dot{y}^{2n-1}}$. *Whence the Theorem is manifeſt.*

408. If *R* and *S* be aſſumed to denote any Functions of *y* (that is, any two Quantities expreſſed in Terms of *y* and given Coefficients; then, in order to have the Fluent of $S\times\frac{\overline{\dot{u}\dot{u}\pm\dot{y}\dot{y}}|^{n}}{\dot{y}^{2n-1}}$ a *Maximum* or *Minimum*, when that of $R\dot{u}$ becomes equal to a given Value, it is requiſite that $\frac{S\dot{u}}{R}\times\frac{\overline{\dot{u}\dot{u}\pm\dot{y}\dot{y}}|^{n-1}}{\dot{y}^{2n-1}}$ ſhould be a conſtant

Quantity: This, alſo, is evident from the preceding Demonſtration; and may be of Uſe when the above premiſed Theorem is not ſufficiently general.

PROB. I.

409. *To determine the Nature of the Curve* ACE; *ſo that, the Length of the Arch* AE *being given, the Area* ABE *ſhall be a* Maximum.



Calling (as uſual) the Abſciſſa AD, x; the Ordinate DC, y; and the Arch AC, z, we have $\dot{x} = \sqrt{\dot{z}^2 - \dot{y}^2}$*; and therefore $y\dot{x}$ † $= y \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}|^{\frac{1}{2}} =$ the Fluxion of the Area ADC. Now, ſince, by the Queſtion, the Fluent of $y \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}|^{\frac{1}{2}}$ is to be a *Maximum*, when That of $\dot{z}$ becomes equal to a given Quantity (ACE) let theſe two Fluxions be, reſpectively, compared with $\dfrac{y^r \times \overline{\dot{u}\dot{u} - \dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ and $y^m\,\dot{u}$ (as given in the foregoing Theorem ‡.) By which means, $n = \frac{1}{2}$, $r = 1$, $\dot{u} = \dot{z}$, and $m = 0$; and conſequently $\dfrac{y^{r-m}\,\dot{u} \times \overline{\dot{u}\dot{u} - \dot{y}\dot{y}}|^{n-1}}{\dot{y}^{2n-1}}$ $= y\dot{z} \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}|^{-\frac{1}{2}}$: Which (according to the ſaid Theorem) being, every where, equal to a conſtant Quantity, we ſhall, by putting that Quantity $= a$, and ordering the Equation, get $\dot{z}^2 = \dfrac{a^2\dot{y}^2}{a^2 - y^2}$, and $\dot{x}$ $(\sqrt{\dot{z}^2 - \dot{y}^2}) = \dfrac{y\dot{y}}{\sqrt{a^2 - y^2}}$; and, conſequently, (by taking the Fluent) $x = a - \sqrt{a^2 - y^2}$, or $2ax - xx = y^2$; which is the common Equation of a Circle.

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* Art. 135.

† Art. 112.

‡ Art. 406.

Q. E. I.

COROLLARY.

410. It follows from hence, that the greateſt Area that can poſſibly be contain'd by a Right-line (AE) joining two given Points, and any Curve-line ACE of a given Length, terminating in the ſame Points, will be when the ſaid Curve-line is an Arch of a Circle.

PROB. II.

411. *The Length of the Arch* AE *(ſee the preceding Figure) being given, to determine the Nature of the Curve, ſo that the Solid generated by the Rotation thereof may be a* Maximum.

* Art. 145. Since the Fluent of $y^2 \times \overline{\dot{z}^2 - \dot{y}^2}|^{\frac{1}{2}}$ ($= y^2\dot{x}$ *) is required to be a *Maximum*, when that of $\dot{z}$ has a given Value ACE, every thing will remain as in the laſt Problem; *only*, r muſt here be $= 2$: And therefore (*by the Theorem*) we have $y^2\dot{z} \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}|^{-\frac{1}{2}} = a$. Whence $\dot{z} = \dfrac{a\dot{y}}{\sqrt{a^2 - y^4}}$; and conſequently $\dot{x}$ ($= \sqrt{\dot{z}^2 - \dot{y}^2}$) $= \dfrac{y^2\dot{y}}{\sqrt{a^2 - y^4}}$: Which Values, if b^2 be put $= a$ (in order to have the Powers homologous) will become $\dot{z} = \dfrac{b^2\dot{y}}{\sqrt{b^4 - y^4}}$ and $\dot{x} = \dfrac{y^2\dot{y}}{\sqrt{b^4 - y^4}}$: Whence z and x will be known. *Q. E. I.*

PROB. III.

412. *The Superficies generated by the Arch of a Curve, in its Rotation, about its Axis, being given; to determine the Curve, ſo that the Solid, itſelf, may be a* Maximum.

† Art. 145. Becauſe the Fluent of $y^2 \times \overline{\dot{z}^2 - \dot{y}^2}|^{\frac{1}{2}}$ † is to be a *Maximum*, when that of $y\dot{z}$ becomes equal to a given

Quantity; let the Fluxions here exhibited be therefore compared with $\frac{y^r \times \overline{\dot{u}\dot{u} \pm \dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ and $y^m\dot{u}$ (given in the Theorem.) By means whereof (r being $= 2$, $\dot{u} = \dot{z}$, $n = \frac{1}{2}$, and $m = 1$) we have $y\dot{z} \times \overline{\dot{z}^2 - \dot{y}^2}|^{-\frac{1}{2}} = a$ (a conſtant Quantity *;) which is the very Equation found in *Prob.* 1. belonging to a Circle.

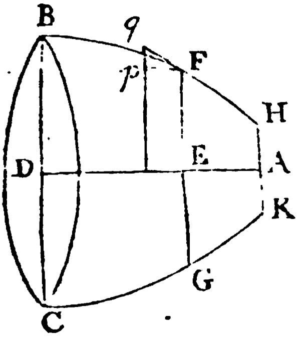
* Art. 406.

If the Solid be ſuppoſed given, and the Superficies a *Minimum*, we ſhall come at the very ſame Concluſion: For, $y^2\dot{x}$ and $y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{1}{2}}$ (which are reſpectively as their Fluxions) being compared with $y^m\dot{u}$ and $\frac{y^r \times \overline{\dot{u}\dot{u} + \dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ we have $m = 2$, $\dot{u} = \dot{x}$, $r = 1$, and $n = \frac{1}{2}$; and therefore $\frac{\dot{x}}{y\sqrt{\dot{x}^2 + \dot{y}^2}}$ equal to a conſtant Quantity: Which being denoted by $\frac{1}{a}$ (ſo that the Terms may be homologous) there comes out $a\dot{x} = y\sqrt{\dot{x}^2 + \dot{y}^2}$; whence $2ax - x^2 = y^2$ (*as before.*)

P R O B. IV.

413. *To determine the Curve* HFB, *from whoſe Revolution a Solid* BK *ſhall be generated; which, moving forward, in a Medium, in the Direction of its Axis* DA, *will be leſs reſiſted than any other Solid of the ſame given Length* DA *and Baſe* BC.

If $AE = x$, $EF = y$, $Fp = \dot{x}$ &c. it is evident, from the Principles of Mechanics, that the reſiſting Force of a Particle of the Medium at F (being as the Square of the Sine of the Angle of Inclination pFq) will be truly repreſented by $\frac{\dot{y}\dot{y}}{\dot{x}\dot{x} + \dot{y}\dot{y}}$ $\left(\frac{\overline{pq}|^2}{\overline{Fq}|^2}\right)$. Moreover, ſince the



the whole Number of Particles acting upon FHKG is proportional to the Area of the Circle FG, or as y^2; the Fluxion hereof ($2y\dot{y}$) drawn into $\frac{\dot{y}\dot{y}}{\dot{x}\dot{x}+\dot{y}\dot{y}}$, will therefore give $\frac{2y\dot{y}^3}{\dot{x}\dot{x}+\dot{y}\dot{y}}$ for the Fluxion of the Reſiſtance upon FHKG.

Now, ſince it is required (by the Queſtion) to have the Fluent of $\frac{y\dot{y}^3}{\dot{x}\dot{x}+\dot{y}\dot{y}}$ (or $\frac{y \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-1}}{\dot{y}^{-3}}$) a *Maximum*, when That of $\dot{x}$ becomes equal to a given Quantity (AD), let theſe two Fluxions be therefore compared with $\frac{y^r \times \overline{\dot{u}\dot{u}+\dot{y}\dot{y}}|^{n}}{\dot{y}^{2n-1}}$ and $y^m\dot{u}$ *. Whence (r being $=1$, $\dot{u}=\dot{x}$, $n=-1$, and $m=0$) we get $\frac{y\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-2}}{\dot{y}^{-3}} = a$ (a conſtant Quantity †); and conſequently $y\dot{y}^3\dot{x} = a \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2$: Whereof the Fluent will be found, by *Art.* 264. That the Curve does not meet its Axis in the extreme Point A, but has an Ordinate AH at that Point (as repreſented in the Figure) is evident from the foregoing Equation. For $\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2$ ($\overline{Fq}|^4$) being, always, greater than $\dot{y}^3\dot{x}$ ($\overline{pq}|^3 \times Fp$), y muſt therefore be greater than a, in the ſame Proportion; and ſo, can never be equal to Nothing.

* Art. 406.

† Art. 406.

Now, as it is demonſtrable that the Angle AHF muſt be $\frac{3}{2}$ of a Right-Angle, AH (the leaſt Value of y) will therefore be $= 4a$ (ſince $\dot{x}$ and $\dot{y}$ are, in this Circumſtance,

ſtance, equal to each other.) But, what a, itſelf, ought to be, muſt be determined from the given Values of AD and BD, and the Reſolution of the foreſaid Equation.

PROB. V.

414. *To determine the Solid of the leaſt Reſiſtance, ſuppoſing the Area of the generating Plane* AHBD, *and its greateſt Ordinate* DB *to be given*; *(ſee the preceding Figure.)*

Since (by the laſt Article) the Fluxion of the Reſiſtance is expreſſed by $\frac{y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{-1}}{\dot{y}^{-3}}$, and that of the Area AEFH by $y\dot{x}$, it is plain (from the premiſed Theorem *) that $\frac{\dot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{-2}}{\dot{y}^{-3}}$ is a conſtant Quantity. * Art. 406.

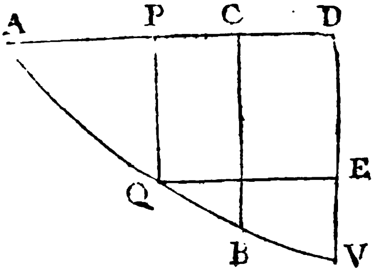
Whence, $\frac{\dot{y}^3\dot{x}}{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^2}$, or its Equal $\frac{\overline{pq}|^3 \times Fp}{\overline{qF}|^4}$, being every where the ſame, the Angle pFq muſt alſo be invariable; and conſequently HFB a Right-line. Therefore the Solid of the leaſt Reſiſtance is *(in this Caſe)* either a whole Cone, or the Fruſtrum of a, greater, Cone. But it is eaſy to ſhew, *that*, when the Area of the generating Plane AB is given ſo ſmall, that the Angle B may be taken equal to the Half of a Right-angle; I ſay, it is demonſtrable, in this Caſe, that the Fruſtrum ſo ariſing will be leſs reſiſted than a whole Cone, or any other Fruſtrum, whereof the Baſe and the Area of the generating Plane are the ſame.

In like manner the Solid of *leaſt Reſiſtance*, when its Bulk and greateſt Diameter are given, may be determined: The Equation of the generating Curve being $\frac{y^{-1}\dot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{-2}}{\dot{y}^{-3}} = \frac{1}{a}$, or $a\dot{x}\dot{y}^3 = y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^2$: Whereof the Solution is given in *Art.* 264.

PROB.

PROB. VI.

415. *To determine the Line, along which a Body, by its own Gravity, will, deſcend, from one given Point* A *to another* B, *in the ſhorteſt Time poſſible.*



Let AD be parallel, and BC perpendicular, to the Horizon, interſecting each other in C; and let QP be any Ordinate to the Curve parallel to BC: Then (calling AP, x; PQ, y *&c.*) the Celerity at Q will be expreſſed by $y^{\frac{1}{2}}$; alſo the Fluxion of the Time of Deſcent thro' AQ will be truly defined by $\frac{\dot{z}}{y^{\frac{1}{2}}}$ *, or its Equal $y^{-\frac{1}{2}} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{1}{2}}$. Here, therefore, the Fluent of $y^{-\frac{1}{2}} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{1}{2}}$ is to be a *Minimum*, when that of $\dot{x}$ arrives to a given Value (AC). Whence, *by the Theorem* †, $y^{-\frac{1}{2}}\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-\frac{1}{2}}$ muſt be $=$ a conſtant Quantity: Which (to have the Terms homologous) let be denoted by $a^{-\frac{1}{2}}$ (or $\frac{1}{\sqrt{a}}$). Then $a^{\frac{1}{2}}\dot{x} = y^{\frac{1}{2}} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{1}{2}}$; whence $\dot{x} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a-y}} = \frac{y\dot{y}}{\sqrt{ay-yy}}$; $\dot{z} = (\sqrt{\dot{x}^2+\dot{y}^2})$ $=$

* Art. 204.

† Art. 405.

$= \frac{a^{\frac{1}{2}}\dot{y}}{\sqrt{a-y}}$; and conſequently $z = 2a - 2a^{\frac{1}{2}}\sqrt{a-y}$.

Therefore, when $y = a$, z is $= 2a$; which two correſponding Values let be denoted by DV and AV; and let QE, parallel to AD, meet DV in E; then VE (VD — ED) being $= a - y$, and VQ (AV — AQ) $= 2a^{\frac{1}{2}}\sqrt{a-y}$, it follows that

VD (a) : VE $(a-y)$ :: VA^2 $(4a^2)$: VQ^2 $(4a \times \overline{a-y})$

Which is one of the moſt remarkable Properties of the Cycloid; the Curve which, therefore, anſwers the Conditions of the Problem.

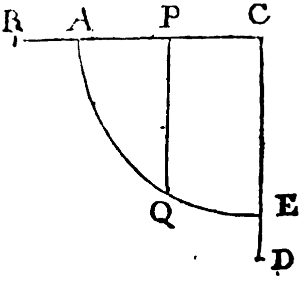
If the Celerity be ſuppoſed as any Function (S) of the Quantity y, the Problem will be reſolved in the ſame manner: The Equation of the Curve being

$$\frac{\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-\frac{1}{2}}}{S} = \frac{1}{a}\,*.$$

* Art. 408.

PROB. VII.

416. *To find the Nature of the Curve* AQE, *along which a heavy Body muſt deſcend from an horizontal Line* RC *to a vertical Line* CD, *ſo that the Area* CAE *may be given, and the Time of the Deſcent a* Minimum.



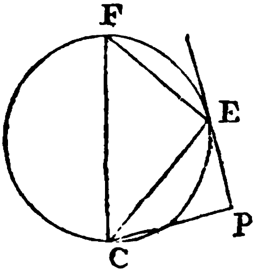
If the Ordinate PQ (parallel to CD) be called y, and the Velocity at Q be denoted by y^n; it is evident that the Fluent of $y^{-n} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{1}{2}}$ $\left(= \frac{\dot{z}}{y^n}\,\dagger\right)$ muſt be a *Minimum* when that of $y\dot{x}$ amounts to a given Value.

† Art. 204.

Therefore (by the Theorem already mention'd ſo often) we have $y^{-n-1}\dot{x}\times\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-\frac{1}{2}}=a^{-n-1}$; and conſequently $\dot{x}=\frac{y^{n+1}\dot{y}}{\sqrt{a^{2n+2}-y^{2n+2}}}$; which, by writing $\frac{1}{2}$ inſtead of n, becomes $\dot{x}=\frac{y^{\frac{3}{2}}\dot{y}}{\sqrt{a^{3}-y^{3}}}$: Whence x will be known. But, if the Celerity was to be every where uniform, then (n being $=0$) we ſhould have $\dot{x}=\frac{y\dot{y}}{\sqrt{a^{2}-y^{2}}}$; and therefore $x=a-\sqrt{a^{2}-y^{2}}$: Which anſwers to a Circle.

LEMMA.

417. *If, upon a Tangent* EP, *from any Point* C *in the Circumference of a Circle* FEC, *a Perpendicular* CP *be let fall, the Chord* (CE) *joining that Point and the Point of Contact, will be a Mean-Proportional between the ſaid Perpendicular* CP *and the Diameter* CF *of the Circle.*



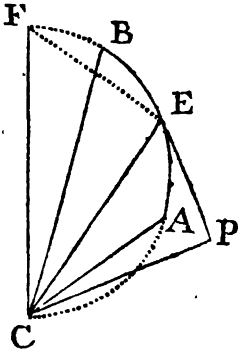
For, the Angles P and CEF being both Right; and alſo CEP = F, the Triangles CPE and CEF are ſimilar: And therefore CP : CE :: CE : CF.

Q. E. D.

PROB. VIII.

418. *In the mixt-lin'd Triangle* ACB, *the Lengths of all the Sides (whereof* CA *and* CB *are Right-lines) are ſuppoſed given; 'tis required to find the Nature of the Curve-ſide* AEB, *ſo that the Area may be a* Maximum.

Put



Put the Arch AE $= z$, and the Ordinate CE $= y$; then, the Fluxion of the Area ACE being $\frac{y}{2}\sqrt{\dot{z}^2-\dot{y}^2}$,* the Fluent of $y\times\overline{\dot{z}\dot{z}-\dot{y}\dot{y}}|^{\frac{1}{2}}$, generated in the Time wherein y, from CA, increaſes to CB, muſt be a *Maximum*: Therefore, *by the Theorem* †, we have $y\dot{z}\times\overline{\dot{z}\dot{z}-\dot{y}\dot{y}}|^{-\frac{1}{2}}$ $= a$ †, or $\frac{\dot{z}}{\sqrt{\dot{z}\dot{z}-\dot{y}\dot{y}}}=\frac{a}{y}$. But, if CP be ſuppoſed perpendicular to the Tangent EP, then will $\frac{\dot{z}}{\sqrt{\dot{z}\dot{z}-\dot{y}\dot{y}}}$ (*Art.* 35.) $= \frac{\text{CE}}{\text{CP}} = \frac{y}{\text{CP}}$; and conſequently $\frac{a}{y}=\frac{y}{\text{CP}}$; or, CP : CE (y) :: CE $(y) : a$: Which Proportion, *by the Lemma*, anſwers to a Circle; whereof the Quantity a is the Diameter.

* Art. 113.

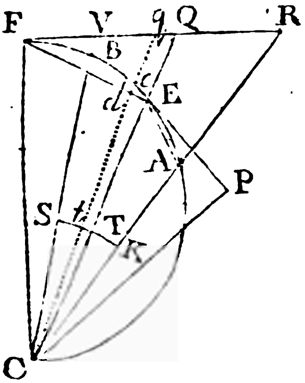
† Art. 406.

Now, that AEB muſt be an Arch of a Circle is alſo evident from *Prob.* 1. but, that the ſame Arch, continu'd out, will paſs thro' the Angle C, does not appear from thence. This is known from above; and is requiſite in finding the particular Circle anſwering to any propoſed *Data*.

PROB. IX.

419. *To find the Path* AEB *which a Body muſt deſcribe in moving uniformly from one given Point* A *to another* B; *ſo that, being every where acted on by a Force, or Virtue, which varies according to the Inverſe-Duplicate-Ratio of the Diſtances from a given Center* C, *the whole Action upon the Body ſhall be a* Minimum.

Putting



Putting AE $=z$, CE $=y$, *de* (indefinitely ſmall) $=\dot{y}$ *, E*e* $=\dot{z}$, and E*d* $(\sqrt{\dot{z}^2-\dot{y}^2})$ $=\dot{u}$; we have $\frac{\dot{z}}{y^2}$ $(=y^{-2}\times\overline{\dot{u}\dot{u}+\dot{y}\dot{y}}|^{\frac{1}{2}})$ for the Meaſure of the Force which acts upon the Body in deſcribing the Particle E*e* $(\dot{z})$: Moreover, if from the Center C, with any given Radius (r) an Arch KT*t*S of a Circle be deſcribed, interſecting CE in T, we ſhall have T*t* (the Meaſure of the Angle EC*e*) $=\frac{r\dot{u}}{y}$. Therefore, ſince the Fluent of $y^{-2}\times\overline{\dot{u}\dot{u}+\dot{y}\dot{y}}|^{\frac{1}{2}}$ is required to be a *Minimum*, and the cotemporary Fluent of $y^{-1}\dot{u}$ (between CA and CB) a given Quantity; it follows, from the Theorem premiſed at the Beginning of the Section, that $y^{-2+1}\dot{u}\times\overline{\dot{u}\dot{u}+\dot{y}\dot{y}}|^{-\frac{1}{2}}$ muſt be equal to a conſtant Quantity $\left(\frac{1}{a}\right)$ and conſequently $\frac{\dot{u}}{\overline{\dot{u}\dot{u}+\dot{y}\dot{y}}|}$ $\left(=\frac{\sqrt{\dot{z}^2-\dot{y}^2}}{\dot{z}}\right)=\frac{y}{a}$: Which is the very Equation found in the preceding Problem. Therefore, if thro' the three given Points A, B, and C, the Circumference of a Circle be deſcribed, the Arch thereof terminated by A and B will be the Path of the Body. *Q. E. I.*

* Art. 134.

COROLLARY.

420. If FR be a Tangent to the Circle, at the Extremity of the Diameter CF, and CA and CE be pro-

duced to meet it in R and Q, it follows that the whole Action upon the Body, in deſcribing the Arch AE, will be proportional to the correſponding Part RQ of the ſaid Tangent. For, if C*e* be, alſo, produced to meet FR in *q*, and EF be drawn, it is plain that the Triangles CEF and CFQ, as alſo CE*e* and C*q*Q, are ſimilar: Whence it will be, CE (y) : CF (a) :: CF (a) : CQ (or C*q*) $= \frac{aa}{y}$; and CE (y) : E*e* $(\dot{x})$ :: C*q* $\left(\frac{aa}{y}\right)$: Q*q* $= \frac{aa\dot{x}}{yy}$: Which (a being conſtant) is as $\left(\frac{\dot{x}}{yy}\right)$ the Force that acts upon the Body in deſcribing E*e* $(\dot{x})$. And, as this every where holds, the whole Action in deſcribing AE muſt therefore be proportional to RQ. Which Force (it is eaſy to prove) will be to that exerted on the Body in moving through the Chord AE, as the Chord to the Arch.

PROB. X.

421. *To determine the Path in which a Body may move from one given Point* A *to another* B, *in the ſhorteſt Time poſſible; ſuppoſing the Velocity to be, every where, proportional to any Power* (y^p) *of the Diſtance from a given Center* C. (See the laſt Figure.)

Here every thing will remain as in the preceding Problem; *only* y^{-p} muſt be wrote inſtead of y^{-2}. Therefore we have $y^{-p+1} \times \dot{u} \times \overline{\dot{u}\dot{u} + \dot{y}\dot{y}}|^{-\frac{1}{2}} =$ a conſtant Quantity: Which Quantity (to have the Terms homologous) let be denoted by $\frac{b}{a^p}$; then, by Reduction,

$$\frac{by^{p-1}}{a^p} = \frac{\dot{u}}{\sqrt{\dot{u}\dot{u} + \dot{y}\dot{y}}} \left(= \frac{\text{E}d}{\text{E}e}\right) = \frac{\text{CP}}{\text{CE}} = \frac{\text{CP}}{y}:$$

And conſequently CP $= \frac{by^p}{a^p}$. Hence, if $p = 0$, or the

Velocity be conſtant; then CP being every where $= b$, the Body muſt, in this Caſe, deſcribe a Right-line. But, if $p = 1$, then CP being $= \frac{by}{a}$; the Curve will be a logarithmic Spiral, whoſe Center is C[*]: Except in that particular Caſe, where CA = CB, when it degenerates to a Circle.

Laſtly, if $p = 2$, the Curve will be a Circle (*by the preceding Lemma*) whoſe Diameter is $\frac{aa}{b}$, and whoſe Periphery paſſes through the given Point C.

After the ſame manner, the Value of CP (upon which the Nature of the Curve depends) may be determined, when the Velocity is expounded by any given Function (*S*) of the Diſtance (*y*) from the Center of Force: And (by writing S in the room of y^n [†] *&c.*) will come out CP $= \frac{bS}{c}$; where *b* and *c* repreſent conſtant Quantities.

When the Velocity is That which the Body may acquire, in deſcending through BE, by a centripetal Force expreſſed by y^p, then the Value of *S* (the Meaſure of that Velocity) being interpreted by $\sqrt{d^{p+1} - y^{p+1}}$ [‡] (where CB$=d$) we therefore have CP$=\frac{b\sqrt{d^{p+1}-y^{p+1}}}{c}$ for the Equation of the Curve of the ſwifteſt Deſcent, according to this laſt *Hypotheſis* of a centripetal Force varying as any Power *p* of the Diſtance.

422. Beſides the Problems already reſolved in this Section, there are others of the ſame Nature which are confined to more particular Reſtrictions, and require a different Method of Solution.

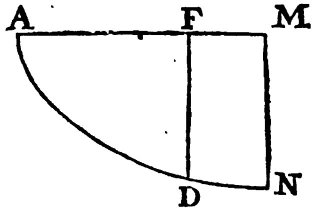
Thus,

[*] Art. 74.

[†] Art. 407.

[‡] Art. 221. and 206.

Thus, if $\mathcal{Q}$, R and S be ſuppoſed to denote any given Powers, or Functions, of the Ordinate (y) of a Curve ANM, and the Nature of the Curve be required, ſo that, when the Fluent of $\mathcal{Q}\dot{x}$ becomes equal to a given Quantity, the Fluent of $R\dot{z}$ may alſo become equal to another given Quantity, and That of $S\dot{z}$, a *Maximum* or *Minimum:* Then, becauſe there is, in this Caſe, a ſecond Equation, or new Condition, beyond what is to be met with in any of the foregoing Problems, the Method of Solution hitherto explained, will, therefore, be inſufficient. But, by a Proceſs ſimilar to that whereby the ſaid Method was demonſtrated (aſſuming, here, three Expreſſions, and three indeterminate Quantities, inſtead of two *) a general Anſwer to this Problem (under all its Reſtrictions) will be obtained: And is exhibited by the Equation, $\frac{\dot{z}}{\dot{x}} = \frac{pR \pm qS}{\mathcal{Q}}$; wherein p and q denote conſtant Quantities.



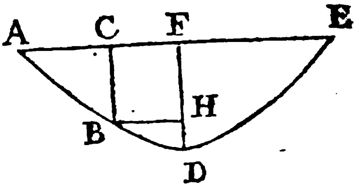
423. Though it ſeems unneceſſary to put down the Invention of this Equation, after what has been hinted above, yet it may not be improper to obſerve, by way of Corollary, that, if $\mathcal{Q} = 1$, $R = 1$, and $S = y^n$, the Equation will then become $\frac{\dot{z}}{\dot{x}} = p \pm qy^n$; expreſſing the Nature of the Curve, when, the whole Abſciſſa (AM) and correſponding Arch (AN) being both given Quantities, the Fluent of $y^n\dot{z}$ is a *Maximum* or *Minimum*, according as the Value of n is poſitive or negative: In both which Caſes, it is very eaſy to perceive, that the Curve muſt be concave to AM, and that the Value of $\frac{\dot{z}}{\dot{x}}$, or its

Equal

* Art. 407.

Equal $p \pm qy^n$, muſt, therefore, decreaſe as y increaſes; whence we may infer that the Sign of qy^n muſt be negative in the former Caſe, and poſitive in the latter.

Ex. Let the Curve ABDE, be the *Catenaria*; formed by a ſlender Chain, or perfectly flexible Cord,



ſuſpended by its two Extremes in the horizontal Line AE: Then, ſince its Center of Gravity muſt be the loweſt poſſible, the Fluent of $y\dot{z}$, when AC = AE, muſt therefore be a *Maximum* * : Whence (n being here $= 1$) our Equation $\left(\frac{\dot{z}}{\dot{x}} = p \pm qy^n\right)$ becomes $\frac{\dot{z}}{\dot{x}} = p - qy$.

* Art. 173.

But, in order to reduce it to a more convenient Form, let the Diſtance (DF) of the loweſt Point of the Curve from the horizontal-Line AE be put $= b$; then, when y (BC) becomes $= b$, $\dot{x}$ will be $= \dot{z}$; and therefore the Equation, in that Circumſtance, is $1 = p - qb$; whence $p = 1 + qb$, and conſequently $\frac{\dot{z}}{\dot{x}} = 1 + qb - qy = 1 + q \times \overline{b - y}$: Which, by putting $b - y$ (DH) $= s$ and $a = \frac{1}{q}$ is reduced to $\frac{\dot{z}}{\dot{x}} = 1 + \frac{s}{a}$: From whence $a^2\dot{z}^2$ $(= \overline{a+s}|^2 \times \dot{x}^2) = \overline{a+s}|^2 \times \overline{\dot{z}^2 - \dot{s}^2}$; and conſequently BD $= \sqrt{2as + ss}$.

For another Example (wherein the Exponent n will be negative) let the required Curve be That along which a Body may deſcend, by its own Gravity, from one given Point A to another B, in leſs Time than through any other Line of the ſame Length. In which Caſe, the Fluent of $\dot{z}y^{-\frac{1}{2}}$ being a *Minimum*, when x and z become equal to given Quantities, our Equation (by writing $-\frac{1}{2}$ for n) will here become $\frac{\dot{z}}{\dot{x}} = p + qy^{-\frac{1}{2}}$: From whence exterminating x, or $\dot{z}$, by means of the Equation $\dot{x}^2 + \dot{y}^2 = \dot{z}^2$, the Fluent may alſo be determined.

SECTION XI.

The Reſolution of Problems of various Kinds.

PROB. I.

424. *ANY hyperbolical Logarithm* (y) *being given; it is propoſed to find the natural Number anſwering thereto.*

If the Number ſought be denoted by $1 + x$, we ſhall (*by Art.* 126.) have $\dot{y} = \frac{\dot{x}}{1 + x}$, or $\dot{y} + x\dot{y} - \dot{x} = 0$. Let $Ay + By^2 + Cy^3$ *&c.* $= x$; then $A\dot{y} + 2By\dot{y} + 3Cy^2\dot{y}$ *&c.* $= \dot{x}$, and our Equation will become

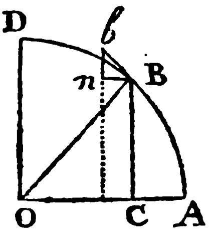
$$\left.\begin{array}{l} \dot{y} + Ay\dot{y} + By^2\dot{y} + Cy^3\dot{y} \text{ \&c.} \\ -A\dot{y} - 2By\dot{y} - 3Cy^2\dot{y} - 4Dy^3\dot{y} \text{ \&c.} \end{array}\right\} = 0.$$

Whence, by comparing the homologous Terms, we get $A = 1$, $B = \frac{A}{2} = \frac{1}{2}$, $C = \frac{B}{3} = \frac{1}{2 \cdot 3}$, $D = \frac{C}{4} = \frac{1}{2 \cdot 3 \cdot 4}$ *&c.* Therefore $1 + y + \frac{y^2}{2} + \frac{y^3}{2 \cdot 3} + \frac{y^4}{2 \cdot 3 \cdot 4} + \frac{y^5}{2 \cdot 3 \cdot 4 \cdot 5}$ *&c.* is $(= 1 + x)$ the Number ſought.

PROB. II.

425. *The Radius* AO *and any Arch* AB *of a Circle* ABD *being given; to find the Sine* BC, *and Co-ſine* OC *of that Arch.*

Let AO (BO) $= r$, AB $= z$, AC $= x$, BC $= y$,



B$b = \dot{z}$, B$n = \dot{x}$, and $bn = \dot{y}$: Becauſe of the ſimilar Triangles OBC and Bnb, it will be

$$\text{OB}\ (r) : \text{BC}\ (y) :: \text{B}b\ (\dot{z}) : \text{B}n\ (\dot{x})$$

$$\text{And OB}\ (r) : \text{OC}\ (r-x) :: \text{B}b\ (\dot{z}) : bn\ (\dot{y})$$

From which we have

$$y\dot{z} = r\dot{x}$$

$$\text{And } r\dot{y} = r\dot{z} - x\dot{z}.$$

Let $x = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5$ &c.

And $y = az + bz^2 + cz^3 + dz^4 + ez^5$ &c.

Then, by Subſtitution and Transpoſition, our two Equations will become

$$\left.\begin{array}{l} * \ + az\dot{z} + bz^2\dot{z} + cz^3\dot{z} + dz^4\dot{z}\ \text{\&c.} \\ -rA\dot{z} - 2rBz\dot{z} - 3rCz^2\dot{z} - 4rDz^3\dot{z} - 5rEz^4\dot{z}\ \text{\&c.} \end{array}\right\} = 0$$

And

$$\left.\begin{array}{l} ra\dot{z} + 2rbz\dot{z} + 3rcz^2\dot{z} + 4rdz^3\dot{z} + 5rez^4\dot{z}\ \text{\&c.} \\ -r\dot{z} + Az\dot{z} + Bz^2\dot{z} + Cz^3\dot{z} + Dz^4\dot{z}\ \text{\&c.} \end{array}\right\} = 0$$

From which, by equating the homologous Terms, we get $A = 0$, $a = 2rB$, $b = 3rC$, $b = 4rD$, $d = 5rE$ &c.

Alſo $a = 1$, $b = -\frac{A}{2r}$, $c = -\frac{B}{3r}$, $d = -\frac{C}{4r}$ &c.

There

Therefore $2rB = 1$, $3rC = -\frac{A}{2r}$, $4rD = -\frac{B}{3r}$, $5rE = -\frac{C}{4r}$, *&c.* and conſequently $B = \frac{1}{2r}$, $C = 0$, $D = -\frac{B}{3 \cdot 4r^2} = -\frac{1}{2 \cdot 3 \cdot 4r^3}$, $E = 0$, $F = -\frac{D}{5 \cdot 6r^2} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot r^5}$ *&c.*
Whence, alſo b $(=3rC) = 0$, c $(=4rD) = -\frac{1}{2 \cdot 3r^2}$ *&c. &c.*

Hence it is evident that y $(= az + bz^2 + cz^3$ *&c.*$)$ $= z - \frac{z^3}{2 \cdot 3r^2} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5r^5} - \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7r^6}$ + *&c.* And that x $(= Az + Bz^2 + Cz^3$ *&c.*$)$ $= \frac{z^2}{2r} - \frac{z^4}{2 \cdot 3 \cdot 4r^3} + \frac{z^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6r^5}$ — *&c.* ‡

PROB. III.

426. *To find the Value of* x, *when* x^x *is a* Minimum.

The Logarithm of x^x is $= x \times l.\, x$; whoſe Fluxion $\dot{x} \times l.\, x + \dot{x}$ being $= 0$, we have $l : x = -1$. But (*by Prob.* 1.) the Number whoſe hyp. Log. is y will be $1 + y + \frac{y^2}{2} + \frac{y^3}{2 \cdot 3} + \frac{y^4}{2 \cdot 3 \cdot 4}$ *&c.* Therefore, by writing -1 inſtead of y, we have $x = 1 - 1 +$

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‡ *The Subſtance of this Solution (being the moſt neat and artful I have ſeen to that uſeful Problem) I had from a Letter ſign'd* ——— Needler; *which was put into my Hands by a Friend, who receiv'd it from the late Dr.* Halley, *to whom it was wrote.*

$\frac{1}{2} - \frac{1}{2\cdot3} + \frac{1}{2\cdot3\cdot4} - \frac{1}{2\cdot3\cdot4\cdot5}$ &c. $= 0,367878$ &c.

PROB. IV.

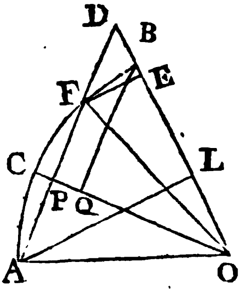
427. *To divide a given Number (a) ſo that the continual Product of all its Parts may be a* Maximum.

It is evident (*from Art.* 23.) that all the Parts muſt be equal: If, therefore, any one of them be denoted by x, their Number will be $\frac{a}{x}$, and we ſhall have $\overline{x}|^{\frac{a}{x}}$ a *Maximum:* And therefore its Logarithm $\frac{a}{x} \times L.x$ a *Maximum* alſo: And its Flaxion $-\frac{a\dot{x}}{x^2} \times L.x$
* Art. 22. and 126. $-\frac{a\dot{x}}{x^2} = 0$*: Whence H-$L.x = 1$, and conſequently
† Art. 424. $x = 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot3} + \frac{1}{2\cdot3\cdot4}$ &c. $= 2.71828$ &c. Therefore the next inferior, or ſuperior, Number to 2,71828 &c. that will exactly meaſure the given Number a, is the required Value of each Part: Thus, let $a = 10$; then becauſe $\frac{10}{2.71828 \text{ \&c.}} = 4$ nearly, the Number of Parts, in this Caſe, will be 4, and the Value of each $= \frac{10}{4} = 2.5$.

PROB. V.

428. *To divide a given Angle* AOB *into two Parts* AOC *and* BOC, *ſo that the Product of any given Powers,* $AP^n \times BQ^m$, *of their Sines* AP *and* BQ *may be a* Maximum.

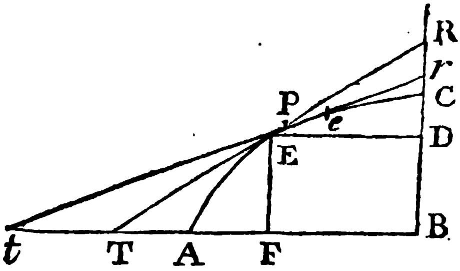
Let AP, produced, cut the Radius OB in D, and the Arch AB in F; likewiſe let FE and AL be perpendicular to OB, and join O, F: Putting AO$=r$, AP$=x$ and BQ$=y$. Then, becauſe $x^n y^m$ is to be a *Maximum*, we have $nx^{n-1}\dot{x} \times y^m + x^n \times my^{m-1}\dot{y}=0$; and conſequently $ny\dot{x} = -mx\dot{y}$.



Moreover, ſince the Fluxion of the Arch AC is $=\frac{r\dot{x}}{\sqrt{r^2-x^2}}$ and that of BC $= \frac{r\dot{y}}{\sqrt{r^2-y^2}}$ (*Art.* 142.) we alſo have $\frac{r\dot{y}}{\sqrt{r^2-y^2}} + \frac{r\dot{x}}{\sqrt{r^2-x^2}} = 0$, or $\frac{\dot{y}}{\sqrt{r^2-y^2}} = \frac{-\dot{x}}{\sqrt{r^2-x^2}}$; which multiply'd by the former Equation, *&c.* gives $\frac{ny}{\sqrt{r^2-y^2}} = \frac{mx}{\sqrt{r^2-x^2}}$, or $n \times \frac{y\sqrt{r^2-x^2}}{\sqrt{r^2-y^2}} = mx$: Whence, becauſe OQ ($\sqrt{r^2-y^2}$) : QB ($y$) :: OP ($\sqrt{r^2-x^2}$) : PD $= \frac{y\sqrt{r^2-x^2}}{\sqrt{r^2-y^2}}$, we have $n \times$ PD ($=mx$) $= m \times$ AP; and therefore PD : AP :: $m : n$; whence (by Compoſition and Diviſion) AD : DF :: $m+n : m-n$: But (*by ſim. Triang.*) AD : DF :: AL : EF; conſequently $m+n : m-n$ :: AL : FE; that is, as the Sum of the Indices of the two propoſed Powers is to their Difference, ſo the Sine of the whole given Angle to the Sine of the Difference of its two, required, Parts. This Proportion is given in Words, at length, becauſe it will be found of frequent Uſe in the Solution of mechanical Problems.

PROB. VI.

429. *To ſhew that the leaſt Triangle that can be deſcribed about, and the greateſt Parallelogram in, a given Curve* ABC, *concave to its Axis, will be when the Subtangent* FT *is equal to the Baſe* BF *of the Parallelogram, or half the Baſe* BT *of the Triangle.*



It appears from *Art.* 25. and is demonſtrable by common Geometry, that the greateſt Parallelogram that can be inſcrib'd in the Triangle BTR (ſuppoſing the Poſition of TR to remain the ſame) will be *that* whoſe Baſe BF is half the Baſe BT of the Triangle: Therefore, as a greater Figure cannot poſſibly be inſcribed in the Curve BAC than in the Triangle BTR circumſcribing it, the greateſt Parallelogram that can be inſcribed, either in the Triangle or the Curve, muſt be That above ſpecified.

But now, to make it alſo appear that the Triangle BTR is a *Minimum* when FT=BF; let B*tr* be any other circumſcribing Triangle, and let the two Tangents TER and *ter* interſect each other in P. Then, ER being = ET, it is plain that RP is leſs than PT, and P*r* (leſs than PR leſs than PT) leſs than P*t*: Therefore, the Sides PR and P*r* of the Triangle RP*r* being leſs than the Sides, PT and P*t* of the Triangle TP*t*, and the oppoſite Angles RP*r* and TP*t* equal to each other, it follows that the Triangle PR*r* is leſs than TP*t*; and conſequently, by adding the Trapezium BTP*r* to both, it appears that BTR is leſs than B*tr*.

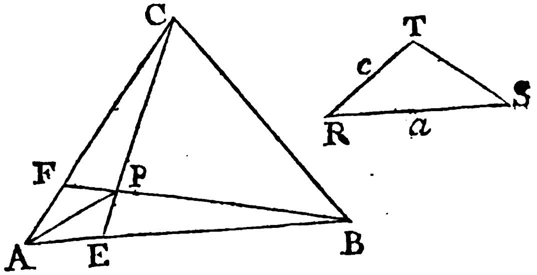
COROLLARY.

430. Hence the greateſt inſcribed Parallelogram is half the leaſt circumſcribing Triangle.

In the ſame Way it may be proved, that the greateſt inſcribed Cylinder, and the leaſt circumſcribing Cone, in, and about, the Solid generated by Revolution of a given Curve, will be when the Sub-tangent is equal to twice the Altitude of the Cylinder, or $\frac{2}{3}$ of the Altitude of the Cone: And that the two Figures will be to each other in the Ratio of 4 to 9.

PROB. VII.

431. *Three Points* A, B, C *being given, to find the Poſition of a fourth Point* P, *ſo that, if Lines be drawn from thence to the three former, the Sum of the Products* $a \times AP$, $b \times BP$, *and* $c \times CP$ *(where a, b and c denote given Numbers) ſhall be a* Minimum.

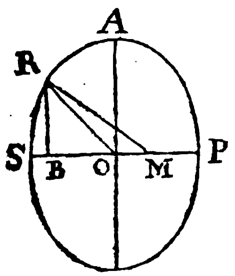


If CP and BP be produced to E and F, it will appear from *Art.* 35. and 36. that the Sine of BPE muſt be to that of APE, as *a* to *b*; and the Sine of CPF (BPE) to that of APF, as *a* to *c*. Therefore, the Sines of the three Angles BPE, APE, and APF (which Angles, taken all together, make two Right-ones) being in the given Ratio of *a*, *b* and *c*, it follows, that, if a Triangle RST be conſtructed, whoſe Sides RS, ST and RT are in the ſaid Ratio of *a*, *b* and *c*, the Angles T, R and S oppoſite thereto, will be reſpectively equal to the fore-mention'd Angles BPE, APE, and APF. From whence, all the Angles at the Point P being given, the Poſition of that Point is given by common Geometry.

But it is obſervable, that, when one of the three given Quantities a, b, c (ſuppoſe a) is equal to, or greater than, the Sum of the other two, a Triangle cannot then be formed whoſe Sides are proportional to the ſaid Quantities: In that Caſe the Point P will fall in the Point (A) correſponding to the greateſt Quantity (a). For, it is plain that $b\times AB$ is leſs than $b\times BP + b\times AP$; and that $c\times AC$ is leſs than $c\times CP+c\times AP$; whence, by adding the Leſs to the Leſs, and the Greater to the Greater, it alſo appears that $b\times AB+c\times AC$ muſt be leſs than $b\times BP+c\times CP+\overline{b+c}\times AP$ leſs than $b\times BP+c\times CP+a\times AP$; becauſe a (by Hypotheſis) is equal to, or greater than, $b+c$.

PROB. VIII.

432. *To determine in what Latitude a Right-line perpendicular to the Surface of the Earth, and Another drawn, from the ſame Point, to the Center, make the greateſt Angle, poſſible, with each other; the Ratio of the Axis and the Equatoreal Diameter being ſuppoſed given.*



Let AE repreſent the Equatoreal Diameter, and SP the Axis of the Earth (taken as an oblate Spheroid) alſo let RO and RM repreſent the two Lines ſpecified in the Problem, whereof let the latter (perpendicular to ARS) meet SP in M; and let RB be perpendicular to SP.

It is evident, from the Property of the Ellipſis, that $SP^2 : AE^2 :: BO : BM$. And (by Trigonometry) BO : BM :: Tang. BRO : Tang. BRM; whence, by Equality,

lity, $SP^2 : AE^2 :: \text{Tang. } BRO : \text{Tang. } BRM$; therefore, by Compoſition and Diviſion, $AE^2 + SP^2 : AE^2 - SP^2 :: \text{Tang. } BRM + \text{Tang. } BRO : \text{Tang. } BRM - \text{Tang. } BRO$. But, *the Sum of the Tangents of any two Angles is to their Difference, as the Sine of the Sum of thoſe Angles to the Sine of their Difference*[*]; whence it follows that $AE^2 + SP^2 : AE^2 - SP^2 :: \text{Sine. } \overline{BRM + BRO} : \text{Sine. } \overline{BRM - BRO}\,(ORM)$.

Now, ſince the Ratio of the two firſt Terms is conſtant, or in every Part of the Ellipſis the ſame, it is obvious that the Angle ORM, or its Sine, will be the greateſt poſſible, when its Antecedent (the Sine of $\overline{BRM + BRO}$) is the greateſt poſſible, that is when $BRM + BRO =$ a Right-Angle and its Sine = Radius. Therefore, in the propoſed Circumſtance, when ORM is a *Maximum*, our laſt Proportion will become $AE^2 + SP^2 : AE^2 - SP^2 :: \text{Radius} : \text{Sine of } ORM$: And half the Angle, ſo found, added 45°, will give (BRM) the Complement of the required Latitude; becauſe $BRM + BRO$ (or $2BRM - ORM$) being $= 90°$, it is evident that $2BRM = 90 + ORM$, and conſequently $BRM = 45° + \frac{1}{2} ORM$.

PROB. IX.

433. *Of all the Semi-cubical Parabolas, to determine that, whereof, the Length of the Curve being given, the Area ſhall be a* Maximum.

The general Equation is $ax^2 = y^3$: Moreover, the Area is univerſally $= \frac{3y^{\frac{5}{2}}}{5a^{\frac{1}{2}}}$, and the Length of the Curve $= \frac{\overline{4a+9y}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}} - \frac{8a}{27}$ (*ſee Art.* 137.) Let the laſt of theſe be put $= c$, and, by ordering the Equation, you will get $y = \frac{a^{\frac{1}{3}} \times \overline{27c + 8a}|^{\frac{2}{3}} - 4a}{9}$: Whence, $\frac{3y^{\frac{5}{2}}}{5a^{\frac{1}{2}}}$ (and conſequently $\frac{y}{a^{\frac{1}{5}}}$) being a *Maximum*, it is evident that $\frac{a^{\frac{1}{3}} \times \overline{27c + 8a}|^{\frac{2}{3}} - 4a}{a^{\frac{1}{5}}}$, or its Equal $a^{\frac{2}{15}} \times \overline{27c + 8a}|^{\frac{2}{3}} - 4a^{\frac{4}{5}}$ muſt likewiſe be a *Maximum*: Which, put into Fluxions and reduced, gives $a = c \times \frac{9 + 3\sqrt{21}}{32}$; Whence x and y will alſo be found.

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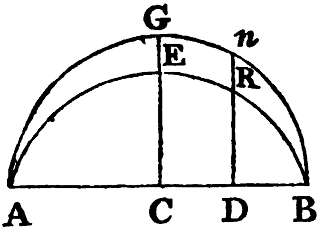
[*] *Vid. p.* 56. *of my Trigonometry.*

PROB. X.

434. *To determine the Ratio of the Periphery of any given Ellipſis to that of its circumſcribing Circle.*

Call the Semi-tranſverſe Axis CB, a; the Semi-conjugate CE, c; any Ordinate Dr, y; and its Diſtance



CD from the Center, x: Then (by the Nature of the Curve) y being $= \frac{c}{a}\sqrt{aa - xx}$, we have $\dot{y} = \frac{-cx\dot{x}}{a\sqrt{aa - xx}}$; and conſequently $\dot{z}$ $(\sqrt{\dot{x}^2 + \dot{y}^2}) = \frac{\dot{x}\sqrt{a^4 - \overline{a^2 - c^2} \times x^2}}{a\sqrt{aa - xx}}$: Which by making $d =$

$\frac{aa-cc}{aa}$ will be reduced to $\dot{z} = \frac{\dot{x}\sqrt{aa-dxx}}{\sqrt{aa-xx}} = \frac{a\dot{x}}{\sqrt{aa-xx}} \times \overline{1 - \frac{dx^2}{2a^2} - \frac{d^2x^4}{2.4a^4} - \frac{3d^3x^6}{2.4.6a^6}\ \&c.}$ (by throwing the Numerator into a Series) whereof the *whole* Fluent, when x becomes $= a$, will be z (ERB) $= A \times \overline{1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6} - \frac{3.3.5.5.7d^4}{2.2.4.4.6.6.8.8}}$ &c. (*by Art.* 286.) where A denotes the Length of the Arch G*n*B, or ¼ of the Periphery of the circumſcribing Circle.

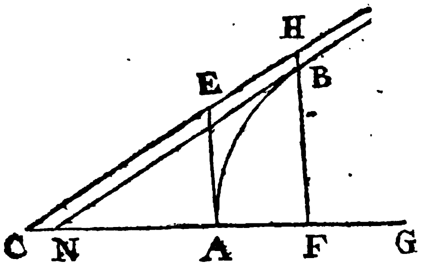
Hence it follows that the Periphery of the Ellipſis is to that of its circumſcribing Circle, as $1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6}$ &c. or as $1 - \frac{d}{2.2} \times A + \frac{1.3d}{4.4} \times B + \frac{3.5d}{6.6} \times C + \frac{5.7d}{8.8} \times D$ &c. to Unity: Where A, B, C, D &c. denote the preceding Terms, under their proper Signs.

PROB. XI.

435. *To determine the Difference between the Length of the Arch of a Semi-hyperbola infinitely produced, and its Aſymptote.*

Call the Semi-tranſverſe Axis (AC) a; the Semi-conjugate (or its Equal AE); b the Diſtance (CF) of any Ordinate from the Center, x; the Ordinate itſelf, y; and the Arch correſponding, z: Then, from the Nature of the Curve we have $y = \frac{b\sqrt{x^2-a^2}}{a}$; whence

$\dot{y} =$



$\dot{y} = \frac{bx\dot{x}}{a\sqrt{x^2-a^2}}$; and conſequently $\dot{z}\ (=\sqrt{\dot{x}^2+\dot{y}^2}) =$ $\frac{\dot{x}\sqrt{\frac{aaxx+bbxx}{aa}-a^2}}{\sqrt{xx-aa}}$: Which, making $d^2 = \frac{a^2}{a^2+b^2}$ $\left(=\frac{\text{CA}^2}{\text{CE}^2}\right)$ and $u = \frac{a}{x}$ will be transformed to $\dot{z} =$ $-\frac{a\dot{u}}{du^2} \times \frac{\overline{1-dduu}|^{\frac{1}{2}}}{\overline{1-uu}|^{\frac{1}{2}}}$; whereof the upper Surd, expanded, is $= 1 - \frac{d^2u^2}{2} - \frac{d^4u^4}{8}$ &c. And therefore $\dot{z} =$ $\frac{a}{d}$ into $\frac{-\dot{u}}{u^2\sqrt{1-uu}} + \frac{d^2\dot{u}}{2\sqrt{1-uu}} + \frac{d^4u^2\dot{u}}{8\sqrt{1-uu}} +$ $\frac{3d^6u^4\dot{u}}{8.6\sqrt{1-uu}} + \frac{3.5\,d^8u^6\dot{u}}{8.6.8\sqrt{1-uu}}$ &c. Now the Fluent of the firſt Term hereof, $\frac{a}{d}$ into $\frac{-\dot{u}}{u^2\sqrt{1-uu}}$ $\left(=\frac{x\dot{x}}{d\sqrt{x^2-a^2}}\right)$ is univerſally expreſſed by $\frac{\sqrt{x^2-a^2}}{d}$, or its Equal $\frac{\text{BF} \times \text{CE}}{\text{AE}}$: Which, if BN be parallel to the Aſymptote EC, will (becauſe AE : CE ::

CE :: BF : BN) be alſo truly repreſented by BN: And this Line BN, when x or z becomes infinite, will coincide with the Aſymptote. Therefore the Fluent of the remaining Terms is the Difference ſought: Which Fluent, when $u = 1$, or $y = 0$ (putting A for $\frac{1}{4}$ of the Periphery of the Circle whoſe Radius is Unity) will be

$$= aA \times \overline{\frac{d}{2} + \frac{d^3}{2.2.4} + \frac{3 \cdot 3d^5}{2.2.4 \cdot 4.6.} + \frac{3 \cdot 3 \cdot 5 \cdot 5d^7}{2.2.4.4.6.6.8} + \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7d^9}{2.2.4.4.6.6.8.8.10} \&c.}$$

(*by Art.* 286.) but $= 0$ when $u = 0$ (or y is infinite). Therefore the Exceſs of the Aſymptote above the Curve is truly exhibited by the preceding Series. *Q. E. I.*

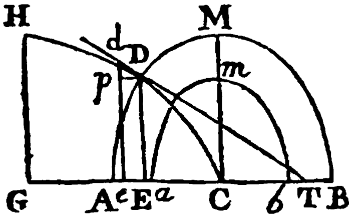
If a be taken $= 1$, and $b = 0$, then d will become $= 1$: And therefore, the Curve in this Caſe falling into its Axis AG, we have

$$A \times \overline{\frac{1}{2} + \frac{1}{2.2.4} + \frac{3 \cdot 3}{2.2.4.4.6} + \frac{3 \cdot 3 \cdot 5 \cdot 5}{2.2.4.4.6.6.8} \&c.} = CA,$$

or Unity. Whence it appears that the Sum of the Series $\frac{1}{2} + \frac{1}{2.2.4} + \frac{3 \cdot 3}{2.2.4.4.6}$ is the Reciprocal of $\frac{1}{4}$ of the Periphery of the Circle whoſe Radius is Unity. And, from the Problem preceding the laſt, it will likewiſe appear, that the Sum of the Series $1 - \frac{1}{2.2} - \frac{3}{2.2.4.4} - \frac{3 \cdot 3 \cdot 5 \cdot}{2.2.4.4.6.6}$ &c. will be denoted by the ſame Quantity; and conſequently that theſe two Serieſes are equal to each other. From the Addition and Subtraction of which and their Multiples, various other Serieſes may be produced, whoſe Sums are explicable by means of the Periphery of a Circle.

PROB. XII.

436. *To determine the Nature of the Curve* CDH, *which will interſect any Number of ſimilar and concentric Ellipſes* AMB, *amb* &c. *at Right-Angles.*



Let the Tangent DT, which is a Normal to the Ellipſis AMB, meet the Axis AB in T; and, ſuppoſing AC, CM, *a*C, C*m* &*c.* to be the principal Semi-diameters of their reſpective Ellipſes, let the given Ratio of AC² to CM² (or of *a*C² to C*m*² &*c.*) be that of 1 to *n*: Putting CE $= x$, ED $= y$, D*p* (E*e*) $= \dot{x}$, and *dp* $= \dot{y}$.

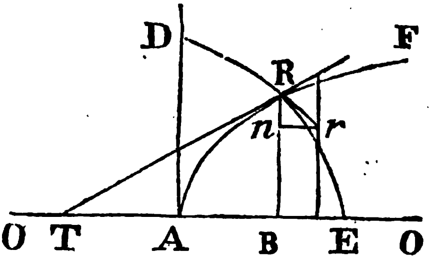
It is a known Property of the Ellipſis that AC² : CM² :: CE : ET; therefore ET $= nx$: Moreover ET (nx) : D*p* ($\dot{x}$) :: ED (y) : *pd* ($\dot{y}$) by ſimilar Triangles) whence $\frac{\dot{x}}{nx} = \frac{\dot{y}}{y}$, or $\frac{\dot{x}}{x} = \frac{n\dot{y}}{y}$; whereof the Fluent

* Art. 126. is $L:x - L:a = nL:y - nL:a$ * (where *a* denotes any conſtant Quantity at Pleaſure.) Hence we alſo have $L:\frac{x}{a} = n \times L:\frac{y}{a} = L:\frac{y^n}{a^n}$, and conſequently $\frac{x}{a} = \frac{y^n}{a^n}$, or $a^{n-1}x = y^n$.

PROB. XIII.

437. *To find the Equation of a Curve* ERD *that will cut any Number of Ellipſes, or Hyperbolas, having the ſame Center* O *and Vertex* A, *at Right-Angles.*

Let RT be a Tangent to any one of the propoſed Conic Sections ARF, at the Interſection R, meeting the



the Axis AO in T; and put AO $=a$, OB $=x$, BR $=y$, $nr=\dot{x}$, R$n=-\dot{y}$: Then (*per Conics*) BT $=\frac{a^2-x^2}{x}$, in the Ellipſis, and $=\frac{x^2-a^2}{x}$, in the Hyperbola: Whence, by reaſon of the ſimilar Triangles TBR, and Rrn, it will be $\frac{a^2 \backsim x^2}{x}$ (BT) : y (BR) :: $-\dot{y}$ (Rn) : $\mp \dot{x}$ (rn): Therefore $+ yy = \frac{a^2\dot{x} - x^2\dot{x}}{x} = \frac{a^2\dot{x}}{x} - x\dot{x}$, and conſequently $+ \frac{y^2}{2} + d^2 = a^2 \times \text{L} : \frac{x}{a} - \frac{1}{2}x^2$. Where d denotes a conſtant Quantity, depending on the given Value of AE.

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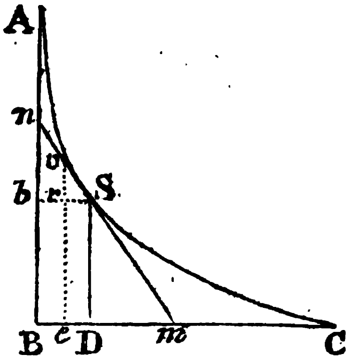
P R O B. XIV.

438. *Let two Points* n *and* m *move, at the ſame time, from two given Poſitions* B *and* C, *with equal Celerities, along two Right-lines* BA *and* BC *perpendicular to each other: 'Tis propoſed to determine the Curve* ASC, *to which a Right-line joining the ſaid Points ſhall, always, be a Tangent.*

Let DS and sv be parallel to BA, and Srb perpendicular thereto: Putting BC $=a$, CD $=x$, SD $=y$, Sr $=\dot{x}$, and $rv=\dot{y}$. Therefore (*by ſim. Triangles*) $\dot{y}:\dot{x}$

$:: y : \frac{y\dot{x}}{\dot{y}} = \mathrm{D}m$, and $\dot{x} : \dot{y} :: a-x$ (Sb) : $\frac{\overline{a-x}\times\dot{y}}{\dot{x}} =$ bn: Whence Cm (CD—Dm) $= x - \frac{y\dot{x}}{\dot{y}}$, and B$n$ (Bb $+ bn) = y + \frac{\overline{a-x}\times\dot{y}}{\dot{x}}$: Which two laſt Values, becauſe the Velocities of the Bodies are equal, muſt alſo be equal to each other, that is, $x - \frac{y\dot{x}}{\dot{y}} = y + \frac{\overline{a-x}\times\dot{y}}{\dot{x}}$: Hence, by making $\dot{x}$ conſtant, and taking the Fluxion of the whole Equation, we get $\dot{x} - \frac{\dot{y}\dot{x}\dot{y} - y\dot{x}\ddot{y}}{\dot{y}^2} = \dot{y} - \frac{\dot{x}\dot{y} - \overline{\dot{a-x}}\times\ddot{y}}{\dot{x}}$; or $\frac{\overline{\dot{a-x}}\times\ddot{y}}{\dot{x}} = \frac{y\dot{x}\ddot{y}}{\dot{y}^2}$; from which there ariſes $\overline{a-x}\times\dot{y}^2 = y\dot{x}^2$, and $\frac{\dot{y}}{\sqrt{y}} = \frac{\dot{x}}{\sqrt{a-x}}$: Where, the Fluent on both Sides being taken, we have $2\sqrt{y} = 2\sqrt{a} - 2\sqrt{a-x}$, and conſequently $x = 2\sqrt{ay} - y$: Which Equation pertains to the common Parabola.



Otherwiſe more univerſally, thus.

439. Put Cm = v and Bn = w, and let theſe Quantities (inſtead of being equal) have any given Relation to each other. Then, ſince the abſolute Celerity of m is expreſſed by $\dot{v}$, its angular Celerity, in a Direction perpendicular to Sm, by which the Line Sm tends to revolve about the Point of Contact S as a Center, will be truly defined by $\frac{\textit{Sine of Bmn}}{\textit{Radius}} \times \dot{v}$ (*Art.* 35.) In the ſame manner the angular Celerity of n, about the Point S, will be defined by $\frac{\textit{Sin. Bmn}}{\textit{Rad.}} \times \dot{w}$. Now, as theſe Celerities muſt be to each other as the Diſtances Sm and Sn from the Center S (or directly as the Radii) we have Sm : Sn (:: DS : bn) :: *Sin. Bmn* × $\dot{v}$: *Sin. Bnm* × $\dot{w}$; whence, becauſe *Sin. Bmn* : *Sin. Bnm* :: *Bn* (w) : *Bm* $(a-v)$ we alſo have DS : bn :: $w \times \dot{v} : \overline{a-v} \times \dot{w}$: Therefore, by Compoſition, DS : (DS + bn) w :: $w\dot{v} : w\dot{v} + \overline{a-v} \times \dot{w}$, and conſequently DS $= \frac{w^2\dot{v}}{w\dot{v} + \overline{a-v} \times \dot{w}}$: Whence bn (w — SD) $= \frac{\overline{a-v} \times w\dot{w}}{w\dot{v} + \overline{a-v} \times \dot{w}}$; and BD $\left(= \text{S}b = \frac{bn \times Bm}{Bn}\right)$ $= \frac{\overline{a-v}|^2 \times \dot{w}}{w\dot{v} + \overline{a-v} \times \dot{w}}$: From whence the Curve itſelf will be given.

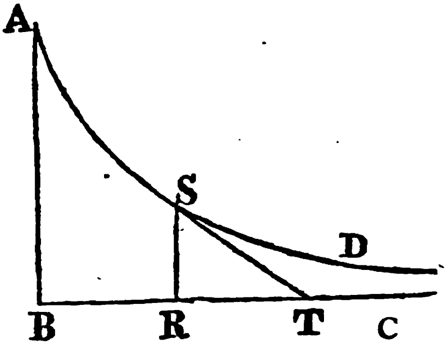
If v and w be taken equal to each other (as above) then SD (y) will become $= \frac{w^2}{a}$, and BD $= \frac{\overline{a-w}|^2}{a}$ $= a - 2w + \frac{w^2}{a}$; in which laſt, if for w its Equal

$\sqrt{ay}$ be ſubſtituted, we ſhall have BD $= a - 2\sqrt{ay} + y$; and conſequently CD ($a -$ BD) $= 2\sqrt{ay} - y$, the very ſame as before.

PROB. XV.

440. *Suppoſing a Body* T *to proceed, uniformly, along a Right-Line* BC, *and another Body* S, *in purſuit of the ſame, always directly towards it, with a Celerity which is to that of* T, *in any given Ratio, of* 1 *to* n; *it is propoſed to find the Equation of the Curve* ASD *deſcribed by the latter.*

Let the Tangent AB, which makes Right-Angles with BC, be put $= a$, BR $= x$, RS $= y$, and AS $= z$:



Then the Subtangent RT being $= \frac{y\dot{x}}{-\dot{y}}$, we have BT $= x + \frac{y\dot{x}}{-\dot{y}}$: Moreover, ſince the Diſtances BT and AS gone over in the ſame Time, are as the Celerities n and 1, we alſo have BT ($= n \times$ AS) $= nz = x + \frac{y\dot{x}}{-\dot{y}}$: Whence, in Fluxions (making $\dot{y}$ conſtant) $\frac{-y\ddot{x}}{\dot{y}} = n\dot{z}$; and conſequently $\frac{-n\dot{y}}{y} \left(= \frac{\ddot{x}}{\dot{z}}\right) = \frac{\ddot{x}}{\sqrt{\dot{y}^2 + \dot{x}^2}}$:

The Fluent of which (*by Art.* 126.) is $-n \times \text{Log.}\, y = \text{Log.}\, \frac{\dot{x} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$: But when $y = a$, $\dot{x}$ is $= 0$, and then the Equation becomes $-n \times \text{Log.}\, a = 0$; therefore the Fluent, duly corrected, is $n \times \text{Log.}\, a - n \times \text{Log.}\, y = \text{Log.}\, \frac{\dot{x} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$, or $\text{Log.}\, \frac{a^n}{y^n} = \text{Log.}\, \frac{\dot{x} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$. Whence it is evident that $\frac{a^n}{y^n} = \frac{\dot{x} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$, and $\frac{a^n \dot{y}}{y^n} - \dot{x} = \sqrt{\dot{y}^2 + \dot{x}^2}$; from which, by ſquaring both Sides, $2\dot{x}$ is found $= \frac{a^n \dot{y}}{y^n} - \frac{y^n \dot{y}}{a^n}$; whoſe Fluent is $2x = -\frac{a^n y^{1-n}}{1-n} + \frac{a^{-n} y^{n+1}}{n+1}$. But when $y = a$, x is $= 0$, and then, $0 = -\frac{a}{1-n} + \frac{a}{n+1} = -\frac{2na}{1-nn}$; therefore the Fluent corrected is $2x = -\frac{a^n y^{1-n}}{1-n} + \frac{a^{-n} y^{n \times 1}}{n+1} + \frac{2na}{1-n^2}$. *Q. E. I.*

Otherwiſe (without ſecond Fluxions.)

441. Put ST $= P$ and RT $= Q$. Then ſince the abſolute Velocity of the Body *S* is denoted by Unity, that with which the Ordinate SR is carry'd towards the Body *T* will be denoted by $\frac{Q}{P} \times 1$ or $\frac{Q}{P}$ (*by Art.* 35.) which ſubtracted from n the Velocity of *T*, leaves $n - \frac{Q}{P}$ for the relative Celerity with which *T* recedes from

R: After the ſame Manner, if from $\frac{Q}{P} \times n$ the Celerity of T in the Direction ST produced, there be taken (1) the Celerity of S in the ſame Direction, the Remainder, $\frac{nQ}{P} - 1$, will be the Celerity with which T recedes from S: Therefore, the Fluxions of Quantities being as the Celerities of their Increaſe, we have $n - \frac{Q}{P} : \frac{nQ}{P} - 1 :: \dot{Q} : \dot{P}$; and conſequently $\overline{nQ - P} \times \dot{Q} = \overline{nP - Q} \times \dot{P}$. But, ſince the Quantities P and Q are concerned exactly alike, the Equation thus derived will, in all probability, become more ſimple, by ſubſtituting for their Sum and Difference: Let therefore $P + Q = s$, and $P - Q = v$, or, which is the ſame, let $P = \frac{s+v}{2}$, and $Q = \frac{s-v}{2}$: Then, by Subſtitution, we ſhall have $\frac{ns - nv - s - v}{2} \times \frac{\dot{s} - \dot{v}}{2} = \frac{ns + nv - s + v}{2} \times \frac{\dot{s} + \dot{v}}{2}$; which contracted, &c. becomes $\overline{1+n} \times v\dot{s} = \overline{1-n} \times s\dot{v}$, or $\overline{1+n} \times \frac{\dot{s}}{s} = \overline{1-n} \times \frac{\dot{v}}{v}$; whoſe Fluent (corrected) is $\overline{1+n} \times \text{Log. } s = \overline{1-n} \times \text{Log. } v + 2n \times \text{Log. } a$, or Log. $s^{1+n} =$ Log. $a^{2n} v^{1-n}$. Whence $s^{1+n} = a^{2n} v^{1-n}$, and conſequently $s^{1+n} \times v^{1+n} = a^{2n} v^2$: But sv $(= \overline{\text{ST} + \text{RT}} \times \overline{\text{ST} - \text{RT}} = \text{RS}^2) = y^2$ therefore $s^{1+n} \times v^{1+n} = y^{2n+2} = a^{2n} v^2$, and $v = \frac{y^{n+1}}{a^n}$; whence $s \left(= \frac{y^2}{v}\right) = \frac{a^n}{y^{n-1}}$, ST $\left(\frac{s+v}{2}\right) \frac{a^n}{2y^{n-1}} + \frac{y^{n+1}}{2a^n}$, RT $\left(\frac{s-v}{2}\right)$

=

$=\frac{a^n}{2y^{n-1}}-\frac{y^{n+1}}{2a^n}$. But RS (y) : RT $\left(\frac{a^n}{2y^{n-1}}-\frac{y^{n+1}}{2a^n}\right)$ :: $\dot{y}:\dot{x}$; whence $2\dot{x}=\frac{a^n\dot{y}}{y^n}-\frac{y^n\dot{y}}{a^n}$, and $2x=-\frac{a^n y^{1-n}}{1-n}+\frac{a^{-n}y^{n+1}}{n+1}+\frac{2na}{1-nn}$, the very ſame as before.

COROLLARY.

442. If the Velocity of *S* be greater than that of *T* (or n be leſs than Unity) the two Bodies will concur when the latter has moved over a Diſtance expreſſed by $\frac{na}{1-n^2}$; becauſe, when y becomes $=0$, $2x$ is barely $=\frac{2na}{1-n^2}$. But if the Velocity of *S* be leſs than that of *T*, it is plain that *S* can never come up with *T*: But its neareſt Approach will be when $y=\overline{\frac{n-1}{n+1}}\Big|^{\frac{1}{2n}}\times a$: For, ſince ST is univerſally $=\frac{a^n}{2y^{n-1}}+\frac{y^{n+1}}{2a^n}$, let the Fluxion of this Expreſſion be taken and put equal to Nothing; and y will be found as above exhibited.

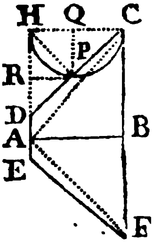
If the Celerities of *S* and *T*, inſtead of being uniform, vary according to a given Law; then, denoting the former by *A* and the latter by *B*, the Equation of the Curve will be $\frac{\dot{x}}{\sqrt{\dot{y}^2+\dot{x}^2}}=-\frac{B\dot{y}}{Ay}$: And if the Fluent of $-\frac{B\dot{y}}{Ay}$ be explicable by a Logarithm, as L. N; then, the Fluent of $\frac{\dot{x}}{\sqrt{\dot{y}^2+\dot{x}^2}}$ being L. $\frac{\dot{y}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$ *, we

* Art. 126.

we ſhall have $N = \frac{\dot{y} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$; which, ordered, gives $\dot{x} = \frac{N\dot{y}}{2} - \frac{\dot{y}}{2N}$: Whence x will be found.

PROB. XVI.

443. *To determine the Fruſtum* CDEF *of a Triangular-Priſm, of a given Baſe* CF *and Altitude* BA; *which, moving in a Medium, in the Direction of its Length* BA, *ſhall be reſiſted the leaſt poſſible.*



Draw CH parallel to BA meeting ED, produced, in H: Moreover, let HP, PQ and PR be perpendicular to CD, CH and DH reſpectively.

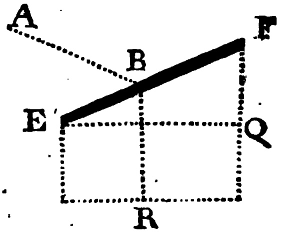
Since the Number of reſiſting Particles acting upon DC is as DH, and the Force of each as $\left(\frac{DR^2}{DP^2}\right)$ the Square of the Sine of the Angle of Incidence DPR, the whole Reſiſtance ſuſtained by DC will therefore be expreſſed by $\frac{DH \times DR^2}{DP^2}$, or DR, which is equal to it (by the Similarity of the Triangles DHP and DPR) Whence the Reſiſtance upon ADC is truly expreſſed by AR (AD + DR) and is a *Minimum* when its Defect (PQ) below the given Quantity AH (or BC) is a *Maximum*: But PQ is a *Maximum* when CQ and HQ are equal; becauſe, the Angle CPH being Right, a Semi-circle deſcribed upon CH will always paſs through the Point P; and it is well known that the greateſt Ordinate in a Semi-circle is That which divides the Diameter into two equal Parts.

Hence the Angle DCH, when the Reſiſtance upon ADC is a *Minimum*, will be juſt the Half of a Right-Angle, provided BC be given greater than BA; otherwiſe,

wiſe, the whole Priſm CAF will be leſs reſiſted than any Fruſtum CDEF of a greater Priſm.

PROB. XVII.

444. *To determine the Angle* RBE *which a Plane* EBF *muſt make with the Wind blowing in a given Direction* RB, *ſo that the Plane itſelf may be urged in another given Direction* BA *with the greateſt Force poſſible.*



It is known, from the Reſolution of Forces, that the Force whereby the Plane EF is urged in the given Direction BA, by a Particle of Air, acting in the Direction RB, is directly as the Rectangle of the Sines of the Angles (ABE, RBE) which the two given Directions make with the Plane: Therefore, ſince the Number of Particles acting on EF is as the Sine of RBE, it follows that the whole Force, or Effect, of the Wind, in the Direction BA, will be as *S.* ABE × *Squ. S.* RBE; which being a *Maximum*, we have (*by Prob.* 5.) 3 : 1 :: Sine of the whole given Angle RBA : Sine of $\overline{\text{RBE} - \text{ABE}}$. Whence the Angles RBE and ABE are both given. *Q. E. I.*

COROLLARY.

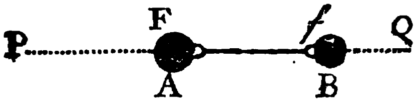
445. If the Angle RBA be a Right one (which is the Caſe with regard to the Sails of a Windmill) then the Sine of $\overline{\text{RBE} - \text{ABE}}$ being $= \frac{1}{3} =$,333 *&c.* we ſhall have RBE − ABE = 19° : 28′; and conſequently

$$\text{RBE}\ \left(\frac{\text{RBA} + \text{ABE}}{2}\right) = 54° : 44'.$$

PROB. XVIII.

446. *If two Bodies* A *and* B, *joined by a String, be urged in oppoſite Directions, towards* P *and* Q, *by any given Forces* F *and* f, *uniformly continued; it is propoſed to find the Tenſion of the String, or the Force whereby the Bodies endeavour to recede from each other.*

Since $F-f$ is the abſolute Force by which the two Bodies are, conſtantly, urged towards P, the *whole* Motion, generated in Both, in any Time T, will therefore be expreſſed by $\overline{F-f}\times T$: Whence, becauſe both Bodies (by reaſon of the String) acquire the ſame Velocity, the Motion generated in A, alone, will be $\frac{A}{A+B}\times\overline{F-f}\times T$, or that Part of the *Whole* defined by $\frac{A}{A+B}$. But the Motion of A, had it not been retarded by the String (or B) would have been $F\times T$; therefore the Loſs of Motion, by the Action upon the String, is $F\times T-\frac{A}{A+B}\times\overline{F-f}\times T$, $=\frac{fA+FB}{A+B}\times T$: Which, divided by the Time T, (wherein that Loſs or Effect is produced) gives $\frac{fA+FB}{A+B}$, for the Tenſion of the Thread, or the Force ſufficient to cauſe the ſaid Loſs or Motion.



The ſame otherwiſe.

447. Becauſe the Force F, was it to act alone, would communicate, by means of the String, the ſame Velocity to B as to A, the Part therefore of the Force F employ'd upon B, by which the String is ſtretch'd, will be $\frac{B}{A+B}\times F$, or $\frac{BF}{A+B}$: And, from the very ſame Argument, if the Force f was to act alone, the Tenſion of the Thread would be $\frac{fA}{A+B}$: Therefore, when both

the Forces act together, the Tenſion will be $\frac{fA+BF}{A+B}$: For it is very plain that, their acting both at the ſame time, no way influences their reſpective Effects on the Thread. *Q. E. I.*

COROLLARY.

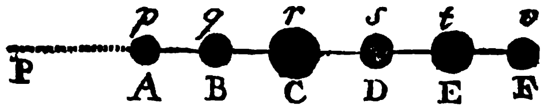
448. If the Forces F and f be reſpectively expounded by the Maſſes, or Weights, of the Bodies A and B; the Tenſion of the Thread will then become $\frac{2AB}{A+B}$. Whence it appears that the Tenſion of a Thread ſliding over a Pin or Pulley, by means of two unequal Weights A and B, ſuſpended at the Ends thereof, is equal to $\frac{2AB}{A+B}$: The Double whereof, or $\frac{4AB}{A+B}$, is the Weight which the Pin or Pulley ſuſtains, while the Bodies are in Motion; becauſe the Thread hangs double, or on both Sides the Pulley.

If ſeveral Bodies A, B, C, D &c. communicating by means of a String or Wire AF, be urged towards a Point P, in the Direction of the String or Wire, by any given Forces p, q, r, s &c. reſpectively, the Tenſion of the Part AB will be

$$= \frac{p \times \overline{B+C+D} \text{ \&c.} - A \times \overline{q+r+s \text{ \&c.}}}{A+B+C+D \text{ \&c.}};$$

of the Part BC

$$= \frac{\overline{p+q} \times \overline{C+D+E} \text{ \&c.} - \overline{A+B} \times \overline{r+s+t} \text{ \&c.}}{A+B+C+D \text{ \&c.}};$$



of CD

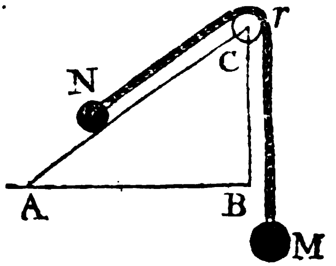
$$= \frac{\overline{p+q+r} \times \overline{D+E+F} - \overline{A+B+C} \times \overline{s+t+v}}{A+B+C+D \text{ \&c.}};$$

&c. &c.

All which eaſily follows from above; and will anſwer alſo in thoſe Caſes where ſome of the Forces are ſuppoſed to act in the contrary Direction, if every ſuch Force be conſidered as a negative Quantity.

PROB. XIX.

449. *Let it be required to raiſe a given Weight* N, *to a given Height* BC, *along an inclin'd Plane* AC, *by means of another given Weight* M, *connected to the former by a flexible Rope* NrM, *moving over a Pulley at* C; *to find the Tenſion of the Rope; alſo the Inclination and Length of the Plane, ſo that the Time of the whole Aſcent may be the leaſt poſſible.*



It is well known that the Force by which N tends to deſcend along the Plane AC, or acts in oppoſition to M (ſuppoſing BC$=a$, and AC $=x$) will be $\frac{aN}{x}$: Therefore $M-\frac{aN}{x}$, or $\frac{xM-aN}{x}$ is the efficacious Force, by which the Bodies are accelerated: But it is likewiſe demonſtrable that the Time of deſcribing any Line by means of a Velocity uniformly accelerated, is in the ſubduplicate Ratio of the Length thereof, directly, and the ſubduplicate Ratio of the accelerating Force, inverſely *: Whence it follows that the Time of deſcribing AC will be repreſented by $\frac{x}{\sqrt{xM-aN}}$: Whoſe Fluxion (or that of its Square) being made equal to Nothing, x will be found $=\frac{2aN}{M}$, or $M : 2N :: a : x$. Hence the

Time

* Art. 203.

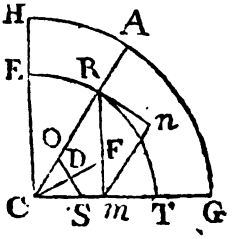
Time of the Aſcent will be the leaſt poſſible, when the Sine of the Plane's Inclination is to the Radius, as the Power (*M*) is to twice the Weight (*N*) to be raiſed.

The Tenſion of the Rope will be determined from the laſt Problem, (by writing N for A, $\frac{aN}{x}$ for F, M for B, and M for f), and comes out $= \frac{MN}{M+N} \times \frac{a+x}{x}$.

Q. E. I.

PROB. XX.

450. *Let* AC *repreſent a Piece of Timber, moveable about a Center* C, *making any Angle* ACG *with the Plane of the Horizon* CG; *to determine the Poſition of a Prop or Supporter* OS, *of a given Length, which ſhall ſuſtain it with the greateſt Facility, in any given Poſition; and alſo what Inclination* AC *will have to the Horizon when the leaſt Force that can ſuſtain it, is greater than the leaſt Force in any other Poſition.*



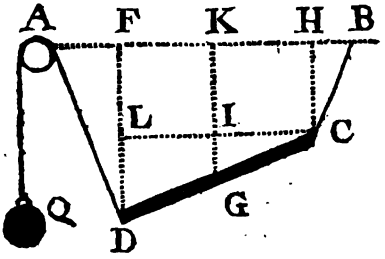
Let R be the Center of Gravity of the Beam AC, and let Rn, Rm and CD be perpendicular to AC, CG and OS reſpectively: Putting SO$=a$, CR $=r$, C$m=x$, and the Weight of the Beam $=w$.

Then, *by the Principles of Mechanics*, we ſhall have, firſt, as Rm : Rn, or as, $r : x :: w : \left(\frac{xw}{r}\right)$ the Force which acting at R, in the Direction Rn, is ſufficient to ſuſtain the Beam AC; ſecondly, as CO : CR $(r) :: \frac{xw}{r}$. (the Quantity laſt found) : $\frac{xw}{CO}$, the Force able to ſupport it, at O, in a perpendicular Direction; and, laſtly, as CD : CO :: $\frac{xw}{CO}$: $\frac{xw}{CD}$, the Force, or Weight, actually ſuſtained by the given Prop SO. Which Force will therefore be the leaſt poſſible when the Perpendicular CD is the greateſt poſſible, let the Angle of Inclination GCA be what it will: But of all Triangles, having the ſame Baſe (OS) and vertical Angle (SCO) the Iſoſceles one is known to have the greateſt Perpendicular: Therefore the Triangle CSO will be Iſoſceles, and the Angles S and O equal to each other, when the Weight ſuſtain'd by the Prop OS is a *Minimum*.

But, now, to give a Solution to the latter Part of the Problem, or to find (ſuppoſing the Angles S and O to be equal) when $\frac{x}{CD} \times w$ is a *Maximum*, let CD produced meet mR in F; and then, becauſe of the ſimilar Triangles CDS and CmF, we ſhall have CD : x (Cm) :: SD ($\frac{1}{2}a$) : mF, or $\frac{x}{CD} = \frac{mF}{\frac{1}{2}a}$; and conſequently $\frac{x}{CD} \times w = \frac{mF}{\frac{1}{2}a} \times w$: But, ſince CF biſects the Angle mCR, we alſo have, $r+x$ (CR + Cm) : x (Cm) :: $\sqrt{r^2-x^2}$ (Rm) : Fm = $\frac{x\sqrt{r^2-x^2}}{r+x} = x\sqrt{\frac{r-x}{r+x}}$: Whence the Force $\frac{mF}{\frac{1}{2}a} \times w$, acting upon the Supporter, is likewiſe truly expreſſed by $\frac{wx}{\frac{1}{2}a}\sqrt{\frac{r-x}{r+x}}$: Whereof the Fluxion being taken and put equal to Nothing &c. we get $x = \frac{r\sqrt{5}-r}{2}$: Therefore CR : C$m$ (:: 1 : $\frac{\sqrt{5}-1}{2}$) :: Radius : Coſine of RCG = 51° : 50′, the Inclination required.

PROB. XXI.

451. *To determine the Poſition of a Beam* CD, *moveable about one End* C *as a Center, and ſuſtained at the other End* D *by a given Weight* Q, *appended to a Cord* QAD *paſſing over a Pulley at a given Point* A.



Let G be the Center of Gravity of the Beam; alſo let DF, GK and CH be perpendicular to the Plane of the Horizon, and CL and AH parallel to the ſame: Putting AH $=a$, CH $=b$, CD $=c$, CG $=d$, DL $=x$, CL $=y$, and the Weight of the Beam $=w$. Then AF $=a-y$, DF $=b+x$, and AD $(\sqrt{\mathrm{AF}^2+\mathrm{DF}^2}) = \sqrt{a^2-2ay+y^2+b^2+2bx+x^2}$; which (becauſe $y^2+x^2=c^2$) will alſo be $=\sqrt{a^2+b^2+c^2+2bx-2ay} = \sqrt{f^2+2bx-2ay}$ (by putting $f^2=a^2+b^2+c^2$) whoſe Fluxion, $\frac{b\dot{x}-a\dot{y}}{\sqrt{f^2+2bx-2ay}}$, multiply'd by Q is the Momentum of the Weight Q, ſuppoſing the Beam to to be in Motion. Moreover, becauſe DC : DL :: CG : GI, we have GI $=\frac{dx}{c}$; whoſe Fluxion, $\frac{d\dot{x}}{c}$, multiply'd by w, is the Momentum of the Beam itſelf in a vertical Direction.

Wherefore making theſe *Momenta* equal to each other (according to the Principles of Mechanics) we get $\frac{b\dot{x}-a\dot{y}}{\sqrt{f^2+2bx-2ay}} \times Q = \frac{d\dot{x}}{c} \times w$, and conſequently $\overline{b\dot{x}-a\dot{y}} \times cQ = dw\dot{x}\sqrt{f^2+2bx-2ay}$: But, ſince

$y^2 + x^2 = c^2$, we have $2y\dot{y} + 2x\dot{x} = 0$, or $-\dot{y} = \frac{x\dot{x}}{y}$: And therefore (by Subſtitution) $\overline{b\dot{x} + \frac{ax\dot{x}}{y}} \times cQ = dw\dot{x}\sqrt{f^2 + 2bx - 2ay}$, or $\overline{by + ax} \times cQ = dwy \times \sqrt{f^2 + 2bx - 2ay}$: From whence, and the foregoing Equation $x^2 + y^2 = c^2$, both x and y may be determined.

The ſame otherwiſe.

452. It is evident, from Mechanics, that the Force which, acting in the Direction DF, would ſuſtain the End D, is to the whole Weight w, as CG to CD; and therefore is $= \frac{\text{CD}}{\text{CG}} \times w$: It is likewiſe known that two Forces acting in the different Directions DF and DA, ſo as to have the ſame Effect in ſuſtaining DC, or cauſing It to move about the Point C, muſt be to each other, inverſely, as the Sines of the Angles of Incidence FDC and ADC. Therefore we have $S.\,\text{FDC} : S.\,\text{ADC} :: Q : \frac{\text{CD}}{\text{CG}} \times w$; from which given Ratio of the Sines, the Angles themſelves will be found, by an algebraic Proceſs independent of Fluxions.

COROLLARY.

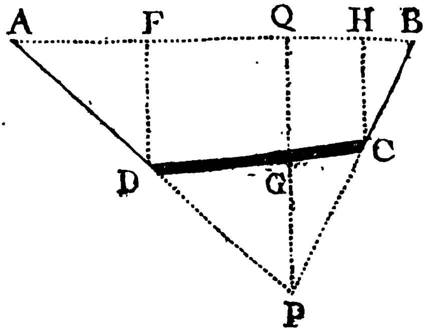
453. If the Poſition of CD be ſuppoſed given, and the Tenſion of AD (or the Weight Q) be required: Then, from the foregoing Proportion, we ſhall have $Q = \frac{S.\,\text{FDC}}{S.\,\text{ADC}} \times \frac{\text{CG}}{\text{CD}} \times w$. Which will alſo expreſs the Tenſion of AD when the End C is ſuſtained by a Cord BC inſtead of a Pin at C: Whence it follows that the Tenſions of two Cords AD and BC, ſuſtaining a Beam or Rod CD, at its Extremes D and C, are expreſſed by $\frac{S.\,\text{FDC}}{S.\,\text{ADC}} \times \frac{\text{CG}}{\text{CD}} \times w$, and $\frac{S.\,\text{HCD}}{S.\,\text{BCD}} \times \frac{\text{DG}}{\text{CD}} \times w$; and

therefore are to each other as $\frac{CG}{S.\ ADC}$ to $\frac{DG}{S.\ BCD}$, or as *S*. BCD × CG to *S*. ADC × DG reſpectively; becauſe the Sine of FDC and that of its Supplement HCD are equal to each other.

PROB. XXII.

454. *To determine the Poſition of a Beam* DC, *ſuſpended at its Extremes by two Cords* AD *and* BC *of given Lengths, from two given Points* A *and* B *in the ſame horizontal Line* AB.

Let G be the Center of Gravity of the Beam, and let DF and CH be perpendicular to AB.



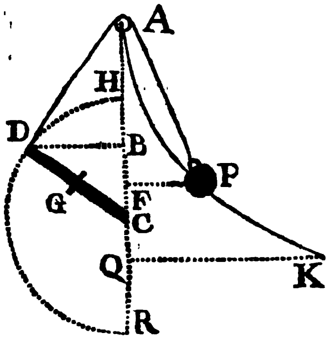
It appears, from the Corol. to the laſt Problem, that the Tenſion of AD is to that of BC, as $\frac{CG}{S.\ ADC}$ to $\frac{DG}{S\ BCD}$; whence (by the Reſolution of Forces) the Force of AD, in a Direction parallel to the Horizon, is to the Force of BC, in the oppoſite Direction, as $\frac{CG}{S.\ ADC} \times \frac{S.\ ADF}{Rad.}$ to $\frac{DG}{S.\ BCD} \times \frac{S.\ BCH}{Rad.}$. Which Forces, that the Beam may remain *in Equilibrio*, muſt conſequently be equal to each other; and therefore $\frac{S.\,BCD}{S.\,ADC} = \frac{S.\,BCH}{S.\,ADF} \times \frac{DG}{CG}$. But now, to determine the Angles themſelves, from this Equation and the given Lengths of AB, BC *&c.* let AD and BC be produced to meet each other in P, and let PQ, perpendicular to AB, be drawn; putting $AB = a$, $AD = b$, $BC = c$, $DC = d$, $DG = f$, $CG = g$, $AP = x$, and $BP = y$.

Then, becauſe AB : AP+BP :: AP—BP : AQ—BQ $= \frac{AP^2 - BP^2}{AB}$, we have $AQ = \frac{1}{2} AB + \frac{AP^2 - BP^2}{2AB} = \frac{AB^2 + AP^2 - BP^2}{2AB}$; and conſequently the Co-ſine of A (= Sine ADF) to the Radius $1 = \frac{AB^2 + AP^2 - BP^2}{2AB \times AP}$: Whence, from the ſame Argument, it is evident that the Co-ſine of B (= Sine BCH) will be expreſſed by $\frac{AB^2 + BP^2 - AP^2}{2AB \times BP}$; and That of APB by $\frac{AP^2 + BP^2 - AB^2}{2AP \times BP}$; And alſo by $\frac{PD^2 + PC^2 - DC^2}{2PD \times PC}$; which two laſt Quantities being equal to each other, we have $PD \times PC \times \overline{AP^2 + BP^2 - AB^2} = AP \times BP \times \overline{PD^2 + PC^2 - DC^2}$; that is $\overline{x-b} \times \overline{y-c} \times \overline{x^2 + y^2 - a^2} = xy \times \overline{\overline{x-b}^2 + \overline{y-c}^2 - d^2}$. Moreover, ſince PC : PD :: S. ADC (or PDC) : S. BCD (or PCD) we alſo have $\frac{PD}{PC} = \frac{S.\,BCD}{S.\,ADC} = \frac{S.\,BCH}{S.\,ADF} \times \frac{DG}{CG}$ (by the firſt Equation); whence $CG \times PD \times S.\,ADF = DG \times PC \times S.\,BCH$; that is $CG \times PD \times \frac{AB^2 + AP^2 - BP^2}{2AB \times AP} = DG \times PC \times \frac{AB^2 + BP^2 - AP^2}{2AB \times BP}$, or

$CG \times PD \times BP \times \overline{AB^2+AP^2-BP^2} = DG \times PC \times AP \times \overline{AB^2+BP^2-AP^2}$, which, in algebraic Terms, is $gy \times \overline{x-b} \times \overline{a^2+x^2-y^2} = fx \times \overline{y-c} \times \overline{a^2+y^2-x^2}$. From whence and the preceding Equation the Values of x and y will be known.

P R O B. XXIII.

455. *Suppoſing a Beam* CD, *moveable about one End* C, *as a Center, to be ſuſtained at the other End* D *by means of a given Weight* P, *hanging at a Rope paſſing over a Pulley at a given Point* A, *vertical to* C; *it is propoſed to find the Curve* APK *along which the Weight muſt aſcend, or deſcend, ſo as to be, every where, a juſt Counterpoiſe to the Beam.*



From the Center C, with the Radius CD, let a Semi-circle HDR be deſcribed, and let DB and PF be perpendicular to the vertical Line AHCR; alſo let $CD = a$, $CA = b$, $AH = c$, $AF = x$, $PF = y$, $HB = z$, and the Length of the Rope $DAP = m$; likewiſe let HQ (h) be the given Value of x (AF) when D coincides with H.

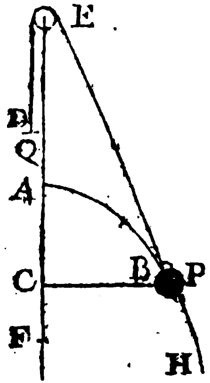
Becauſe the Weight and the Beam are always in *Equilibrio*, by Hypotheſis, their *Momenta*, and conſequently their Velocities, in a vertical Direction, muſt be every where in a conſtant Ratio; and therefore the Diſtance QF ($h-x$) aſcended by the Weight *P*, will be, to the Diſtance HB deſcended by the End of the Beam D likewiſe in a conſtant Ratio: Let this Ratio be that of b to any given Quantity d, that is, let $h-x : z :: b : d$, and we ſhall have $dh - dx = bz$: Moreover, we have AD^2 $(CD^2+AC^2-2AC \times BC) = a^2+b^2-2b \times \overline{a-z} = \overline{b-a}^2+2bz = c^2+2bz = c^2-2dh+2dx$: Whence AP $(m - AD) = m -$

$\sqrt{cc-2db+2dx}$, and therefore, y^2 $(AP^2 - AF^2) = \overline{m-\sqrt{cc-2db+2dx}}|^2 - x^2$. *Q. E. I.*

After the ſame manner a Curve may be found, along which a Weight deſcending, ſhall be every where in *Equilibrio* with another Weight aſcending thro' the Arch of a given Curve.

PROB. XXIV.

456. *To find the Equation of a Curve* ABH, *along which a given Weight* P, *ſuſpended by a String* PED *paſſing over a Pulley* E, *muſt deſcend, ſo that the Tenſion of the String may vary according to any given Law.*



Let EC be perpendicular, and CP parallel, to the Plane of the Horizon; alſo let AE $=a$, AC $=x$, CB $=y$, EP $=v$, and let the Tenſion of the String (or the Force acting at the End D) be denoted by any variable, or conſtant, Quantity Q.

Therefore, becauſe the Celerity of the Weight P, in a vertical Direction, is to its Celerity, in the Direction EP produced, (or the Celerity of the other End D) as $\dot{x}$ to $\dot{v}$, it is evident that the Weight itſelf muſt be to the tending Force Q, inverſely in that Ratio, and conſequently $P\dot{x} = Q\dot{v}$.

Furthermore, becauſe EC $= a+x$ and BC$^2 =$ BE$^2 -$ EC^2, we have $y^2 = v^2 - \overline{a+x}|^2$: From which Equations, when the Relation of P and Q is given, the Curve itſelf will alſo be known.

Thus, for Example, let the Ratio of P to Q, be conſtant, or that of m to n, then $m\dot{x}$ being $= n\dot{v}$, we have (by taking the Fluent) $mx + na = nv$; whence $v = a + \frac{mx}{n}$; and therefore y^2 $(= a^2 + \frac{2max}{n} + \frac{m^2x^2}{n^2}$

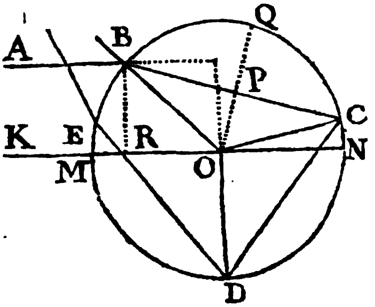
$-a^2-2ax-x^2) = \frac{m-n}{n} \times 2ax + \frac{m^2-n^2}{n^2} \times x^2$: Which is the Equation of an Hyperbola.

Again, for a ſecond Example, let the tending Force $\mathcal{Q}$ be to the Weight P, as DE^n to $\mathrm{AC}^m \times c^{n-m}$, or as $\overline{b-v}|^n : x^m c^{n-m}$ (ſuppoſing $b = \mathrm{PED}$ and $c =$ any given Line AF.) Therefore, ſince $\mathcal{Q} = \frac{\overline{b-v}|^n}{c^{n-m} x^m} \times P$, and $\frac{\overline{b-v}|^n}{c^{n-m} x^m} \times P\dot{v}\ (= \mathcal{Q}\dot{v}) = P\dot{x}$, we have $\overline{b-v}|^n \times \dot{v} = c^{n-m} x^m \dot{x}$, and ſo $\frac{\overline{b-a}|^{n+1} - \overline{b-v}|^{n+1}}{n+1} = \frac{c^{n-m} x^{m+1}}{m+1}$; whence $\overline{b-v}|^{n+1} = \overline{b-a}|^{n+1} - \frac{\overline{n+1} \times c^{n-m} x^{m+1}}{m+1}$, and v (EP) $= b - \overline{\overline{b-a}|^{n+1} - \frac{\overline{n+1} \times c^{n-m} x^{m+1}}{m+1}}|^{\frac{1}{n+1}}$. From which the Relation of x and y, or the Value of BC, is alſo known.

But if $m = 0$, and $n = 1$, (which will be the Caſe when the Force acting at D is equal to that by which a Beam or Rod is made to move about a Center, as in the laſt Problem) v will then become, barely, $= b - \overline{\overline{b-a}|^2 - 2cx}|^{\frac{1}{2}}$, and therefore $y^2\ (= v^2 - \overline{a+x}|^2) = \overline{b - \sqrt{\overline{b-a}|^2 - 2cx}}|^2 - \overline{a+x}|^2$: Therefore ABH is, in this Caſe, a Line of the fourth Order.

PROB. XXV.

457. *Suppoſing a Ray of Light* ABCD *to be refracted at the Surface of a given Sphere* MQND, *and afterwards reflected any given Number* (*n*) *of Times, within the Sphere; to determine the Diſtance of the Incident Ray* AB *from the Axis* MN, *ſo that the Arch* MBCDE, *intercepted by the given Point* M *and the emerging Ray at* E, *may be a* Minimum.



Let the Radius OB = 1, the Sine of Incidence BR = x, and the Sine of Refraction OP = y, and let the given Ratio of the two laſt be that of p to q. Since all the Angles of Incidence and Reflexion BCO OCD, CDO &c. are equal, the Arcs BC, CD and DE muſt alſo be equal; and conſequently MBCDE = MB + $\overline{n+1}\times$ BC = MB + $\overline{2n+2}\times$ BQ : Whoſe Fluxion is to be equal to Nothing*. Now the Fluxion of the Arch MB, whoſe Sine is x and Radius Unity, will be $= \frac{\dot{x}}{\sqrt{1-x^2}}$ †; and that of the Arch BQ, whoſe Co-ſine (OP) is y, $= \frac{-\dot{y}}{\sqrt{1-y^2}}$. Hence we have $\frac{\dot{x}}{\sqrt{1-x^2}} - \frac{\overline{2n+2}\times\dot{y}}{\sqrt{1-y^2}} = 0$: But ſince $x : y :: p : q$, y is $= \frac{qx}{p}$ and $\dot{y} = \frac{q\dot{x}}{p}$; and ſo we have $\frac{\dot{x}}{\sqrt{1-x^2}} - \frac{\overline{2n+2}\times q\dot{x}}{\sqrt{p^2-q^2x^2}} = 0$; whence (putting

$m =$

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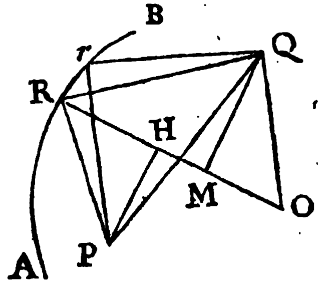
* Art. 22.

† Art. 142.

$m = 2n+2$) x is found $= \frac{1}{q}\sqrt{\frac{m^2q^2-p^2}{m^2-1}}$: From which it is obſervable, that, when mq is leſs than p, or $2n+2$ leſs than $\frac{p}{q}$, the Arch MBCD continually increaſes with BM; and therefore is the leaſt poſſible, when B coincides with M. *Q. E. I.*

PROB. XXVI.

458. *If two Rays of Light* PR *and* Pr, *from a given Point* P, *making an indefinitely ſmall Angle with each other, be reflected at a given Curve Surface* ARB; *'tis propoſed to determine the Concourſe, or Focus,* Q *of the reflected Rays* RQ *and* rQ.



Let RO, perpendicular to the Curve, be the Radius of a Circle having the ſame Curvature with ARB at R; make PH and QM perpendicular to RO, join Q, O; and put RO $=r$, PR$=y$, RH$=v$, and RQ$=z$.

Then, becauſe the Angle of Reflection ORQ is equal to the Angle of Incidence ORP, the Triangles RQM and RPH will be ſimilar, and therefore $y : v :: z : \mathrm{RM} = \frac{vz}{y}$: Whence OQ^2 (RO^2 + RQ^2—2RO × RM)

$$= r^2 + z^2 - \frac{2rvz}{y}.$$

But, ſince this Quantity OQ^2 continues the ſame (by Hypotheſis) whether we regard one Ray or the other (that is, whether y ſtands for PR or Pr) its Fluxion muſt therefore be equal to Nothing; that is, $2z\dot{z} - \frac{2r\dot{v}zy + 2rv\dot{z}y - 2rvz\dot{y}}{y^2} = 0$: Whence $z = \frac{vy\dot{z}}{\frac{y^2\dot{z}}{r} + v\dot{y} - y\dot{v}}$: But (*by Art.* 35.) $\dot{z} = -\dot{y}$; therefore $z = \frac{-vy\dot{y}}{-\frac{y^2\dot{y}}{r} + v\dot{y} - y\dot{v}}$: Moreover (*by Art.* 73.) $r = \frac{y\dot{y}}{\dot{v}}$; therefore

$$z = \frac{-vy\dot{y}}{-y\dot{v} + v\dot{y} - y\dot{v}} = \frac{vy\dot{y}}{2y\dot{v} - v\dot{y}}. \qquad \textit{Q. E. I.}$$

Example 1. Let ARB be an Arch of the Logarithmic Spiral: whoſe Equation is $av = by$ †: And then, $\dot{v}$ being $= \frac{b\dot{y}}{a}$, we ſhall have $z \left(\frac{vy\dot{y}}{2y\dot{v} - v\dot{y}}\right) = y$: Therefore in this Caſe the Incident and Reflected Rays are equal to each other.

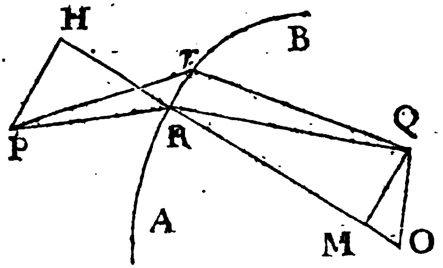
Ex. 2. Let ARB be ſuppoſed to degenerate into a Right-line: In which Caſe v being conſtant, its Fluxion $\dot{v}$ is $= 0$; and therefore $z \left(= \frac{vy\dot{y}}{-v\dot{y}}\right) = -y$: Which being negative, indicates that the Rays do not converge after Reflection, but, on the contrary, diverge from a Point on the contrary Side of ARB, at the Diſtance y. Which is very eaſy to demonſtrate by common Geometry.

PROB. XXVII.

459. *Let two Rays of Light* PR *and* Pr, *from a given Point* P, *be refracted at a given Curve Surface* ARB; *to determine the Focus* Q *of the refracted Rays* RQ *and* rQ.

Let the Lines RO, RH &c. be drawn, and denoted as in the preceding Problem: Moreover, let the Sine of Incidence PRH (to the Radius 1) be repreſented by s, and let it be to the Sine of Refraction ORQ, in the given Ratio of 1 to n.

Then (*by Trigonometry*) 1 : ns (Sine QRM) :: z (RQ) : QM = nsz; and therefore RM = $\sqrt{z^2 - n^2s^2z^2}$



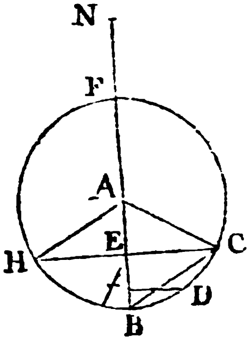
$= z\sqrt{1-n^2s^2}$. From whence, following the Steps of the preceding Problem, we alſo get $OQ^2 = r^2 + z^2 - 2rz\sqrt{1-n^2s^2}$; and its Fluxion $2z\dot{z} - 2r\dot{z}\sqrt{1-n^2s^2} + \frac{2rzn^2s\dot{s}}{\sqrt{1-n^2s^2}} = 0$; or $z\dot{z}\sqrt{1-n^2s^2} - r\dot{z} \times \overline{1-n^2s^2} + n^2rzs\dot{s} = 0$. But (*by Art.* 35.) $\dot{z} = -n\dot{y}$; therefore $-z\dot{y}\sqrt{1-n^2s^2} + r\dot{y} \times \overline{1-n^2s^2} + nrzs\dot{s} = 0$: Moreover (*by Trig.*) 1 (Radius) : s (Sine of PRH) :: y (PR) : $\sqrt{y^2-v^2}$ (PH) whence we have $sy = \sqrt{y^2-v^2}$, $s^2 = 1 - \frac{v^2}{y^2}$, and $s\dot{s} = \frac{-y^2v\dot{v} + v^2y\dot{y}}{y^4} = \frac{v^2\dot{y} - yv\dot{v}}{y^3}$; which Values, of s^2 and $s\dot{s}$, being ſubſtituted in the foregoing Equation, it becomes $-z\dot{y}\sqrt{1-n^2+\frac{n^2v^2}{y^2}} + r\dot{y} \times \overline{1-n^2+\frac{n^2v^2}{y^2}} + nrz \times \frac{v^2\dot{y}-yv\dot{v}}{y^3} = 0$, or $-zy^2\dot{y}\sqrt{\overline{1-nn} \times y^2 + n^2v^2} + ry\dot{y} \times \overline{\overline{1-nn} \times y^2 + n^2v^2} + nrz \times \overline{v^2\dot{y} - yv\dot{v}} = 0$; or (putting $\sqrt{\overline{1-n^2} \times y^2 + n^2v^2} = w$) $\frac{-zy^2w\dot{y}}{r} + w^2y\dot{y} +$

$nzw^2\dot{y} - nyzw\dot{v} = 0$. But (*by Art.* 73.) $r = \frac{y\dot{y}}{\dot{v}}$, therefore $- nyw\dot{v} + w^2y\dot{y} + nzv^2\dot{y} - nyzv\dot{v} = 0$, and conſequently $z = \frac{w^2y\dot{y}}{\overline{wy + nvy} \times \dot{v} - nv^2\dot{y}}$. *Q. E. I.*

From this Solution, that of the preceding Problem is eaſily derived: Alſo from hence the Cauſtic (or the Curve which is the Locus of all the Points Q thus found) will likewiſe be given.

PROB. XXVIII.

460. *To find the Time of the Vibration of a Pendulum in the Arch of a Circle.*



Let AB denote the Pendulum in a vertical Poſition; and from any Point D in the given Arch CBH, wherein the Vibrations are perform'd, draw Df parallel to CH; and let AB $= a$, BE $= c$, Bf $= x$, and BD $= z$: By the Nature of the Circle we have $\dot{z} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$ *: Whence the Fluxion of the Time, being as $\frac{\dot{z}}{\sqrt{Ef}}$ †, will be defined by $\frac{a\dot{x}}{\sqrt{c-x} \times \sqrt{2ax - xx}}$

$$= \frac{a\dot{x}}{\sqrt{cx - xx} \times \sqrt{2a - x}} = \frac{\overline{\tfrac{1}{2}a}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{cx - xx}} \times \overline{1 - \frac{x}{2a}}|^{-\frac{1}{2}}$$

$$= \frac{\overline{\tfrac{1}{2}a}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{cx - xx}} \times \overline{1 + \frac{x}{2 \cdot 2a} + \frac{3x^2}{2 \cdot 4 \cdot 4a^2} + \frac{3 \cdot 5x^3}{2, 4 \cdot 6 \cdot 8a^3} + \frac{3 \cdot 5 \cdot 7x^4}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 16a^4}}$$ &c. Whereof the Fluent, when

* Art. 142.

† Art. 207.

when $x=c$, (or $\overline{cx-x^2}|^{\frac{1}{2}}=0$) is, (*by Art.* 142. *and* 286.) equal to

$$p\sqrt{\tfrac{1}{2}a}\times\overline{1+\frac{c}{2.2.2a}+\frac{3.3c^2}{2.2.4.4.\overline{2a}|^2}+\frac{3.3.5.5c^3}{2.2.4.4.6.6.\overline{2a}|^3}+\frac{3.3.5.5.7.7c^4}{2.2.4.4.6.6.8.8.\overline{2a}|^4}}$$

&c. Which therefore is proportional to the Time of half one Vibration; where p ſtands for the Semi-Periphery of the Circle whoſe Radius is Unity.

Corollary I.

461. Since the Time of the perpendicular Deſcent of a Body through any given Right-line u, computed according to the ſame Method, is as the Fluent of $\frac{\dot{u}}{\sqrt{u}}$ or $2\sqrt{u}$, it follows that the Time of falling along the Diameter BF ($2a$), or the Cord CB *, will * Art. 205. be truly defined by $2\sqrt{2a}$: Which therefore is to the Time of the Deſcent thro' the Arch CDB, as $\frac{4}{p}$ to $1+\frac{c}{2.2.2a}+\frac{3.3c^2}{2.2.4.4.\overline{2a}|^2}$, &c. From whence, as the Time of falling thro' the Diameter BF, is abſolutely given, *by Art.* 202. the true Time of Vibration will alſo be known.

Corollary II.

462. If the Arch in which the Pendulum vibrates be very ſmall, the above Proportion will become, nearly, as 4 to p: From which it appears, that the Time of Deſcent thro' any very ſmall Arch CB is to that along the Chord CB, as the Periphery of any Circle is to four times its Diameter.

Corollary III.

463. Hence, we have a Method for determining how far a Body freely deſcends in a given Time; by knowing

the Time of Vibration, of a given Pendulum: For, if BN be aſſumed for the Space thro' which a Body would deſcend during the Time of one whole Vibration, in the very ſmall Arch CBH; then, the Diſtances deſcended being as the Squares of the * Times, we have, from the laſt Corollary, as $4^2 : \overline{2p}|^2 :: \text{BF}\ (2a) : \text{BN}$, or $1 : \frac{1}{2}p^2 : a : \text{BN}$; that is, as the Square of the Diameter of a Circle is to half the Square of its Periphery, ſo is the Length of the Pendulum, to the Diſtance a Body will freely deſcend, from Reſt, in the Time of one Oſcillation. Thus, for inſtance (becauſe it is found from Experiment that a Pendulum 39,2 Inches long vibrates Seconds) it will be as $1 : 4{,}934\ (= \frac{1}{2}p^2) :: 39{,}2 : 193$ Inches, the Diſtance which a heavy Body will fall in the firſt Second of Time.

* Art. 201.

COROLLARY IV.

464. Moreover, from the foregoing Series, the Time which a Pendulum, vibrating in an exceeding ſmall Arch, will loſe when made to vibrate in a greater Arch of the ſame Circle may alſo be deduced:

For let T be put to denote the Number of Seconds in 24 Hours (or any other given Time) then the Number of Vibrations, performed in that Time will be as

$$\frac{T}{1 + \frac{c}{2 \cdot 2 \cdot 2a} + \frac{3 \cdot 3c^2}{2 \cdot 2 \cdot 4 \cdot \overline{2a}|^2} \ \&c.}$$

; which, therefore, in an exceeding ſmall Arch (where c may be taken as Nothing) will be expreſſed by T: And ſo the Time (t) or Number of Vibrations loſt will be

$$T - \frac{T}{1 + \frac{c}{2 \cdot 2 \cdot 2a} + \frac{3 \cdot 3c^2}{2 \cdot 2 \cdot 4 \cdot 4 \cdot \overline{2a}|^2} \ \&c.} = T \times \overline{\frac{c}{8a} + \frac{5c^2}{256a^2}} \ \&c.$$

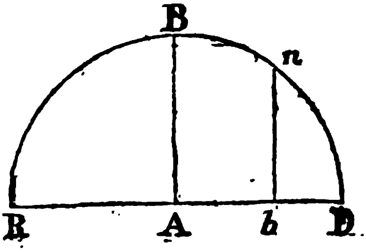
(by dividing by the Denominator.)

Now, if the Number of Degrees deſcribed on each Side of the Perpendicular be repreſented by D, the

Arch itſelf, on each Side, will be $= 3.14159$ &c. $\times a \times \frac{D}{180}$; which, if the Value of D be not more than about 15 or 20 Degrees, will be nearly equal to its Chord, repreſented by $\sqrt{2ac}$ ($= \sqrt{\text{BF} \times \text{BE}}$.) From which Equation we get $\frac{c}{a} = \frac{D^2}{6560}$: This Value, ſubſtituted above, gives $t = T \times \overline{\frac{D^2}{8 \times 6560} + \frac{5D^4}{256 \times \overline{6560}|^2}}$ &c. $= T \times \frac{D^2}{52480}$ nearly: Which, when T is interpreted by 86400 Seconds (or one whole Day) becomes $= 1\frac{1}{2} \times D^2$, nearly: And ſo many are the Seconds which will be loſt *per Diem* in the Arch D. From whence we gather, that if the Pendulum meaſures true Time in any ſmall Arch, whoſe Degrees on each Side the Perpendicular are denoted by A, the Number of Seconds loſt *per Diem* in another Arch whoſe Degrees are B, will be nearly repreſented by $\frac{3}{2} \times \overline{B^2 - A^2}$: Thus, if a Pendulum meaſures true Time, in an Arch of 3 Degrees, it will loſe $10\frac{1}{2}$ Seconds a Day in an Arch of 4 Degrees, and 24″ in an Arch of 5 Degrees.

PROB. XXIX.

465. *To determine the* Meridional Parts *anſwering to any propoſed Latitude, according to* Wright's *Projection, applied to the true ſpheroidal Figure of the Earth.*



Let DAR be the Axis, AB the Semi-equatoreal Diameter, and DBR a Meridian, of the Earth; alſo let *bn* be an Ordinate to the Ellipſis DBR; putting AD (= AR)

$=1$, BA$=d$, A$b=x$, $bn=y$, B$n=z$, and the Meridional Diſtance (in Parts of the Semi-Axis AD) $=u$.

Then, by the Nature of the Ellipſis, we have $y=d\sqrt{1-x^2}$; therefore $\dot{y}=\frac{-dx\dot{x}}{\sqrt{1-x^2}}$; and conſequently $\dot{z}=\sqrt{\dot{x}^2+\frac{d^2x^2\dot{x}^2}{1-xx}}$: Which, by putting $b^2=d^2-1$, will be reduced to $\dot{z}=\frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-x^2}}$. Whence, by the Nature of the Projection, it will be as bn $(d\sqrt{1-x^2})$: AB (d) :: $\dot{z}$ $\left(\frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-xx}}\right)$: $\dot{u}=\frac{\dot{x}\sqrt{1+b^2x^2}}{1-x^2}$; which is the Fluxion of the Quantity required: But we are now to get the ſame thing expreſſed in Terms of the Latitude of the Place n: In order thereto, putting the Sine of that Latitude $=s$, we have, by Trigonometry, as $\dot{z}$ $\left(\frac{x\sqrt{1+b^2x^2}}{\sqrt{1-x^2}}\right)$: $-\dot{y}$ $\left(\frac{dx\dot{x}}{\sqrt{1-x^2}}\right)$ :: Radius (1) : s; and conſequently $s\sqrt{1+b^2x^2}=dx$; from which Equation x is found $=\frac{s}{\sqrt{d^2-b^2s^2}}$: Whence $\dot{x}=\frac{d^2\dot{s}}{\overline{d^2-b^2s^2}\big|^{\frac{3}{2}}}$; alſo $1-x^2=\frac{d^2-b^2s^2-s^2}{d^2-b^2s^2}=\frac{d^2-d^2s^2}{d^2-b^2s^2}$ (becauſe $d^2=1+b^2$) and, laſtly, $\sqrt{1+b^2x^2}$ $\left(=\frac{dx}{s}\right)=\frac{d}{\sqrt{d^2-b^2s^2}}$: Which ſeveral Values being ſubſtituted in that of $\dot{u}$, found above, it will become $\left(=\frac{d^2\dot{s}}{\overline{d^2-b^2s^2}\big|^{\frac{3}{2}}}\times\frac{d}{\sqrt{d^2-b^2s^2}}\times\frac{d^2-b^2s^2}{d^2\times\overline{1-ss}}\right)=\frac{d\dot{s}}{\overline{d^2-b^2s^2}\times\overline{1-ss}}$; which reſolved into two Parts, for the more readily finding the Fluent, gives $\dot{u} = \frac{d\dot{s}}{1-s^2} - \frac{db^2\dot{s}}{d^2-b^2s^2}$: Whereof the Fluent being taken, we have

$$u = \begin{cases} 2\,.\,302585 \text{ \&c.} \times \frac{1}{2}d \times \text{Log.}\,\frac{1+s}{1-s} \\ -\,2\,.\,302585 \text{ \&c.} \times \frac{1}{2}b \times \text{Log.}\,\frac{d+bs}{d-bs} \end{cases}$$

But, as $3{,}14159$ &c. × $2d$ (the Meaſure of the whole Periphery of the Earth at the Equator, in Parts of the Semi-Axis AD) is to 21600 (the Meaſure of the ſame Periphery in Geographical Miles) ſo is the foreſaid Value of u to

$$\left\{\begin{matrix} 3958 \times \text{Log.}\,\frac{1+s}{1-s} \\ -\,\frac{3958b}{b} \times \text{Log.}\,\frac{d+bs}{d-bs} \end{matrix}\right\} \text{ the correſponding Value}$$

of u, in Geographical Miles, or the Meridional Parts required.

COROLLARY.

466. If the Earth be conſidered as differing but little from a Sphere, d will be nearly $=1$, and conſequently $(\sqrt{d^2-1})$ the Value of b, very ſmall: Therefore, in this Caſe, the latter Part of our Fluent $\left(-\frac{3958b}{d} \times \text{Log.}\,\frac{d+bs}{d-bs}\right)$ will become nearly $= 3440b^2s$ (becauſe $\text{Log.}\,\frac{d+bs}{d-bs} = \frac{2bs}{d}\Big) \times \frac{1}{2\,.\,3025 \text{ \&c.}}$ [*]. But if the Earth be taken as a perfect Sphere, this laſt Expreſſion will vaniſh, and ſo the Value of u will become barely $=3958$

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[*] *There is a Miſtake in* p. 43. *and* 44. *of my Diſſertations (by forgetting to divide by the Modulus* 2.3025 &c.) *which may from hence be rectify'd.*

$\times$ Log. $\frac{1+s}{1-s}$. Which Logarithm, it is eaſy to prove, expreſſes twice the artificial Tangent of half the given Latitude increaſed by 45 Degrees (Radius being Unity.) Wherefore, if the Meridional Parts anſwering to any given Latitude, thus found (from a Table of logarithmic Tangents) when the Earth is conſidered as a perfect Sphere, be denoted by M, it follows that the Meridional Parts anſwering to the ſame Latitude, when the Earth is taken as a Spheroid, will be nearly equal to $\overline{M - 3440b^2s}$: Which, becauſe AD (1) : AB $(\sqrt{1+bb})$ :: 230 : 231 *, will (by ſubſtituting the Value of b hence ariſing) be reduced to $M - 30s$. Whence the following Rule.

M

As Radius, to the Sine of the given Latitude, ſo is 30 to a Fourth-Proportional; which ſubtracted from the Meridional Parts when the Earth is taken as a Sphere (found as above) gives the Meridional Parts anſwering to the ſame Latitude, when it is conſidered as an oblate Spheroid.

Thus, for Example, let the given Latitude be 50°: Then, firſt, for the Meridional Parts in the *Sphere*; we muſt, according to the foregoing Preſcript, take the Logarithmic Tangent of 25°+45°, or 70°: Which, by the Table, is found = 0,43893 *&c.* This multiply'd by the conſtant Multiplicator 7916 (= 2 × 3958) produces 3475 for the Meridional Parts in the Sphere: Then by the Rule above, it will be as Radius to the Sine of 50°, ſo is 30 to 23; which ſubtracted from 3475, leaves 3452 for the Meridional Parts anſwering to 50° Latitude, in the Spheroid.

PROB. XXX.

467. *To determine the Paths which Shadows of Objects deſcribe, upon the Plane of the Horizon, during the Sun's apparent diurnal Revolution.*

Let CSODT be the Plane of the Horizon, and AV the perpendicular Height of the Object: Then, ſince the Rays, intercepted by the higheſt Point V, would, in the Sun's diurnal Revolution, form a conical Sur-

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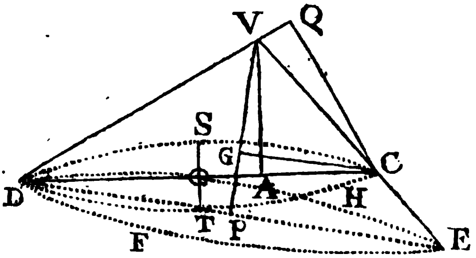
* Art. 397.

face VDFEH about that Point as a Vertex; whoſe Axis PV produced paſſes thro' the Pole of the World; it is evident that the Path of the Shadow, being the Interſection of the Plane of the Horizon with that Surface, muſt be a Conic Section.



Let its two principal Diameters therefore (when an Ellipſis, that is, when the Sun never deſcends below the Horizon) be CD and ST; alſo let DPE and CG be perpendicular to VP the Axis of the Cone, and CQ perpendicular to DV: Putting the Sine of (QVC) twice the Sun's Declination VEP$=f$; the Sine of (DCV) his greater Meridional Altitude $= g$, and that of the leſſer (CDV) $= h$: Then (*by plane Trig.*) $g : 1$ (AV) :: 1 (Radius) : CV $= \frac{1}{g}$; and h (Sine of CDV : $\frac{1}{g}$ (CV) :: f (Sine of DVC) : DC $= \frac{f}{gh}$: Moreover, 1 (Radius) : $\frac{1}{g}$ (CV) :: p (the Sine of the Comp. Decl. GVC) : GC $= \frac{p}{g}$: And in the very ſame Manner it will be found that DP $= \frac{p}{h}$: But GC$\times$DP $=$ OS^2 (*vid. Art.* 41.) whence we have ST (2OS) $= \frac{2p}{\sqrt{gh}}$: From which, and the Tranſverſe Axis (DC $= \frac{f}{gh}$) the Curve itſelf is given.

Q. E. I.

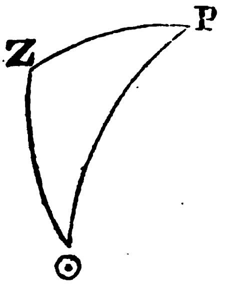
Lemma.

468. *In any ſpherical Triangle, if Radius be ſuppoſed Unity, the Product of the Sines of any two of the Sides drawn into the Co-ſine of the Angle they include, added to the Product of their Co-ſines, is equal to the Co-ſine of the remaining Side.*

This is demonſtrated by the Writers upon Spherics.

PROB. XXXI.

469. *The Elevation of the Pole and the Declination of the Sun being given, to find at what Time of the Day the Azimuth of the Sun increaſes the ſloweſt.*



It is evident that the Time ſought will be when the Fluxion of the Hour Angle P, bears the greateſt Ratio poſſible to That of the Azimuth Z.

Now the Fluxion of the Angle P is to that of Z, univerſally, as Rad. × *S.* ZO : *S.* PO × *Co-ſ.* O (*by Art.* 256. *Caſe* 2.) Conſequently $\frac{S.\ PO \times \text{Co-ſ. } O}{\text{Rad.} \times S.\ ZO}$, or $\frac{\text{Co-ſ. } O}{S.\ ZO}$ is a *Minimum*, in this Caſe, becauſe PO may be conſidered as conſtant.

Let now the Sine of PO be put $=p$, its Co-ſine $=d$, the Co-ſine of PZ $=b$, that of ZO $=x$, and that of O $=y$; then, the Sine of ZO being $=\sqrt{1-x^2}$, we have (*by the Lemma*) $p\sqrt{1-x^2} \times y + dx = b$; whence $y = \frac{b-dx}{p\sqrt{1-x^2}}$ and therefore $\frac{\text{Co-ſ. } O}{S.\ ZO}$ $\left(= \frac{y}{\sqrt{1-x^2}}\right)$ $= \frac{b-dx}{p \times \overline{1-x^2}}$: Which put into Fluxions, and reduced,

duced, gives $x = \frac{b - \sqrt{b^2 - d^2}}{d}$, for the Sine of the Sun's Altitude at the Time required: Whence the Time itſelf is given.

PROB. XXXII.

470. *To determine the Ratio of the Heat received from the Sun in different Latitudes, during the Time of one whole Day, or any Part thereof.*

Let p = the Sine of the Sun's Polar-Diſtance P☉ (*ſee the laſt Fig.*)
d = its Co-ſine, or the Sine of the Declination.
b = the Sine of the Pole's Elevation.
c = its Co-ſine, or the Sine of PZ.
z = the Angle (P) expreſſing the Time from Noon.
x = its Sine, and $\sqrt{1-x^2}$ = its Co-ſine.

Then (by the foregoing Lemma) we ſhall have $pc\sqrt{1-x^2} + bd$ = Co-ſine Z☉ = Sine of the Sun's Altitude.

Now, it is known that the Number of Rays falling in any given Particle of Time, upon a given horizontal Plane, is as that Time and the Sine of the Sun's Altitude conjunctly: Therefore the Number of Rays falling in the Time $\dot{z}$, or $\frac{\dot{x}}{\sqrt{1-xx}}$ (*vid. Art.* 142.) will be defined by $pc\dot{x} + bd\dot{z}$: Whoſe Fluent $pcx + bdz$ is, therefore, as the Heat required.

Where it may be obſerved,

1. That when the Latitude and Declination are of different Kinds, or P☉ is greater than 90 Degrees, the Value of d is to be conſidered as a negative Quantity.

2. That, if the Expreſſion for the Heat found above be divided by the Square of the Sun's Diſtance from the Earth, the Quotient will exhibit the Ratio of the Heat, allowing for the Excentricity of the Earth's Orbit.

COROLLARY I.

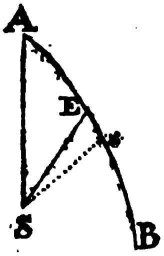
471. If the Place propoſed be at the Equator, the Heat, received in half one diurnal Revolution, will be barely as p; becauſe $b=0$, $c=1$, and $x=1$.

COROLLARY II.

472. But if the Place be at the Pole, then the Heat will be as $d \times \overline{3,14159}$ &c. ſince, in this Caſe, $c=0$, $b=1$, and z ($=$ Semi-Circle) $=3,14159$ &c.

LEMMA.

473. *The Number of Particles of Light, ejected by the Sun, upon the Earth, in a given Time, is proportional to the Angle deſcribed about his Center in that Time.*

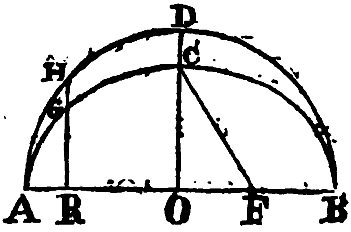


For, let S repreſent the Center of the Sun, AEB the Orbit of the Earth (or That of any other Planet) and let E and r be two Points therein as near as poſſible to each other; Since the Triangle E S r may be taken as rectilineal, its Area, if the Angle E S r be ſuppoſed given, or every where the ſame, will be as $SE \times Sr$, or SE^2: And therefore the Time of deſcribing E r (being always as that Area) is alſo explicable by SE^2: But the Intenſity of the Light, or Heat, at the Diſtance of SE is as $\frac{1}{SE^2}$: Therefore the Intenſity compounded with the Time (or the whole Number of Particles received in that Time) will conſequently be as $\frac{1}{SE^2} \times SE^2$ ($=1$): Which being every where the ſame, the Propoſition is manifeſt.

PROB. XXXIII.

474. *To determine the Ratio of the Heat received from the Sun at the Equator and either of the Poles, during the Time of one whole Year, or any Part thereof.*

If



If the Sine of the Sun's Declination be denoted by d and its Co-ſine by p, the Heat received at the Equator, and the Pole, during half one diurnal Revolution of the Sun, will be as p and $d \times 3,14159$ &c. reſpectively (*by the Corollaries to the preceding Problem*).

Let the Sun's Longitude, conſidered as variable, be now denoted by z, and its Sine by s; and let f be put for the Sine of the Obliquity of the Ecliptic: Then (*per Spherics*) we ſhall have $d = fs$, and conſequently p $(= \sqrt{1 - d^2}) = \sqrt{1 - f^2 s^2}$: Wherefore, ſeeing the Ratio of Heat in the two Places, for one Half-Day, is that of $\sqrt{1 - f^2 s^2}$ to $fs \times 3,14$ &c. let each of theſe Terms be multiplied by $\frac{\dot{s}}{\sqrt{1 - ss}}$ $(= \dot{z})$ * expreſſing the Quantity of Heat falling upon the Earth in the Time of deſcribing $\dot{z}$ (*ſee the foregoing Lemma*) then the Products $\frac{\dot{s}\sqrt{1 - f^2 s^2}}{\sqrt{1 - s^2}}$, and $3,14 f \times \frac{s\dot{s}}{\sqrt{1 - s^2}}$ will be the Fluxions of the required Heat, anſwering to $\dot{z}$.

* Art. 142.

But now to exhibit the Fluents hereof, let ACB be an Ellipſis whoſe greater Semi-Axis AO is = Unity, and its Excentricity FO $= f$; and, ſuppoſing ADB to be a Circle deſcribed about the Ellipſis, let the Arch DH expreſs the Sun's Longitude from the Equinoctial Point; whoſe Sine (OR) being $= s$, its Co-ſine RH will be $= \sqrt{1 - ss}$.

But, by the Property of the Ellipſis, OD (1) OC : $(\sqrt{1 - f^2})$ :: RH $(\sqrt{1 - ss})$: RG $= \sqrt{1 - ff} \times \sqrt{1 - ss}$: Whoſe Fluxion being =

$\frac{\sqrt{1-ff}\times -s\dot{s}}{\sqrt{1-ss}}$, we have $\sqrt{\dot{s}^2+\frac{\overline{1-ff}\times s^2\dot{s}^2}{1-ss}}$ $=\frac{\dot{s}\sqrt{1-f^2s^2}}{\sqrt{1-ss}}=$ the Fluxion of CG. Whence it appears that the Fluent of $\frac{\dot{s}\sqrt{1-f^2s^2}}{\sqrt{1-ss}}$ is truly defined by CG, or CG × AO^2.

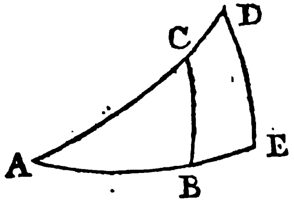
But the Fluent of the other given Fluxion, $3.14f\times\frac{s\dot{s}}{\sqrt{1-ss}}$, will be $=3{,}14f\times\overline{1-\sqrt{1-ss}}=$ ADB × FO × $\overline{OD-RH}$. Therefore the two Fluents, when H and G coincide with A, will be to each other as CA × AO to ADB × FO: Whereof the Antecedent, multiplied by 4, will be as the Heat received at the Equator during one whole Year; and the Conſequent, multiplied by 2, as the Heat at the Pole in the ſame Time (becauſe the Sun ſhines at the Pole only two Quarters of the Year.) Hence the required Ratio, of the Heat received at the Equator and Pole, in one whole Year, will be That of CA × AO to DA × FO; or, in Species, as $1-\frac{f^2}{2.2}-\frac{3f^4}{2.2.4.4}-\frac{3.3.5f^6}{2.2.4.4.6.6}$ * &c. to f; which, in Numbers, is as 959 to 396, or as 17 to 7, nearly.

* Art. 434.

PROB. XXXIV.

475. *To find when that Part of the Equation of Time, ariſing from the Obliquity of the Ecliptic to the Equinoctial, is a* Maximum.

In the right-angled ſpherical Triangle ABC let the Angle A be that made by the Ecliptic AC, and the Equinoctial AB; then the Problem will be, to find when



when the Difference between the Baſe AB and the Hypothenuſe AC is the greateſt poſſible (the Angle A remaining invariable.) Now (*by Art.* 254.) we have *Co-ſ. BC* : *Sin. C* :: Fluxion of AC Fluxion of AB : Alſo (*per Spherics*) *Sin.* C : *Co-ſ.* A :: *Rad.* : *Coſ.* BC $= \frac{\textit{Co-ſ.}\ \mathrm{A} \times \textit{Rad.}}{\textit{Sin.}\ \mathrm{C}}$: Whence, by multiplying the two firſt Terms of the former Proportion by theſe equal Quantities, reſpectively, we get this new Proportion, *viz.* $\overline{\textit{Co-ſ. BC}}|^{2}$: *Co-ſ.* A × *Radius* :: ſo is the Fluxion of AC to That of AB: But, when AC—AB is a *Maximum*, theſe Fluxions become equal; and conſequently $\overline{\textit{Co-ſ. BC}}|^{2} = \textit{Co-ſ. A} \times \textit{Rad.}$ From which Equation BC, and from thence AC, will be known. *Q.E.I.*

The ſame, without Fluxions.

476. It will be (*per Spherics*) *Rad.* : *Co-ſ.* A :: *Tang.* AC : *Tang.* AB; and therefore by Compoſition and Diviſion, *Rad.* + *Co-ſ.* A : *Rad.* — *Co-ſ.* A :: *Tang.* AC + *Tang.* AB : *Tang.* AC — *Tang.* AB :: *Sin.* $\overline{\mathrm{AC}+\mathrm{AB}}$: *Sin.* $\overline{\mathrm{AC}-\mathrm{AB}}$, by the Theorem mentioned in Problem 8th : From which, by following the Steps there laid down, it appears that, *Radius* + *Co-ſ.* A : *Radius* — *Co-ſ.* A :: *Radius* : Sine of $\overline{\mathrm{AC}-\mathrm{AB}}$, when a *Maximum* : Whence (AC + AB being then = 90°) both AC and BC will be given.

COROLLARY.

477. Since, *Radius* + *Co-ſ.* A : *Radius* — *Co-ſ.* A :: *Co-tang.* $\frac{1}{2}$ A : *Tang.* $\frac{1}{2}$ A* :: $\overline{\textit{Radius}}|^{2}$: $\overline{\textit{Tang.}\ \tfrac{1}{2}\ \mathrm{A}}|^{2}$;

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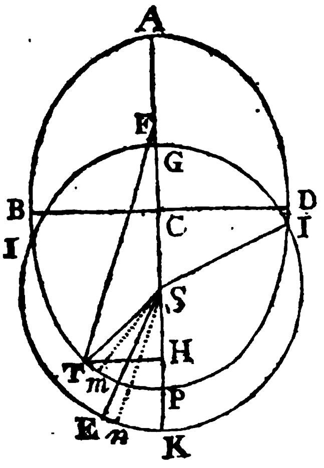
* *Vid. p.* 70. *and* 71. *of my Trigonometry.*

therefore $\overline{\mathit{Radius}}|^2 : \overline{\mathit{Tang.}\ \tfrac{1}{2}A}|^2 :: \mathit{Radius} : \mathit{Sine}$ of $\overline{AC - AB}$, Or, *Radius* : *Tang.* ½ A :: *Tang.* ½ A : the *Sine* of the greateſt Equation: Which, ſuppoſing the Angle A to be 23° 29′, comes out 2° : 28′ : 34″ : anſwering, in Time, to 9 Minutes : 54 Seconds.

P R O B. XXXV.

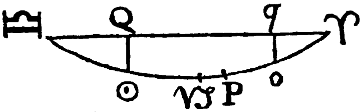
478. *To determine when the abſolute Equation of Time, ariſing from the Inequality of the Sun's apparent Motion, and the Obliquity of the Ecliptic, conjunctly, is a* **Maximum.**



Let ABPD be the Ellipſis in which the Earth revolves about the Sun in the Focus S; let F be the other Focus, and T the Place of the Earth in its Orbit at the Time required. Moreover, about S, as a Center, let a Circle GEKI be deſcribed, whoſe Diameter GK is a Mean Proportional between the two Axes AP and BD of the Ellipſis; ſo that the Area thereof may be equal to That of the Ellipſis: And, ſuppoſing S*m* to be indefinitely near to ST, let ES*n* be a Sector of the ſaid Circle, equal to the Area TS*m*.

Then, the Time in which the Earth moves through the Arch T*m* being to the Time of one intire Revolution, as the Area TS*m*, or ES*n*, is to the whole Ellipſis, or the equal Circle GEKF; and theſe Areas ES*n*, and GEKI being in the Ratio of the Arch E*n* to the whole Periphery GEKI; it is evident that E*n*, or the Angle ESn, will expreſs the Increaſe of the *Mean Longitude*, in the foreſaid Time of deſcribing the Arch Tm: And that this Angle or Increaſe, by reaſon of the Equality of the Areas ESn and TSm, will be to the Angle TSm, expreſſing the correſponding Increaſe of the *True Longitude*, as ST^2 to SE^2. Therefore, if the former be denoted by *M*, the latter will be repreſented by $\frac{SE^2}{ST^2} \times M$. But now to get a proper Expreſſion for the Value of this Increaſe of the *True Longitude*, in Algebraic Terms; let FT be drawn, and alſo TH, perpendicular to AP: Putting AC ($=CP$) $=a$, $CB=b$, CS ($=CF$) $=c$, $ST=z$, and the Co-ſine of (TSP) the Earth's Diſtance from its *Perihelion* (to the Radius 1) $=x$: Then FT being ($=AP-ST$) $=2a-z$ (by the Property of the Ellipſis) and $SH=xz$ (*by Trig.*) we have $\overline{FT+ST} \times \overline{FT-ST}$ ($2a \times \overline{2a-2z}$ $= FS \times 2CH$ ($2c \times 2 \times \overline{c+xz}$) by a known Property of Triangles: From which Equation z (ST) is found $= \frac{a^2-c^2}{a+cx} = \frac{b^2}{a+cx}$: And this Value, with that of ES^2 ($=ab$) being ſubſtituted in the Increaſe of the True Longitude, found above, we thence get $\frac{a \times \overline{a+cx}^2}{b^3} \times M$ for the Meaſure of that Increaſe; where *M* denotes the Increment of the *Mean Motion* correſponding.

This being obtained, let ♎ ♑ ♈ (in the annexed Figure) repreſent the Southern Semi-Circle of the Ecliptic, P the Place of the Perihelion, ♑ the Tropic of Capricorn, ☉ the apparent Place of the Sun in the Ecliptic, and Q☉ his Declination, at the Time required: Then it appears, (*from Art.* 475.) that the Increaſe of the *True Longitude* ♎☉, in an indefinitely ſmall Particle of Time, will be to That of the *Right-Aſcenſion* ♎Q, in the ſame Time, as the Square of the Co-ſine of Q☉ is to a Rectangle under the Radius and the Co-ſine of the Angle ♎: Therefore, the former, being expreſſed by $\frac{a\times\overline{a+cx}|^2}{b^3}\times M$, the latter is truly repreſented by $\frac{a\times\overline{a+cx}|^2}{b^3}\times M\times\frac{\text{Rad.}\times\text{Coſ.}\ \text{♎}}{\overline{\text{Co-ſ.}\ \odot Q}|^2}$: Which, in the required Circumſtance, when the propoſed Equation (or the Difference between the Sun's



Mean Motion and *Right Aſcenſion*) is a *Maximum*, muſt conſequently be equal to (*M*) the corresponding Increaſe of *Mean Motion*; and therefore $\frac{a\times\overline{a+cx}|^2}{b^3}=\frac{\overline{\text{Co-ſ.}\ \odot Q}|^2}{\text{Rad.}\times\text{Co-ſ.}\ \text{♎}}$.

But, to obtain the Value of the latter Part of this Equation, alſo, in Algebraic Terms, let the Sine and Co-ſine of (♈P) the Diſtance of the *Perihelion* from ♈, be denoted by m and n reſpectively; then, the Co-ſine of P☉ being (as above) expreſſed by x, and its Sine by $\sqrt{1-xx}$, we ſhall thence get $nx+m\sqrt{1-xx}$=Co-ſine of ☉♈ = Sine of ♎☉ (*by the Elem. of Trig.*) But (putting the Sine of the Angle ♎ $=p$ and its Co-ſine $=q$) we have (*per Spherics*) Radius (1) : Sine ♎☉ $(nx+m\sqrt{1-xx}::p:pnx+pm\sqrt{1-xx}$=Sine of Q☉; from whence $\overline{\text{Co-ſ.}Q\odot}|^2=1-\overline{pnx+pm\sqrt{1-xx}}|^2$: Which Value, with That of the Co-ſine of the Angle ♎, being ſubſtituted above, we, at length, get $\frac{a\times\overline{a+cx}|^2}{b^3}=$

$\frac{\overline{1 - pnx + pm\sqrt{1 - xx}}|^2}{q}$; from which Equation the Value of x may be determined.

The foregoing Equation, it may be obſerved, gives the Time of the *Maximum* which precedes the Winter Solſtice; but if the *Maximum* following that Solſtice be ſought; it is but changing the Sign of m, and then you will have $\frac{a \times \overline{a + cx}|^2}{b^3} = \frac{\overline{1 - pnx - pm\sqrt{1 - xx}}|^2}{q}$, anſwering in this Caſe. And from the negative Roots of this, and the preceding, Equation, the Times of the other *Maxima* after, and before, the Summer Solſtice will alſo be obtained. *Q. E. I.*

COROLLARY.

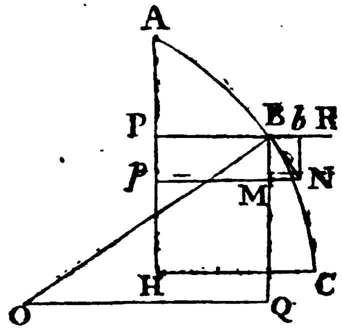
479. It is evident that the Equation of the Earth's Orbit (or that Part of the Equation of Time ariſing from the Inequality of the Sun's apparent Motion) will be a *Maximum*, when the Center of the Earth is in the Interſection I of the Ellipſis and the Circle; where the *Mean Motion* and *True Longitude* increaſe with the ſame Celerity.

PROB. XXXVI.

480. *To determine the Law of the Denſity of a Medium and the Curve deſcribed therein, by Means of an uniform Gravity, ſo that the Projectile may, every where, move with the ſame Velocity.*

It appears, from Art. 367. that $\sqrt{\frac{\dot{y}\dot{y}}{\ddot{x}}}$ is a general Expreſſion for the Celerity in the Direction of the Ordinate PBR; whence $\frac{\dot{z}}{\dot{y}} \times \sqrt{\frac{\dot{y}\dot{y}}{\ddot{x}}}$, or its Equal, $\frac{\dot{z}}{\sqrt{\ddot{x}}}$, muſt be the true Meaſure of the abſolute Celerity,

lerity, in the Direction BN: Which being a conſtant Quantity (by Hypotheſis) its Square muſt alſo be con-



ſtant; and ſo, we have $\frac{\dot{s}\dot{s}}{\ddot{x}} = a$, and conſequently $\dot{x}\dot{x} + \dot{y}\dot{y}\ (= \dot{s}\dot{s}) = a\ddot{x}$.

But, in order to the Solution of the Equation thus given, make $u : 1 :: \dot{x} : \dot{y}$, or $\dot{x} = u\dot{y}$; then $\ddot{x} = \dot{u}\dot{y}$, and, by Subſtitution, $u^2\dot{y}^2 + \dot{y}^2 = a\dot{u}\dot{y}$: Hence, $\dot{y}$ being $= \frac{a\dot{u}}{uu+1}$, and $\dot{x} = \frac{au\dot{u}}{uu+1}$, we get $y = a \times$ *Arch*, whoſe Tangent is u * (and Secant $\sqrt{1+uu}$); and $x = \frac{1}{2}a \times$ Hyp. Log. $\overline{1+uu} = a \times$ Hyp. Log. $\sqrt{1+uu}$ †. Therefore, as the *Hyp. Log.* of $\sqrt{1+uu}$ is $= \frac{x}{a}$, the Common Logarithm of $\sqrt{1+uu}$ will be $= \frac{0{,}4342944\,\&c. \times x}{a}$; and conſequently $y = a \times$ *Arch*, whoſe Radius is Unity, and Log. Secant $\frac{0.4342944\,\&c. \times x}{a}$.

Moreover, with reſpect to the Denſity of the Medium; if the abſolute Force of Gravity, in the Direction QB, be

* Art. 142.

† Art. 126.

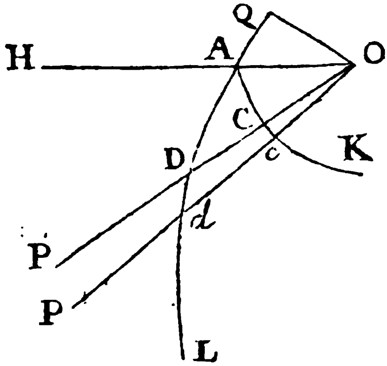
be denoted by Unity, its Efficacy in the Direction BN, whereby the Body is accelerated, will be expreſſed by $\frac{\dot{x}}{\dot{z}}$, or its Equal $\frac{u}{\sqrt{1+uu}}$: Which, as the Velocity is ſuppoſed to remain every where the ſame, muſt alſo expreſs the Force of the Reſiſtance, in the oppoſite Direction, or the true Meaſure of the required Denſity. This, therefore, if M be put for the abſolute Number whoſe Hyperbolical Logarithm is Unity, may be had in Terms of x, and will be $\overline{1-\overline{M}|^{\frac{-2x}{a}}}\Big|^{\frac{1}{2}}$: Becauſe Hyp. Log. $\overline{M}|^{\frac{x}{a}}$ $\left(=\frac{x}{a}\right)$ being $=$ Hyp. Log. $\sqrt{1+uu}$, we have $\sqrt{1+uu}=\overline{M}|^{\frac{x}{a}}$; whence $u=\overline{M^{\frac{2x}{a}}-1}\Big|^{\frac{1}{2}}$, and conſequently $\frac{u}{\sqrt{1+uu}}=\overline{1-\overline{M}|^{\frac{-2x}{a}}}\Big|^{\frac{1}{2}}$. *Q. E. I.*

PROB. XXXVII.

481. *Let a Line, or an inflexible Rod* OP *(conſidered without regard to Thickneſs) be ſuppoſed to revolve about one of its Extremes* O, *as a Center, with a Motion regulated according to any given Law; whilſt a Ring, or Ball, carried about with it, and tending to the Center* O *with any given Force, is ſuffered to move or ſlide freely along the ſaid Line or Rod: It is propoſed to determine the Velocity of the Ring, and its Preſſure upon the Rod, in any propoſed Poſition, together with the Nature of the Curve* ADL *deſcribed by means of that compound Motion.*

Let ODP be any Poſition of the revolving Line, and D the correſponding Poſition of the Body: Moreover, ſuppoſing ACK to be the Circumference of a

Circle deſcribed from the Center O, through the given Point A, let the Meaſure of the angular Celerity of that *Line*, in the ſaid Circumference ACK, be repre-



ſented by u; alſo let v denote the Celerity of the Ring at D in the Direction DP; and w the true Meaſure of the centripetal Force: Call OA, a; OD, x; and AC, z; and let the given Values of u and v, at A, be denoted by b and c reſpectively. Then it will be, as $a : x :: u : \left(\frac{ux}{a}\right)$ the paracentric Velocity of the Body, at D; whoſe Square, divided by the Diſtance OD, gives $\frac{u^2x}{a^2}$, for the true Meaſure of the Centrifugal Force* ariſing from the Revolution of the Rod: From which the centripetal Force w being deducted, the Remainder, $\frac{xu^2}{a^2} - w$, is the true Force whereby the Velocity in the Line OP is accelerated. Therefore (*by Art.* 218.) we have $v\dot{v} = \overline{\frac{xu^2}{a^2} - w} \times \dot{x} = \frac{u^2x\dot{x}}{a^2} - w\dot{x}$.

Moreover, becauſe the Fluxion of the Time is expreſſed either by $\frac{\dot{x}}{v}$ or by $\frac{\dot{z}}{u}$, theſe two Values muſt,

* Art. 211.

therefore, be equal to each other, and conſequently $v = \frac{u\dot{x}}{\dot{z}}$: From which, and the preceding Equation (when u and w are exhibited in Terms of x or z) the required Relation of v, x and z will alſo become known— But now, in order to determine the Action of the Rod upon the Ring; let OdP be indefinitely near to ODP, interſecting ADL and ACK in d and c; and put Od = $x + \overset{\prime}{x}$. Then, becauſe a Body, acted on by no other Force beſides That tending to the Center, about which it revolves, deſcribes Areas proportional to the Times *, and the angular Celerity of a Ray revolving with the Body, is, in that Caſe, as the Square of the Diſtance of the Body from the Center, inverſely (*vid. Art.* 478.) it follows, that, if the Rod was to ceaſe to act upon the Ring, at the Poſition ODP, the angular Celerity at c, would then be $\frac{x^2}{\overline{x+\overset{\prime}{x}}\rvert^{2}} \times u$, inſtead of $u + \overset{\prime}{u}$. Therefore the Exceſs of $u + \overset{\prime}{u}$ above $\frac{x^2}{\overline{x+\overset{\prime}{x}}\rvert^{2}} \times u$, which is $= \overset{\prime}{u} + \frac{2u\overset{\prime}{x}}{x} - \frac{3u\overset{\prime}{x}^2}{x^2}$ &c. is the Increaſe of the ſaid angular Celerity, at the Diſtance OC, ariſing from the Action of the Rod. Therefore it will be, as OC (a) : OD (x) :: the ſaid Increaſe to $\left(\frac{x\overset{\prime}{u}}{a} + \frac{2u\overset{\prime}{x}}{a} - \frac{3u\overset{\prime}{x}^2}{ax} \text{ \&c.}\right)$ the Alteration of the Ring's paracentric Velocity, ariſing from the ſame Cauſe. Which, divided by $\left(\frac{\overset{\prime}{x}}{v}\right)$ the Time wherein *It* is produced, gives $\frac{xv\overset{\prime}{u}}{a\overset{\prime}{x}} + \frac{2uv}{a} -$

* Art. 224.

$\frac{3uvu}{ax}$ &c. for the Meaſure of the Force, by which *It* is produced. From whence, by ſubſtituting $\frac{\dot{u}}{\dot{x}}$ in the Room of $\frac{u}{t}$, and neglecting all the Terms after the two

* Art. 134. firſt (in order to have the limiting Ratio *) we get $\frac{xv\dot{u}}{a\dot{x}} + \frac{2uv}{a}$. Therefore it will be, as $\frac{xv\dot{u}}{a\dot{x}} + \frac{2uv}{a}$ to

† Art. 212. $\frac{bb}{a}$ †, or as $\frac{xv\dot{u}}{bb\dot{x}} + \frac{2uv}{bb}$ to Unity, ſo is the Action of the Rod upon the Ring, to the (given) Centrifugal Force at A (or the Force that would retain a Body in the Circle ACK, with the Velocity *b*.) *Q. E. I.*

COROLLARY I.

482. If the angular Motion be uniform, the Equations found above, will become $v\dot{v} = \frac{b^2x\dot{x}}{a^2} - w\dot{x}$, and $v = \frac{b\dot{x}}{\dot{z}}$. From the latter of which, by taking the Fluxion, we have $\dot{v} = \frac{b\ddot{x}}{\dot{z}}$; whence (by Subſtitution) $\frac{b^2\dot{x}\ddot{x}}{\dot{z}\dot{z}} = \frac{b^2x\dot{x}}{aa} - w\dot{x}$, and conſequently $\ddot{x} - \frac{x\dot{z}^2}{a^2} = -\frac{w\dot{z}^2}{b^2}$; from the Solution of which, the Relation of *x* and *z* will be given. And then, the Value of $v \left(\frac{b\dot{x}}{\dot{z}}\right)$ being alſo known, the Action upon the Rod, which in this Caſe is barely $= \frac{2bv}{a} \left(= \frac{2b^2\dot{x}}{a\dot{z}}\right)$ will be given likewiſe,

being to $\left(\frac{bb}{a}\right)$ the centrifugal Force in the Circle ACK, as $\frac{2\dot{x}}{\dot{x}}$ to Unity.

COROLLARY II.

483. But if the Angular Celerity be proportional to any Power (x^m) of the Diſtance, and the Centripetal Force w be, alſo, ſuppoſed to vary according to ſome Power (x^n) of the ſame Diſtance: Then, putting p to denote the Centripetal, and q the Centrifugal, Force, at the given Point A; the Value of w will, here, be expounded by $\frac{x^n}{a^n}\times p$, and That of u by $\frac{x^m}{a^m}\times b$: And therefore, the paracentric Velocity of the Ring at D being $= \frac{x^m}{a^m}\times b\times\frac{x}{a}\left(=\frac{bx^{m+1}}{a^{m+1}}\right)$ it will be as $\frac{bb}{a} : \frac{b^2x^{2m+2}}{xa^{2m+2}} :: q : \frac{x^{2m+1}}{a^{2m+1}}\times q$, the Centrifugal Force at D *. Hence * Art. 212. $v\dot{v} = \frac{qx^{2m+1}\dot{x}}{a^{2m+1}} - \frac{px^n\dot{x}}{a^n}$; whereof the (corrected) Fluent is $\frac{1}{2}vv - \frac{1}{2}cc = \frac{qx^{2m+2}}{\overline{2m+2}\times a^{2m+1}} - \frac{px^{n+1}}{\overline{n+1}\times a^n} - \frac{qa}{2m+2} + \frac{pa}{n+1}$: From whence v is found =

$$\sqrt{cc - \frac{qa}{m+1} + \frac{2pa}{n+1} + \frac{qx^{2m+2}}{\overline{m+1}\cdot a^{2m+1}} - \frac{2px^{n+1}}{\overline{n+1}\cdot a^n}}$$

and $\dot{z}\left(=\frac{u\dot{x}}{v} = \frac{bx^m\dot{x}}{a^m v}\right) =$

$$\frac{bx^m\dot{x}}{a^m\sqrt{cc - \frac{qa}{m+1} + \frac{2pq}{n+1} + \frac{qx^{2m+2}}{\overline{m+1}\cdot a^{2m+1}} - \frac{2px^{n+1}}{\overline{n+1}\cdot a^n}}}$$

Moreover, by ſubſtituting for u, and its Fluxion, we get $\frac{xv\dot{u}}{a\dot{x}}+\frac{2uv}{a}=\overline{m+2}\times\frac{bx^m v}{a^{m+1}}$, expreſſing the Action of the Rod upon the Ring: Which, therefore, when m is expounded by -2, will intirely vaniſh: And, in that Caſe, $\dot{z}$ will become $=$

$$\frac{a^2 b\dot{x}}{x\sqrt{cc+qa+\frac{2pa}{n+1}\times x^2-qa^3-\frac{2px^{n+3}}{\overline{n+1}\cdot a^n}}};$$

expreſſing the Nature of the Trajectory deſcribed by means of a Centripetal Force, varying according to any Power (x^n) of the Diſtance. But this Equation will be rendered ſomewhat more commodious, by ſubſtituting the Values of b and c: For, if OQ (perpendicular to the Tangent at A) be denoted by h, it will be, h: $\sqrt{a^2-b^2}$ (AQ) :: b (the Celerity in the Direction AC)

* Art. 35. to $c=\frac{b\sqrt{a^2-b^2}}{b}=$ the Celerity in the Direction AH *.

† Art. 211. Therefore, b being $=\sqrt{aq}$ †, we have $c^2=\frac{a^3q}{bb}-aq$;

and $\dot{z}=\frac{a^2\dot{x}}{x\sqrt{\frac{aa}{bb}+\frac{2p}{\overline{n+1}\cdot q}\times x^2-a^2-\frac{2px^{n+3}}{\overline{n+1}\cdot qa^{n+1}}}}$:

Which Equation is the ſame, in effect, with that given in *Art.* 242. by a different Method.

COROLLARY III.

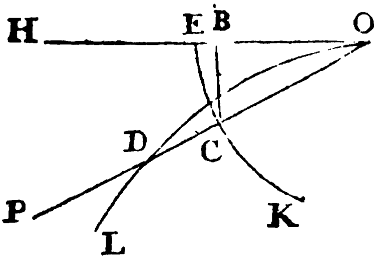
484. If the Angular Celerity be ſuppoſed uniform, and the Ring to have no other Motion along the Rod than what it acquires from its Centrifugal Force; then c, m and p being all of them equal to Nothing, $\dot{z}$ will here become, barely $=\frac{b\dot{x}}{\sqrt{-qa+\frac{qx^2}{a}}}=\frac{a\dot{x}}{\sqrt{x^2-a^2}}$: And

therefore $z = a \times$ Hyp. Log. $\frac{x + \sqrt{xx - aa}}{a}$. Hence if the Number whoſe Hyp. Log. is $\frac{z}{a}$ be denoted by N, we ſhall have $\frac{x + \sqrt{xx - aa}}{a} = N$: From which x is found $= a \times \overline{\frac{N}{2} + \frac{1}{2N}}$; whence $\dot{x}$ is, alſo, had $= \frac{a\dot{N}}{2} - \frac{a\dot{N}}{2N^2} = \frac{N\dot{z}}{2} - \frac{\dot{z}}{2N}$ (becauſe $\frac{\dot{N}}{N} = \frac{\dot{z}}{a}$). Therefore, it will be (*by Corol.* 1.) as Unity is to $\frac{N}{2} - \frac{1}{2N}$, ſo is the Angular Velocity (b) in the Arch ACK to the Velocity with which the Body recedes from the Center of Motion: And ſo, likewiſe, is the Centrifugal Force in that Arch to half the Preſſure upon the Rod—By taking $z =$ the whole Periphery, or $\frac{z}{a} = 2 \times \overline{3 . 145}$ &c. N will come out $= 535.5$, and $x = 267.7 \times a$: From whence it appears that the Diſtance of the Ring from the Center at the End of one intire Revolution will be almoſt 268 times as great as at firſt.

COROLLARY IV.

485. If a Body be ſuppoſed to deſcend from the Point O, (*ſee the next Fig.*) by the Force of its own Gravity, along an inclin'd Plane OCP; whilſt the Plane itſelf moves uniformly about that Point, from an horizontal Poſition OEH; then the Place, and the Preſſure of the Body upon the Plane, in any given Poſition OCP, may, alſo, be derived from the Equations in *Corollary* 1. For let CB (perpendicular to OH) be put $= y$; and let the Ratio of the Centrifugal Force in the Circle ECK, to the Force of Gravity (*given by Art.* 217.) be as r to *Unity*: Then, as the Meaſure of the former Force is expreſſed by $\frac{bb}{a}$,

That of the latter muſt be repreſented by $\frac{bb}{ra}$; and, conſequently, its Efficacy in the Direction PO, by $\frac{bby}{raa}\left(=\frac{bb}{ra}\times\frac{\mathrm{CB}}{\mathrm{OC}}\right)$: Which Value being ſubſtituted for $-w$, in the aforeſaid Corollary, we have $\ddot{x} - \frac{x\dot{z}^2}{aa} = \frac{y\dot{z}^2}{raa}$. But now, in order to the Solution of this



Equation, put the Radius OC (a) = 1 (that the Operation may be as ſimple as poſſible) alſo, inſtead of y, let its Equal $z - \frac{z^3}{2\cdot 3} + \frac{z^5}{2\cdot 3\cdot 4\cdot 5}$ * &c. be ſubſtituted, and let x be aſſumed $= Az^3 + Bz^5 + Cz^7 + Dz^9$ &c.

Then, by proceeding as is taught in *Art.* 267. the Value of x will come out $= \frac{1}{r}$ into $\frac{z^3}{2\cdot 3} + \frac{z^7}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \frac{z^{11}}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}$ + &c. Whence the Velocity $\left(\frac{b\dot{x}}{\dot{z}}\right)$ in the Plane, is, alſo, found $= \frac{b}{r}$ into $\frac{z^2}{2} + \frac{z^6}{2\cdot 3\cdot 4\cdot 5\cdot 6}$ &c. Which, therefore, is to

* Art. 425.

to (*b*) the angular Velocity of the Plane, in the Arch ECK, as $\frac{z^2}{2} + \frac{z^4}{2\cdot3\cdot4\cdot5\cdot6} +$ &c. to *r*. Moreover, the Centrifugal Force in the ſaid Arch being denoted by *r* (the Force of Gravity being Unity) it will likewiſe be (by the above-mentioned *Corol.*) as $1 : \frac{2\dot{x}}{\dot{x}} :: r : \left(\frac{2r\dot{x}}{\dot{x}} =\right)$ $z^2 + \frac{z^6}{3\cdot4\cdot5\cdot6} + \frac{z^{10}}{3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10} +$ &c. = the Force ſufficient to keep the Body upon the Plane. But the Force of Gravity in a Direction perpendicular to the Plane (the Weight of the Body being repreſented by *Unity*) is $\frac{OB}{OC} = 1 - \frac{z^2}{2} + \frac{z^4}{2\cdot3\cdot4}$ * &c. From which deducting the Quantity laſt found, there reſts $1 - \frac{3z^2}{2} + \frac{z^4}{2\cdot3\cdot4} - \frac{3z^6}{2\cdot3\cdot4\cdot5\cdot6}$ &c. for the true Preſſure of the Body upon the Plane. By putting of Which equal to Nothing, z^2 will be found = 0.67715; anſwering to an Angle (EOC) of 47° : 9′: Which Angle is therefore the Inclination, when the Force of Gravity is no longer ſufficient to keep the Body upon the Plane.

* Art. 425.

Though the Value of *x*, given above, is found by an Infinite Series, yet the Sum of that Series is eaſily exhibited by the Meaſures of Angles and Ratios. For, putting *N* to denote the Number whoſe hyperbolical Logarithm is *z*,

We have †

$$\begin{cases} 1 + z + \frac{z^2}{2} + \frac{z^3}{2\cdot3} + \frac{z^4}{2\cdot3\cdot4} \text{ \&c.} = N \\ 1 - z + \frac{z^2}{2} - \frac{z^3}{2\cdot3} + \frac{z^4}{2\cdot3\cdot4} \text{ \&c.} = \frac{1}{N}: \end{cases}$$

† Art. 424.

Half the Difference of which two Equations is $z + \frac{z^3}{2\cdot3} + \frac{z^5}{2\cdot3\cdot4\cdot5} + \frac{z^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7}$ &c. $= \frac{N}{2} - \frac{1}{2N}$:

From which taking $z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \frac{z^7}{2.3.4.5.6.7}$ &c. $= y$; and dividing the remainder by $2r$, there reſults $\left(\frac{1}{r} \times \overline{\frac{z^3}{2.3} + \frac{z^7}{2.3.4.5.6.7}\text{ \&c.}}\right) \frac{1}{2r} \times \overline{\frac{N}{2} - \frac{1}{2N}} - y$, for the true Value of x. Which, if required, may be expreſſed independent of r; by putting d for the Diſtance through which a Body, freely, deſcends in the firſt Second of Time, and taking b to denote the Velocity of the Plane, (*per Second*) in the Arch ECK: For then, the Ratio of the Centrifugal Force, in the ſaid Arch, to the Force of Gravity (or That of r to 1) being as $\frac{bb}{1}$ $\left(= \frac{bb}{\text{OC}}\right)$ to $2d$ *, we ſhall have $r = \frac{bb}{2d}$, and conſequently $x = \frac{d}{bb} \times \overline{\frac{N}{2} - \frac{1}{2N}} - y$.

* Art. 211.

By Computations, not very unlike Thoſe above, the Motion of the Moon's *Apogee*, and the principal Equations of the Lunar Orbit may be exhibited; by means of proper Approximations, derived from the general Equations in *Art.* 481. But this is a Conſideration that would require a Volume of itſelf, to treat it, from firſt Principles, with all the Attention and Perſpicuity ſuitable to the Importance of the Subject. I ſhall conclude this Work with the following ſhort Table of *Hyperbolical Logarithms*, drawn up and communicated by my ingenious Friend Mr. *John Turner*: Whereof the Uſe, in finding Fluents, will ſufficiently appear from the foregoing Pages. In the ſaid Table we have given the Hyperbolical Logarithms of every whole Number and hundredth Part of an Unit, from 1 to 10 (which Form is beſt adapted to the Purpoſes above-mentioned) by Help whereof, and the following Obſervations the Hyperbolical Logarithm of any Number, not exceeding ſeven Places of Figures, may be found with very little Trouble.

1°. *If the Number given be between* 1 *and* 10 *(ſo as to fall within the Limits of the Table.)*

Then take from it the next inferior Number in the Table, and divide the Remainder by the ſaid inferior Number increaſed by half the Remainder; and let the Quotient be added to the Logarithm of the ſaid inferior Number, the Sum will be the Logarithm ſought.

Thus, let the Hyperbolical Logarithm of 3.45678 be required; then the Operation will ſtand thus: 3.45339).005678(.0016442: Which added to 1.2383742, the Log. of 3.45, gives 1.2400184 for the Logarithm ſought.

2°. *When the Number propoſed exceeds* 10.

Find the Logarithm thereof, ſuppoſing all the Figures after the Firſt to be Decimals; then to the Logarithm, ſo found, let 2.3025851, 4.6051702, or 6.9077553 *&c.* be added, according as the whole Number conſiſts of 2, 3, or 4 *&c.* Places: The Sum will be the Logarithm ſought.

Thus, the Hyperbolical Logarithm of 345.678 will be found to be 5.8451886: For That of 3.45678 being 1.2400184; the ſame, added to 4.6051702, gives the very Quantity above exhibited. The Reaſon of which, as well as of the Operation in the preceding Caſe, is evident from the Nature and Conſtruction of Logarithms.

N	Logarithm	N	Logarithm	N	Logarithm
1.01	.0099503	1.34	.2926696	1.67	.5128236
1.02	.0198026	1.35	.3001045	1.68	.5187937
1.03	.0295588	1.36	.3074846	1.69	.5247285
1.04	.0392207	1.37	.3148107	1.70	.5306282
1.05	.0487902	1.38	.3220834	1.71	.5364933
1.06	.0582689	1.39	.3293037	1.72	.5423242
1.07	.0676586	1.40	.3364722	1.73	.5481214
1.08	.0769610	1.41	.3435897	1.74	.5538851
1.09	.0861777	1.42	.3506568	1.75	.5596157
1.10	.0953102	1.43	.3576744	1.76	.5653138
1.11	.1043600	1.44	.3646431	1.77	.5709795
1.12	.1133287	1.45	.3715635	1.78	.5766133
1.13	.1222176	1.46	.3784364	1.79	.5822156
1.14	.1310283	1.47	.3852624	1.80	.5877866
1.15	.1397619	1.48	.3920420	1.81	.5933268
1.16	.1484200	1.49	.3987761	1.82	.5988365
1.17	.1570037	1.50	.4054651	1.83	.6043159
1.18	.1655144	1.51	.4121096	1.84	.6097655
1.19	.1739533	1.52	.4187103	1.85	.6151856
1.20	.1823215	1.53	.4252677	1.86	.6205764
1.21	.1906203	1.54	.4317824	1.87	.6259384
1.22	.1988508	1.55	.4382549	1.88	.6312717
1.23	.2070141	1.56	.4446858	1.89	.6365768
1.24	.2151113	1.57	.4510756	1.90	.6418538
1.25	.2231435	1.58	.4574248	1.91	.6471032
1.26	.2311117	1.59	.4637340	1.92	.6523251
1.27	.2390169	1.60	.4700036	1.93	.6575200
1.28	.2468600	1.61	.4762341	1.94	.6626879
1.29	.2546422	1.62	.4824261	1.95	.6678293
1.30	.2623642	1.63	.4885800	1.96	.6729444
1.31	.2700271	1.64	.4946962	1.97	.6780335
1.32	.2776317	1.65	.5007752	1.98	.6830968
1.33	.2851789	1.66	.5068175	1.99	.6881346
1.34	.2926696	1.67	.5128236	2.00	.6931472

N	Logarithm	N	Logarithm	N	Logarithm
2.01	.6981347	2.34	.8501509	2.67	.9820784
2.02	.7030974	2.35	.8544153	2.68	.9858167
2.03	.7080357	2.36	.8586616	2.69	.9895411
2.04	.7129497	2.37	.8628899	2.70	.9932517
2.05	.7178397	2.38	.8671004	2.71	.9969486
2.06	.7227059	2.39	.8712933	2.72	1.0006318
2.07	.7275485	2.40	.8754687	2.73	1.0043015
2.08	.7323678	2.41	.8796267	2.74	1.0079579
2.09	.7371640	2.42	.8837675	2.75	1.0116008
2.10	.7419373	2.43	.8878912	2.76	1.0152306
2.11	.7466879	2.44	.8919980	2.77	1.0188473
2.12	.7514160	2.45	.8960880	2.78	1.0224509
2.13	.7561219	2.46	.9001613	2.79	1.0260415
2.14	.7608058	2.47	9042181	2.80	1.0296194
2.15	.7654678	2.48	.9082585	2.81	1.0331844
2.16	.7701082	2.49	.9122826	2.82	1.0367368
2.17	.7747271	2.50	.9162907	2.83	1.0402766
2.18	.7793248	2.51	.9202827	2.84	1.0438040
2.19	.7839015	2.52	.9242589	2.85	1.0473189
2.20	.7884573	2.53	.9282193	2.86	1.0508216
2.21	.7929925	2.54	.9321640	2.87	1.0543120
2.22	.7275071	2.55	.9360933	2.88	1.0577902
2.23	.8020015	2.56	.9400072	2.89	1.0612564
2.24	.8064758	2.57	.9439058	2.90	1.0647107
2.25	.8109302	2.58	.9477893	2.91	1.0681530
2.26	.8153648	2.59	.9516578	2.92	1.0715836
2.27	.8197798	2.60	.9555114	2.93	1.0750024
2.28	.8241754	2.61	.9593502	2.94	1.0784095
2.29	.8285518	2.62	.9631743	2.95	1.0818051
2.30	.8329091	2.63	.9669838	2.96	1.0851892
2.31	.8372475	2.64	.9707789	2.97	1.0885619
2.32	.8415671	2.65	.9745596	2.98	1.0919233
2.33	.8458682	2.66	.9783261	2.99	1.0952733
2.34	.8501509	2.67	.9820784	3.00	1.0986123

N	Logarithm	N	Logarithm	N	Logarithm
3.01	1.1019400	3.34	1.2059707	3.67	1.3001916
3.02	1.1052568	3.35	1.2089603	3.68	1.3029127
3.03	1.1085626	3.36	1.2119409	3.69	1.3056264
3.04	1.1118575	3.37	1.2149127	3.70	1.3083328
3.05	1.1151415	3.38	1.2178757	3.71	1.3110318
3.06	1.1184149	3.39	1.2208299	3.72	1.3137236
3.07	1.1216775	3.40	1.2237754	3.73	1.3164082
3.08	1.1249295	3.41	1.2267122	3.74	1.3190856
3.09	1.1281710	3.42	1.2296405	3.75	1.3217558
3.10	1.1314021	3.43	1.2325605	3.76	1.3244189
3.11	1.1346227	3.44	1.2354714	3.77	1.3270749
3.12	1.1378330	3.45	1.2383742	3.78	1.3297240
3.13	1.1410330	3.46	1.2412685	3.79	1.3323660
3.14	1.1442227	3.47	1.2441545	3.80	1.3350010
3.15	1.1474024	3.48	1.2470322	3.81	1.3376291
3.16	1.1505720	3.49	1.2499017	3.82	1.3402504
3.17	1.1537315	3.50	1.2527629	3.83	1.3428648
3.18	1.1568811	3.51	1.2556160	3.84	1.3454723
3.19	1.1600209	3.52	1.2584609	3.85	1.3480731
3.20	1.1631508	3.53	1.2612978	3.86	1.3506671
3.21	1.1662709	3.54	1.2641266	3.87	1.3532544
3.22	1.1693813	3.55	1.2669475	3.88	1.3558351
3.23	1.1724821	3.56	1.2697605	3.89	1.3584091
3.24	1.1755733	3.57	1.2725655	3.90	1.3609765
3.25	1.1786549	3.58	1.2753627	3.91	1.3635373
3.26	1.1817271	3.59	1.2781521	3.92	1.3660916
3.27	1.1847899	3.60	1.2809338	3.93	1.3686394
3.28	1.1878434	3.61	1.2837077	3.94	1.3711807
3.29	1.1908875	3.62	1.2864740	3.95	1.3737156
3.30	1.1939224	3.63	1.2892326	3.96	1.3762440
3.31	1.1969481	3.64	1.2919836	3.97	1.3787661
3.32	1.1999647	3.65	1.2947271	3.98	1.3812818
3.33	1.2029722	3.66	1.2974631	3.99	1.3837912
3.34	1.2059707	3.67	1.3001916	4.00	1.3862943

N	Logarithm	N	Logarithm	N	Logarithm
4.01	1.3887912	4.34	1.4678743	4.67	1.5411590
4.02	1.3912818	4.35	1.4701758	4.68	1.5432981
4.03	1.3937663	4.36	1.4724720	4.69	1.5454325
4.04	1.3962446	4.37	1.4747630	4.70	1.5475625
4.05	1.3987168	4.38	1.4770487	4.71	1.5496879
4.06	1.4011829	4.39	2.4793292	4.72	1.5518087
4.07	1.4036429	4.40	1.4816045	4.73	1.5539252
4.08	1.4060969	4.41	1.4838746	4.74	1.5560371
4.09	1.4085449	4.42	1.4861396	4.75	1.5581446
4.10	1.4109869	4.43	1.4883995	4.76	1.5602476
4.11	1.4134230	4.44	.4906543	4.77	1.5623462
4.12	1.4158531	4.45	1.4929040	4.78	1.5644405
4.13	1.4182774	4.46	1.4951487	4.79	1.5665304
4.14	1.4206957	4.47	1.4973883	4.80	1.5686159
4.15	1.4231083	4.48	1.4996230	4.81	1.5706971
4.16	1.4255150	4.49	1.5018527	4.82	1.5727739
4.17	1.4279160	4.50	1.5040774	4.83	1.5748464
4.18	1.4303112	4.51	1.5062971	4.84	1.5769147
4.19	1.4327007	4.52	1.5085119	4.85	1.5789787
4.20	1.4350845	4.53	1.5107219	4.86	1.5810384
4.21	1.4374626	4.54	1.5129269	4.87	1.5830939
4.22	1.4398351	4.55	1.5151272	4.88	1.5851452
4.23	1.4422020	4.56	1.5173226	4.89	1.5871923
4.24	1.4445632	4.57	1.5195132	4.90	1.5892352
4.25	1.4469189	4.58	1.5216990	4.91	1.5912739
4.26	1.4492691	4.59	1.5238800	4.92	1.5933085
4.27	1.4516138	4.60	1.5260563	4.93	1.5953389
4.28	1.4539530	4.61	1.5282278	4.94	1.5973653
4.29	1.4562867	4.62	1.5303947	4.95	1.5993875
4.30	1.4586149	4.63	1.5325568	4.96	1.6014057
4.31	1.4609379	4.64	1.5347143	4.97	1.6034198
4.32	1.4632553	4.65	1.5368672	4.98	1.6054298
4.33	1.4655675	4.66	1.5390154	4.99	1.6074358
4.34	1.4678743	4.67	1.5411590	5.00	1.6094379

N	Logarithm	N	Logarithm	N	Logarithm
5.01	1.6114359	5.34	1.6752256	5.67	1.7351891
5.02	1.6134300	5.35	1.6770965	5.68	1.7369512
5.03	1.6154200	5.36	1.6789639	5.69	1.7387102
5.04	1.6174060	5.37	1.6808278	5.70	1.7404661
5.05	1.6193882	5.38	1.6826882	5.71	1.7422189
5.06	1.6213664	5.39	1.6845453	5.72	1.7439687
5.07	1.6233408	5.40	1.6863989	5.73	1.7457155
5.08	1.6253112	5.41	1.6882491	5.74	1.7474591
5.09	1.6272778	5.42	1.6900958	5.75	1.7491998
5.10	1.6292405	5.43	1.6919391	5.76	1.7509374
5.11	1.6311994	5.44	1.6937790	5.77	1.7526720
5.12	1.6331544	5.45	1.6956155	5.78	1.7544036
5.13	1.6351056	5.46	1.6974487	5.79	1.7561323
5.14	1.6370530	5.47	1.6992786	5.80	1.7578579
5.15	1.6389967	5.48	1.7011051	5.81	1.7595805
5.16	1.6409365	5.49	1.7029282	5.82	1.7613002
5.17	1.6428726	5.50	1.7047481	5.83	1.7630170
5.18	1.6448050	5.51	1.7065646	5.84	1.7647308
5.19	1.6467336	5.52	1.7083778	5.85	1.7664416
5.20	1.6486586	5.53	1.7101878	5.86	1.7681496
5.21	1.6505798	5.54	1.7119944	5.87	1.7698546
5.22	1.6524974	5.55	1.7137979	5.88	1.7715567
5.23	1.6544112	5.56	1.7155981	5.89	1.7732559
5.24	1.6563214	5.57	1.7173950	5.90	1.7749523
5.25	1.6582280	5.58	1.7191887	5.91	1.7766458
5.26	1.6601310	5.59	1.7209792	5.92	1.7783364
5.27	1.6620303	5.60	1.7227666	5.93	1.7800242
5.28	1.6639260	5.61	1.7245507	5.94	1.7817091
5.29	1.6658182	5.62	1.7263316	5.95	1.7833912
5.30	1.6677068	5.63	1.7281094	5.96	1.7850704
5.31	1.6695918	5.64	1.7298840	5.97	1.7867469
5.32	1.6714733	5.65	1.7316555	5.98	1.7884205
5.33	1.6733512	5.66	1.7334238	5.99	1.7900914
5.34	1.6752256	5.67	1.7351891	6.00	1.7917594

N	Logarithm	N	Logarithm	N	Logarithm
6.01	1.7934247	6.34	1.8468787	6.67	1.8976198
6.02	1.7950872	6.35	1.8484547	6.68	1.8991179
6.03	1.7967470	6.36	1.8500283	6.69	1.9006138
6.04	1.7984040	6.37	1.8515994	6.70	1.9021078
6.05	1.8000582	6.38	1.8531680	6.71	1.9035985
6.06	1.8017098	6.39	1.8547342	6.72	1.9050881
6.07	1.8033586	6.40	1.8562979	6.73	1.9065751
6.08	1.8050047	6.41	1.8578592	6.74	1.9080600
6.09	1.8066481	6.42	1.8594181	6.75	1.9095425
6.10	1.8082887	6.43	1.8609745	6.76	1.9110228
6.11	1.8099267	6.44	1.8625285	6.77	1.9125011
6.12	1.8115621	6.45	1.8640801	6.78	1.9139771
6.13	1.8131947	6.46	1.8656293	6.79	1.9154509
6.14	1.8148247	6.47	1.8671761	6.80	1.9169226
6.15	1.8164520	6.48	1.8687205	6.81	1.9183921
6.16	1.8180767	6.49	1.8702625	6.82	1.9198594
6.17	1.8196988	6.50	1.8718021	6.83	1.9213247
6.18	1.8213182	6.51	1.8733394	6.84	1.9227877
6.19	1.8229351	6.52	1.8748743	6.85	1.9242486
6.20	1.8245493	6.53	1.8764069	6.86	1.9257074
6.21	1.8261608	6.54	1.8779371	6.87	1.9271641
6.22	1.8277699	6.55	1.8794650	6.88	1.9286186
6.23	1.8293763	6.56	1.8809906	6.89	1.9300710
6.24	1.8309801	6.57	1.8825138	6.90	1.9315214
6.25	1.8325814	6.58	1.8840347	6.91	1.9329696
6.26	1.8341801	6.59	1.8855533	6.92	1.9344157
6.27	1.8357763	6.60	1.8870696	6.93	1.9358598
6.28	1.8373699	6.61	1.8885837	6.94	1.9373017
6.29	1.8389610	6.62	1.8900954	6.95	1.9387416
6.30	1.8405496	6.63	1.8916048	6.96	1.9401794
6.31	1.8421356	6.64	1.8931119	6.97	1.9416152
6.32	1.8437191	6.65	1.8946168	6.98	1.9430489
6.33	1.8453002	6.66	1.8961194	6.99	1.9444805
6.34	1.8468787	6.67	1.8976198	7.00	1.9459101

N	Logarithm	N	Logarithm	N	Logarithm
7.01	1.9473376	7.34	1.9933387	7.67	2.0373166
7.02	1.9487632	7.35	1.9947002	7.68	2.0386195
7.03	1.9501866	7.36	1.9960599	7.69	2.0399207
7.04	1.9516080	7.37	1.9974177	7.70	2.0412203
7.05	1.9530275	7.38	1.9987736	7.71	2.0425181
7.06	1.9544449	7.39	2.0001278	7.72	2.0438143
7.07	1.9558604	7.40	2.0014800	7.73	2.0451088
7.08	1.9572739	7.41	2.0028305	7.74	2.0464016
7.09	1.9586853	7.42	2.0041790	7.75	2.0476928
7.10	1.9600947	7.43	2.0055258	7.76	2.0489823
7.11	1.9615022	7.44	2.0068708	7.77	2.0502701
7.12	1.9629077	7.45	2.0082140	7.78	2.0515563
7.13	1.9643112	7.46	2.0095553	7.79	2.0528408
7.14	1.9657127	7.47	2.0108949	7.80	2.0541237
7.15	1.9671123	7.48	2.0122327	7.81	2.0554049
7.16	1.9685099	7.49	2.0135687	7.82	2.0566845
7.17	1.9699056	7.50	2.0149030	7.83	2.0579624
7.18	1.9712993	7.51	2.0162354	7.84	2.0592388
7.19	1.9726911	7.52	2.0175661	7.85	2.0605135
7.20	1.9740810	7.53	2.0188950	7.86	2.0617866
7.21	1.9754689	7.54	2.0202221	7.87	2.0630580
7.22	1.9768549	7.55	2.0215475	7.88	2.0643278
7.23	1.9782390	7.56	2.0228711	7.89	2.0655961
7.24	1.9796212	7.57	2.0241929	7.90	2.0668627
7.25	1.9810014	7.58	2.0255131	7.91	2.0681277
7.26	1.9823798	7.59	2.0268315	7.92	2.0693911
7.27	1.9837562	7.60	2.0281482	7.93	2.0706530
7.28	1.9851308	7.61	2.0294631	7.94	2.0719132
7.29	1.9865035	7.62	2.0307763	7.95	2.0731719
7.30	1.9878743	7.63	2.0320878	7.96	2.0744290
7.31	1.9892432	7.64	2.0333976	7.97	2.0756845
7.32	1.9906103	7.65	2.0347056	7.98	2.0769384
7.33	1.9919754	7.66	2.0360119	7.99	2.0781907
7.34	1.9933387	7.67	2.0373166	8.00	2.0794415

N	Logarithm	N	Logarithm	N	Logarithm
8.01	2.0806907	8.34	2.1210632	8.67	2.1598687
8.02	2.0819384	8.35	2.1222615	8.68	2.1610215
8.03	2.0831845	8.36	2.1234584	8.69	2.1621729
8.04	2.0844290	8.37	2.1246539	8.70	2.1633230
8.05	2.0856720	8.38	2.1258479	8.71	2.1644718
8.06	2.0869135	8.39	2.1270405	8.72	2.1656192
8.07	2.0881534	8.40	2.1282317	8.73	2.1667653
8.08	2.0893918	8.41	2.1294214	8.74	2.1679101
8.09	2.0906287	8.42	2.1306098	8.75	2.1690536
8.10	2.0918640	8.43	2.1317967	8.76	2.1701959
8.11	2.0930984	8.44	2.1329822	8.77	2.1713367
8.12	2.0943306	8.45	2.1341664	8.78	2.1724763
8.13	2.0955613	8.46	2.1353491	8.79	2.1736146
8.14	2.0967905	8.47	2.1365304	8.80	2.1747517
8.15	2.0980182	8.48	2.1377104	8.81	2.1758874
8.16	2.0992444	8.49	2.1388889	8.82	2.1770218
8.17	2.1004691	8.50	2.1400661	8.83	2.1781550
8.18	2.1016923	8.51	2.1412419	8.84	2.1792868
8.19	2.1029140	8.52	2.1424163	8.85	2.1804174
8.20	2.1041341	8.53	2.1435893	8.86	2.1815467
8.21	2.1053529	8.54	2.1447609	8.87	2.1826747
8.22	2.1065702	8.55	2.1459312	8.88	2.1838015
8.23	2.1077861	8.56	2.1471001	8.89	2.1849270
8.24	2.1089998	8.57	2.1482676	8.90	2.1860512
8.25	2.1102128	8.58	2.1494339	8.91	2.1871742
8.26	2.1114243	8.59	2.1505987	8.92	2.1882959
8.27	2.1126343	8.60	2.1517622	8.93	2.1894163
8.28	2.1138428	8.61	2.1529243	8.94	2.1905355
8.29	2.1150499	8.62	2.1540851	8.95	2.1916535
8.30	2.1162555	8.63	2.1552445	8.96	2.1927702
8.31	2.1174596	8.64	2.1564026	8.97	2.1938856
8.32	2.1186622	8.65	2.1575593	8.98	2.1949998
8.33	2.1198634	8.66	2.1587147	8.99	2.1961128
8.34	2.1210632	8.67	2.1598687	9.00	2.1972245